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ON IDEMPOTENT MODIFICATIONS OF MV-ALGEBRAS

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Abstract. The notion of idempotent modification of an algebra was introduced by Ježek. He proved that the idempotent modification of a group is subdirectly irreducible. For an MV-algebra $\mathscr{A}$ we denote by $\mathscr{A}', A$ and $\ell(\mathscr{A})$ the idempotent modification, the underlying set or the underlying lattice of $\mathscr{A}$, respectively. In the present paper we prove that if $\mathscr{A}$ is semisimple and $\ell(\mathscr{A})$ is a chain, then $\mathscr{A}'$ is subdirectly irreducible. We deal also with a question of Ježek concerning varieties of algebras.

Keywords: MV-algebra, idempotent modification, subdirect reducibility

MSC 2000: 06D35

1. Introduction

The notion of idempotent modification $\mathscr{A}'$ of an algebra $\mathscr{A}$ was introduced by Ježek [8]. It is defined as follows. Suppose that A and F are the underlying set of $\mathscr{A}$ and the set of fundamental operations of $\mathscr{A}$, respectively. The underlying set of $\mathscr{A}'$ is equal to A; the system F' of fundamental operations of $\mathscr{A}'$ consists of operations f', where $f \in F$ and

1) if f is a nullary operation, then $f' = f$;
2) if f is an n-ary operation, $n \in \mathbb{N}$, and if $a_1, \ldots, a_n \in A$, then

$$f'(a_1, \ldots, a_n) = \begin{cases} a_1 & \text{if } a_1 = a_2 = \ldots = a_n, \\ f(a_1, \ldots, a_n) & \text{otherwise.} \end{cases}$$

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Let $\mathscr{C}$ be a class of algebras. Consider the following condition for $\mathscr{C}$.

(c_1) If $\mathscr{A} \in \mathscr{C}$, then $\mathscr{A}'$ is subdirectly irreducible.

The main result of [9] is the following theorem:

(α) (Cf. [9], Theorem 1.) The class of all groups satisfies condition (c_1).

In the mentioned paper, Ježek remarks that it would be interesting to find another variety with the property of Theorem 1.

When we consider the idempotent modification of an MV-algebra, then the following fact must be taken into account. For defining the notion of an MV-algebra, different systems of axioms have been applied in literature (cf., e.g., Chang [2], Cignoli, D'Ottaviano and Mundici [3], Dvurečenskij and Pulmannová [4], Glushankof [6], Cattaneo and Lombardo [1]). An operation which is considered as fundamental in one of these systems can be taken as a derived operation in another system. In all cases, by means of the fundamental operations we can define binary operations $\vee$ and $\wedge$ on the corresponding underlying set A of the MV-algebra $\mathscr{A}$ such that $(A; \vee, \wedge)$ turns out to be a lattice.

By defining the idempotent modification, the question which operations are considered to be fundamental is essential.

In the approach of the present paper, we will apply the axioms from [2] with the distinction that we add the operations $\vee$ and $\wedge$ to the system of fundamental operations. For the detailed formulation, cf. Section 2 below.

We prove the following result

(β) Let $\mathscr{C}_1$ be the class of all MV-algebras $\mathscr{A}$ such that $\mathscr{A}$ is semisimple and the underlying lattice $(A; \vee, \wedge)$ is a chain. Then $\mathscr{C}_1$ satisfies condition (c_1).

We remark that $\mathscr{C}_1$ fails to be a variety. There exists an infinite set of mutually nonisomorphic MV-algebras belonging to $\mathscr{C}_1$.

In the last section of the paper we deal with the suggestion proposed by Ježek. We construct a variety $\mathscr{V}$ such that for each algebra $\mathscr{A} \in \mathscr{V}$, the idempotent modification $\mathscr{A}'$ of $\mathscr{A}$ is subdirectly irreducible. Applying $\mathscr{V}$, an infinite system of varieties having the analogous property can be defined.

2. Preliminaries

The notion of an MV-algebra was introduced by Chang [2] as an algebraic description of many valued logics. It was investigated by several authors using different systems of axioms.

We recall the system of axioms from [2]. Suppose that A is a nonempty set, $\oplus$ and $\odot$ are binary operations, $\neg$ is a unary operation, and $0, 1$ are nullary operations (i.e., constants) on A. By means of these operations we define binary operations $\vee$ and $\wedge$ on A putting

(1) $x \vee y = (x \odot \neg y) \oplus y$,
(2) $x \wedge y = (x \oplus \neg y) \odot y$.

2.1. Definition. The algebraic structure $\mathscr{A} = (A; \oplus, \odot, \neg, 0, 1)$ is an MV-algebra if $\vee$, $\wedge$ are binary operations on A defined by (1) and (2) and if the following axioms are satisfied:

Ax. 1. $x \oplus y = y \oplus x$,
Ax. 1′. $x \odot y = y \odot x$,
Ax. 2. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
Ax. 2′. $(x \odot y) \odot z = x \odot (y \odot z)$,
Ax. 3. $x \oplus \neg x = 1$,
Ax. 3′. $x \odot \neg x = 0$,
Ax. 4. $x \oplus 1 = 1$,
Ax. 4′. $x \odot 0 = 0$,
Ax. 5. $x \oplus 0 = x$,
Ax. 5′. $x \odot 1 = x$,
Ax. 6. $\neg(x \oplus y) = \neg x \odot \neg y$,
Ax. 6′. $\neg(x \odot y) = \neg x \oplus \neg y$,
Ax. 7. $x = \neg(\neg x)$,
Ax. 8. $\neg 0 = 1$,
Ax. 9. $x \vee y = y \vee x$,
Ax. 9′. $x \wedge y = y \wedge x$,
Ax. 10. $x \vee (z \vee z) = (x \vee y) \vee z$,
Ax. 10′. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
Ax. 11. $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
Ax. 11′. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

As we have already mentioned in Section 1 above, we modify the method from [2] in such a way that we consider the operations $\vee$ and $\wedge$ as belonging to the fundamental operations of $\mathscr{A}$. In other words, we deal with the algebra $(A; \oplus, \odot, \neg, 0, 1, \vee, \wedge)$ of type $(2, 2, 1, 0, 0, 2, 2)$ and we take as axioms the system from 2.1 augmented by the relations (1) and (2) considered as axioms. Below, the term 'MV-algebra' has always the just mentioned meaning.

It is clear that homomorphic images, subalgebras and direct products remain the same in both formulations.

In 2.2–2.4 we recall some well-known facts on MV-algebras (cf. e.g., [3], [4]).

2.2. The algebraic structure $\ell(\mathscr{A}) = (A; \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1.

2.3. Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For each $x, y \in A$ we put

$$x \oplus y = (x + y) \wedge u, \quad \neg x = u - x, \quad 1 = u,$$
$$x \odot y = \neg(\neg x \oplus \neg y).$$

Then $\mathscr{A} = (A; \oplus, \odot, \neg, 0, 1, \vee, \wedge)$ is an MV-algebra; it will be denoted by $\Gamma(G, u)$.

2.4. Let $\mathscr{A}$ be an MV-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $\mathscr{A} = \Gamma(G, u)$.

In view of 2.3 and 2.4 we conclude that

$$(*) \qquad x \odot y = \neg(\neg x \oplus \neg u)$$

for each MV-algebra.

In what follows, when speaking about an MV-algebra $\mathscr{A}$, we always suppose that G and u are as in 2.4.

The partial order on A (or on G) induced by the operations $\vee$ and $\wedge$ will be denoted by $\leqslant$.

An MV-algebra $\mathscr{A}$ is *semisimple* (or *archimedean*) if for any nonzero elements x_1 and x_2 of A there exists a positive integer n such that $nx_1 \nleqslant x_2$.

Semisimple MV-algebras have been investigated by several authors; cf., e.g., the monograph [3], and the references in this monograph.

We say that an MV-algebra $\mathscr{A}$ is *linearly ordered* if the lattice $(A; \vee, \wedge)$ is a chain.

3. Two-element congruence classes

For an algebra $\mathscr{A}$ with the underlying set A we denote by $\operatorname{Con} \mathscr{A}$ the system of all congruence relations of $\mathscr{A}$; this system is partially ordered in the usual way. Then $\operatorname{Con} \mathscr{A}$ is a complete lattice. Its least element will be denoted by $\sim_0$.

It is well-known that $\mathscr{A}$ is subdirectly reducible if and only if there exists a system $\sim_i\}_{i \in I}$ of elements of $\operatorname{Con} \mathscr{A}$ such that $\bigwedge\limits_{i \in I} \sim_i = \sim_0$ and $\sim_i \neq \sim_0$ for each $i \in I$.

In the opposite case, $\mathscr{A}$ is *subdirectly irreducible*. Thus if $\operatorname{card} A \leqslant 2$, then $\mathscr{A}$ is subdirectly irreducible.

Suppose that $\mathscr{A}$ is an MV-algebra and $\sim \in \operatorname{Con} \mathscr{A}'$. Further, let $\sim_m$ be the greatest element of $\operatorname{Con} \mathscr{A}'$. If $\operatorname{card} A \leqslant 2$, then $\sim \in \{\sim_0, \sim_m\}$. In what follows we assume that $\operatorname{card} A > 2$. For $a \in A$ we put $\overline{a} = \{x \in A : x \sim a\}$.

Lemma 3.1. *Let $a \in A$. Then $\overline{a}$ is a convex sublattice of the lattice $(A; \vee, \wedge)$. If $x, y \in \overline{a}$ and $x \neq y$, then $x \oplus y \in \overline{a}$ and $x \odot y \in \overline{a}$.*

Proof. Since $\vee' = \vee$ and $\wedge' = \wedge$ we conclude that $\sim$ is a congruence of the lattice $(A; \vee, \wedge)$; it is well-known that each congruence class of a lattice is a convex sublattice. Let $x, y \in \overline{a}$, $x \neq y$. Then $x \oplus y = x \oplus' y \sim a \oplus' a = a$, whence $x \oplus y$ belongs to $\overline{a}$. Similarly we verify that $x \odot y$ belongs to $\overline{a}$. $\square$

Let $\mathbb{Z}$ be the additive group of all integers with the natural linear order. Put $u = 2$; then u is a strong unit of the linearly ordered group $\mathbb{Z}$. Consider the MV-algebra $\mathscr{A}_1 = \Gamma(\mathbb{Z}, u)$.

Lemma 3.1.1. *The idempotent modification $\mathscr{A}_1'$ of $\mathscr{A}_1$ is simple.*

Proof. We denote by A_1 the underlying set of $\mathscr{A}_1$; hence $A_1 = \{0, 1, 2\}$. In view of 3.1 it suffices to deal with the partitions

$$\varrho_1 \{\{0\}, \{1, 2\}\}, \quad \varrho_2 = \{\{0, 1\}, \{2\}\}$$

of the set A_1. For $i \in \{1, 2\}$ let $\sim_i$ be the equivalence on A_1 corresponding to ϱ_i.

We have $1 \varrho_1 2$, but the relation $\neg' 1 \ \varrho_1 \ \neg' 2$ fails to be valid. Also, $0 \varrho_2 1$, but $\neg' 0 \ \varrho_2 \ \neg' 1$ does not hold. Hence neither ϱ_1 nor ϱ_2 is a congruence relation on $\mathscr{A}_1'$. Therefore $\mathscr{A}_1'$ is simple. $\square$

In the remaining part of this section we assume that the lattice $(A; \vee, \wedge)$ is a chain. It is well-known that in this case the lattice ordered group G is linearly ordered. We will be interested in two-element congruence classes of the congruence $\sim$.

Suppose that $a \in A$ and that $\overline{a}$ is a two-element set, i.e., $\overline{a} = \{a, b\}$ with $a \neq b$. Then in view of 4.1, $\{a, b\}$ must be a chain and $a \oplus b \in \{a, b\}$. Without loss of generality we can assume that $a < b$. We have $a \oplus b \geqslant b$, thus

$$b = a \oplus b = (a + b) \wedge u.$$

If $a + b \geqslant u$, then $(a + b) \wedge u = u$, hence $b = u$. If $a + b < u$, then $(a + b) \wedge u = a + b$, thus $a + b = b$ and so $a = 0$. We obtain

Lemma 3.2. *Assume that $\overline{a} = \{a, b\}$ is a two-element set and $a < b$. Then we have either $a = 0$ or $b = u$.*

Lemma 3.3. *Let $\overline{a}$ be as in 3.2 and let $a = 0$. If $b = u$, then $\overline{a} = A$. If $b + b = u$, then A is a three element set, namely, $A = \{a, b, u\}$.*

P r o o f. The first assertion is obvious. Suppose that $b+b = u$. Since the interval $[0, b + b]$ of the lattice $(A; \vee, \wedge)$ is isomorphic to the interval $[0, b]$ and $[0, b] = \{0, b\}$, we get $[b, b + b] = \{b, b + b\} = \{b, u\}$. Because the interval $[0, u]$ is a chain we obtain that $A = [0, u] = \{0, b, u\}$ with $0 < b < u$. $\square$

We remark that in the case $u = 0$ and $b + b = u$ we have the same situation as in Lemma 3.1.1. Thus in this case, the algebra $\mathscr{A}'$ is subdirectly irreducible.

Again, let $a = 0$ and let us now suppose that $b + b \neq u$. We cannot have $b + b > u$, since this relation would yield $\operatorname{card}[b, b + b] > 2$, which is impossible. Let us apply the usual notation $b + b = 2b$, $b + b + b = 3b$.

The interval $[2b, 3b]$ of G is a two-element set, hence we cannot have $3b > u$; thus either $3b = u$ or $3b < u$.

Suppose that $3b = u$. Hence $2b = \neg b$ and then $b \neq \neg b$. We get

$$u = b \oplus \neg b = b \oplus' \neg b \sim 0 \oplus' \neg b = 0 \oplus \neg b = \neg b.$$

This yields that $A = \{0, b, 2b, u\}$ and $\sim$ has exactly two congruence classes, namely $\{0, b\}$ and $\{2b, u\}$. If $\sim_1$ is a congruence on $\mathscr{A}'$ such that $\sim_1 \neq \{\sim, \sim_0, \sim_m\}$, then the partition of A corresponding to $\sim_1$ must have the form $\{\{0\}, \{b, 2b\}, \{u\}\}$. In view of $b \sim_1 2b$ and in view of 3.2 we arrive at a contradiction. Hence we have

Lemma 3.4. *Let $\overline{a}$ be as in 3.2, $a = 0$ and $3b = u$. Then A is a four-element set and $\mathscr{A}'$ is subdirectly irreducible.*

We return to the assumption as above with the distinction that we suppose that $3b < u$. In this case we have $b \neq 2b$, $0 \neq 2b$, hence

$$0 \oplus' 2b = 0 \oplus 2b = 2b, \quad b \oplus' 2b = b \oplus 2b = b + 2b = 3b.$$

Since $0 \sim b$ we get $2b \sim 3b$. Also, $2b \neq \neg b$.

If $3b \neq \neg b$, then

$$2b \oplus' \neg b = 2b \oplus \neg b = 2b + (\neg b) = b,$$
$$3b \oplus' \neg b = 3b \oplus \neg b = 3b + (\neg b) = 2b,$$

hence $b \sim 2b$, which is a contradiction.

If $3b = \neg b$, then

$$3b \oplus' \neg b = 3b,$$
$$3b \oplus' \neg b \sim 2b \oplus' \neg b = b,$$

thus $b \sim 3b$; again, we arrive at a contradiction.

Summarizing, we obtain

Lemma 3.5. *Let $\mathscr{A}$ be an MV-algebra such that the lattice $(A; \vee, \wedge)$ is a chain. Let $\sim \in \operatorname{Con} \mathscr{A}$, $a \in A$ and assume that $\overline{a} = \{a, b\}$, $a < b$. Then some of the following conditions is satisfied:*

(i) $b = u$ *(i.e.,* $\operatorname{card} A = 2$*);*
(ii) *A is a three-element set, i.e., $A = \{0, b, u\}$, and $\mathscr{A}'$ is subdirectly irreducible;*
(iii) *A is a four-element set, $A = \{0, b, 2b, u\}$ and $\mathscr{A}'$ is subdirectly irreducible.*

Again, let us apply the assumptions and the notation as in 3.2. Suppose that $b = u$. Now we can apply the analogous method as above with the distinction that instead of dealing with the operation $\oplus'$ we deal with the operation $\odot'$. We obtain a result analogous to 3.5. Thus we have

Proposition 3.6. *Let $\mathscr{A}$ be an MV-algebra such that the lattice $(A; \vee, \wedge)$ is a chain. Let $\sim \in \operatorname{Con} \mathscr{A}'$ and suppose that there exists $a \in A$ with $\operatorname{card} \overline{a} = 2$. Then some of the following conditions is satisfied:*

(i) $\operatorname{card} A = 2$*;*
(ii) $\operatorname{card} A = 3$ *and $\mathscr{A}'$ is subdirectly irreducible;*
(iii) $\operatorname{card} A = 4$ *and $\mathscr{A}'$ is subdirectly irreducible.*

It is easy to verify that if $\mathscr{A}$ and $\mathscr{B}$ are linearly ordered MV-algebras with $\operatorname{card} A = \operatorname{card} B = 4$, then $\mathscr{A} \simeq \mathscr{B}$.

4. Subdirect irreducibility

In this section we assume that the MV-algebra under consideration is linearly ordered. Our aim is to prove the assertion (β) from Section 1. In view of the results of Section 3 it suffices to consider an MV-algebra $\mathscr{A}$ with $\operatorname{card} A \geqslant 5$ and a congruence $\sim$ of $\mathscr{A}'$ such that $\sim_0 \neq \sim \neq \sim_m$. Then according to 3.6, for each $a \in A$ we have either $\operatorname{card} \overline{a} = 1$ or $\operatorname{card} \overline{a} \geqslant 3$. Since $\sim \neq \sim_0$, there exists $a \in A$ with $\operatorname{card} A \geqslant 3$.

From the properties of the operation $\odot$ we obtain by simple calculation

Lemma 4.1. *If $x, y \in A$ and $x < y$, then $0 = x \odot \neg x < y \odot \neg x$.*

Lemma 4.2. *Let a, b, c be mutually distinct elements of A, $c \neq u$, $\overline{a} = \overline{b} = \overline{c}$. Then there exists $c' \in A$ such that $c < c'$ and $\overline{c'} = \overline{a}$.*

Proof. Denote $b \oplus' c = c'$. We have $c' = b \oplus c$ and in view of 3.1, $\overline{c'} = \overline{a}$. Since $\mathscr{A}$ is linearly ordered, we get $c' = (b + c) \wedge u > c$. $\square$

Lemma 4.3. *There exists $b_0 \in A$ such that $0 < b_0$ and $\overline{b_0} = \overline{0}$.*

Proof. There exists $x \in A$ with $\operatorname{card} \overline{x} \geqslant 3$. Thus there are $a, b, c \in \overline{x}$ with $a < b < c$.

1) Assume that $a \neq \neg a$ and $b \neq \neg a$. Put $b_0 = b \odot' \neg a$. Hence $b = b \odot \neg a$ and in view of 4.1, $b_0 > 0$. Further

$$b_0 \sim a \odot' \neg a = a \odot \neg a = 0.$$

2) Assume that $a \neq \neg a$ and $b = \neg a$. Then $c \neq \neg a$. Put $b_0 = c \odot' \neg a$. Similarly as in 1), we get $b_0 > 0$ and $b_0 \sim 0$.

3) Assume that $a = \neg a$. Then $b \neq \neg b$. Suppose that $c \neq \neg b$. Put $b_0 = c \odot' \neg b$. We obtain $b_0 > 0$ and $b_0 \sim 0$.

4) Assume that $a = \neg a$ and $c = \neg b$. Then we have $b \neq \neg b$. Since $u \neq \neg b$, we get $c \neq u$. Thus in view of 4.2, there exists $c_1 \in A$ with $c_1 > c$, $c_1 \sim a$. We obtain $c_1 \neq \neg b$. Put $b_0 = c_1 \odot' \neg b$. Then $b_0 > 0$ and $b_0 \sim 0$. $\square$

Lemma 4.4. *There exist $b_1, c_1 \in A$ such that $0 < b_1 < c_1$ and $0 \sim b_1 \sim c_1$.*

Proof. In view of 4.3, there exists $b_0 \in \overline{0}$ with $b_0 > 0$. Hence $\operatorname{card} \overline{0} \neq 1$. Then $\operatorname{card} \overline{0} \geqslant 3$. Thus there is $c_0 \in \overline{0}$ such that $c_0 \notin \{0, b_0\}$. Now it suffices to apply the fact that $\overline{a}$ is linearly ordered. $\square$

Proposition 4.5. *Assume that $\mathscr{A}$ is an MV-algebra which is linearly ordered and semisimple. Then the algebra $\mathscr{A}'$ is simple.*

Proof. Let $\sim$ be a congruence of $\mathscr{A}'$ such that $\sim \neq \sim_0$. We have to verify that $\sim = \sim_m$. The case $\operatorname{card} A \leqslant 2$ being trivial, in view of 3.1.1 we can assume that $\operatorname{card} A > 3$.

Since A is semisimple, the corresponding unital group G is archimedean. Also, G is linearly ordered. Let b_1 and c_1 be as in Lemma 4.4.

Consider the element $b_1 + c_1$ of G. If $b_1 + c_1 \geqslant u$, then $b_1 \oplus c_1 = (b_1 + c_1) \wedge u = u$, thus in view of 3.1 we have $\overline{0} = \overline{u}$ and so $\sim = \sim_m$.

Further, assume that $b_1 + c_1 < u$. Denote $b_1 + c_1 = d_0$ and $d_0 + nc_1 = d_n$ for $n \in \mathbb{N}$. We have $b_1 \oplus c_1 = d_0$, thus $d_0 \in \overline{0}$.

Since G is archimedean and linearly ordered there exists $n_1 \in \mathbb{N}$ such that

$$d_{n_1-1} < u \leqslant d_{n_1}.$$

1) Assume that $n_1 = 1$. We have $d_1 = d_0 + c_1$ and $d_0 > c_1$, thus

$$d_0 \oplus' c_1 = d_0 \oplus c_1 = (d_0 + c_1) \wedge u = u. \tag{1}$$

From $d_0, c_1 \in \overline{0}$ we get $d_0 \oplus c_1 \in \overline{0}$, hence $\overline{u} = \overline{0}$ and $\sim = \sim_m$.

2) Assume that $n_1 > 1$. By the same method as in 1) and by induction we verify that $d_{n_1-1} \in \overline{0}$, $d_{n_1-1} > c_1$. Taking d_{n_1-1} instead of d_0 in (1) and applying steps analogous to those in 1) we again get $\overline{u} = \overline{0}$, hence $\sim = \sim_m$. □

The assertion (β) from Section 1 is a corollary of Proposition 4.5.

5. On the variety $\mathscr{V}$

Let (α) be as in Section 1. This section deals with Ježek's remark concerning the existence of further varieties with the property as in (α).

Let $\mathscr{V}$ be the collection of all algebras having the form $\mathscr{A} = (A; f, g, h, 0, 1)$, where A is a nonempty set and $\mathscr{A}$ is of the type $(3, 3, 3, 0, 0)$, such that for each $x, y \in A$ the relations

$$\begin{aligned} f(x, y, x) &= 0, \quad & g(x, y, x) &= 1, \\ h(0, x, y) &= x, \quad & h(1, x, y) &= y \end{aligned}$$

are valid. Then $\mathscr{V}$ is a variety.

Under the terminology as in Section 1, let $\mathscr{A}'$ be the idenpotent modification of $\mathscr{A}$.

First suppose that $0 = 1$. Then for each $x, y \in A$ we have

$$x = h(0, x, y) = h(1, x, y) = y,$$

hence A is a one-element set. Thus $\mathscr{A}'$ is subdirectly irreducible.

Further, suppose that $0 \neq 1$. Then $\operatorname{card} A \geqslant 2$. Let $\sim$ be a congruence relation on $\mathscr{A}'$, $\sim \neq \sim_0$. Thus there exist $x, y \in A$ such that $x \neq y$ and $x \sim y$. We obtain

$$\begin{aligned} x = f'(x, x, x) &\sim f'(x, y, x) = 0, \\ x = g'(x, x, x) &\sim g'(x, y, x) = 1, \end{aligned}$$

whence $0 \sim 1$ for each nontrivial congruence of $\mathscr{A}$. This yields that $\mathscr{A}'$ is subdirectly irreducible. Therefore we get

Proposition 5.1. *Let $\mathscr{A}$ be an algebra belonging to the variety $\mathscr{V}$. Then the idempotent modification of $\mathscr{A}$ is subdirectly irreducible.*

It is easy to verify that there exists a proper class of mutually nonisomorphic algebras belonging to the variety $\mathscr{V}$.

Let $\mathscr{A}$ be as above and $n \in \mathbb{N}$, $n \geqslant 4$. Let f_n be an n-ary operation on A; we set $\mathscr{B} = (A; f, g, h, f_n, 0, 1)$. Suppose that, e.g., the identity

$$f_n(x_1, x_2, \ldots, x_n) = f_n(x_n, x_2, \ldots, x_{n-1}, x_1)$$

is satisfied in $\mathscr{B}$. The collection of all algebras $\mathscr{B}$ of this form (where $\mathscr{A}$ runs over $\mathscr{V}$) will be denoted by $\mathscr{V}_n$. Then $\mathscr{V}_n$ is a variety and for each element $\mathscr{B}$ of $\mathscr{V}_n$, the idenpotent modification $\mathscr{B}'$ of $\mathscr{B}$ is subdirectly irreducible.

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