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GENERALIZED INDUCED NORMS

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Abstract. Let $\|\cdot\|$ be a norm on the algebra $\mathscr{M}_n$ of all $n \times n$ matrices over $\mathbb{C}$. An interesting problem in matrix theory is that "Are there two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{C}^n$ such that $\|A\| = \max\{\|Ax\|_2 \colon \|x\|_1 = 1\}$ for all $A \in \mathscr{M}_n$?" We will investigate this problem and its various aspects and will discuss some conditions under which $\|\cdot\|_1 = \|\cdot\|_2$.

Keywords: induced norm, generalized induced norm, algebra norm, the full matrix algebra, unitarily invariant, generalized induced congruent

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1. Preliminaries

Throughout the paper $\mathscr{M}_n$ denotes the complex algebra of all $n \times n$ matrices $A = [a_{ij}]$ with entries in $\mathbb{C}$ together with the usual matrix operations. Denote by $\{e_1, e_2, \dots e_n\}$ the standard basis for $\mathbb{C}^n$, where e_i has 1 as its ith entry and 0 elsewhere. We denote by E_{ij} the $n \times n$ matrix with 1 in the (i,j) entry and 0 elsewhere.

For $1 \leqslant p \leqslant \infty$ the ℓ_p-norm on $\mathbb{C}^n$ is defined as follows:

$$\ell_p(x) = \ell_p\biggl(\sum_{i=1}^n x_i e_i\biggr) = \begin{cases} \Bigl(\sum\limits_{i=1}^n |x_i|^p\Bigr)^{1/p} & 1 \leqslant p < \infty, \\ \max\{|x_1|, \dots, |x_n|\} & p = \infty. \end{cases}$$

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is said to be unitarily invariant if $\|x\| = \|Ux\|$ for all unitaries U and all $x \in \mathbb{C}^n$.

By an algebra norm (or a matrix norm) we mean a norm $\|\cdot\|$ on $\mathscr{M}_n$ such that $\|AB\| \leqslant \|A\|\|B\|$ for all $A, B \in \mathscr{M}_n$. An algebra norm $\|\cdot\|$ on $\mathscr{M}_n$ is called unitarily

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invariant if $\|UAV\| = \|A\|$ for all unitaries U and V and all $A \in \mathscr{M}_n$. See Chapter IV of [2] for more information.

Example 1.1. The norm $\|A\|_\sigma = \sum_{i,j=1}^{n} |a_{ij}|$ is an algebra norm, but the norm $\|A\|_m = \max\{|a_{i,j}|\colon\ 1 \leqslant i,\, j \leqslant n\}$ is not an algebra norm, since $\left\|\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}^2\right\|_m > \left\|\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\|_m^2$.

Remark 1.2. It is easy to show that for each norm $\|\cdot\|$ on $\mathscr{M}_n$, the scaled norm $\max\Big\{\|AB\|/\|A\|\|B\|\colon\ A, B \neq 0\Big\}\|\cdot\|$ is an algebra norm; cf. [1].

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{C}^n$. Then for each $A\colon\ (\mathbb{C}^n, \|\cdot\|_1) \to (\mathbb{C}^n, \|\cdot\|_2)$ we can define $\|A\| = \max\{\|Ax\|_2\colon\ \|x\|_1 = 1\}$. If $\|\cdot\|_1 = \|\cdot\|_2$, then $\|I\| = 1$, and there are many examples of $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $\|I\| \neq 1$. This shows that given $\|\cdot\|$ on $\mathscr{M}_n$, we cannot deduce in general that there is a norm $\|\cdot\|_1$ on $\mathbb{C}^n$ with $\|A\| = \max\{\|Ax\|_1\colon\ \|x\|_1 = 1\}$. Let us recall the concept of g-ind norm as follows.

Definition 1.3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{C}^n$. Then the norm $\|\cdot\|_{1,2}$ on $\mathscr{M}_n$ defined by $\|A\|_{1,2} = \max\{\|Ax\|_2\colon\ \|x\|_1 = 1\}$ is called the generalized induced (or g-ind) norm constructed via $\|\cdot\|_1$ and $\|\cdot\|_2$. If $\|\cdot\|_1 = \|\cdot\|_2$, then $\|\cdot\|_{1,1}$ is called induced norm.

Example 1.4. $\|A\|_C = \max\Big\{\sum_{i=1}^{n} |a_{i,j}|\colon\ 1 \leqslant j \leqslant n\Big\}$, $\|A\|_R = \max\Big\{\sum_{j=1}^{n} |a_{i,j}|\colon\ 1 \leqslant i \leqslant n\Big\}$ and the spectral norm $\|A\|_S = \max\{\sqrt{\lambda}\colon\ \lambda$ is an eigenvalue of $A^*A\}$ are induced by ℓ_1, ℓ_∞ and ℓ_2, respectively.

It is known that the algebra norm $\|A\| = \max\{\|A\|_C, \|A\|_R\}$ is not induced and it is not hard to show that it is not g-ind too; cf. Corollary 3.2.6 of [1].

We need the following proposition which is a special case of a finite dimensional version of the Hahn-Banach theorem [5] in which $*$ denotes the transpose; see Corollary 5.5.15 of [3].

Proposition 1.5. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$ and $y \in \mathbb{C}^n$ be a given vector. There exists a vector $y_0 \in \mathbb{C}^n$ such that $y_0^* y = \|y\|$ and for all $x \in \mathbb{C}^n$, $|y_0^* x| \leqslant \|x\|$.*

In this paper we examine the following nice problems:

(i) Given a norm $\|\cdot\|$ on $\mathscr{M}_n$. Is there any class $\mathscr{A}$ of $\mathscr{M}_n$ such that the restriction of the norm $\|\cdot\|$ to $\mathscr{A}$ is g-ind?
(ii) When is a g-ind norm unitarily invariant?
(iii) If a given norm $\|\cdot\|$ is g-ind via $\|\cdot\|_1$ and $\|\cdot\|_2$, then is it possible to find $\|\cdot\|_1$ and $\|\cdot\|_2$ explicitly in terms of $\|\cdot\|$?

(iv) When are two g-ind norms the same?

(v) Is there any characterization of the g-ind norms which are algebra norms?

2. Main results

We begin with some observations on generalized induced norms.

Let $\|\cdot\|_{1,2}$ be a generalized induced norm on $\mathscr{M}_n$ obtained via $\|\cdot\|_1$ and $\|\cdot\|_2$. Then $\|E_{ij}\|_{1,2} = \max\{\|E_{ij}x\|_2\colon \|x\|_1 = 1\} = \max\{\|x_j e_i\|_2\colon \|(x_1,\ldots,x_j,\ldots,x_n)\|_1 = 1\} = \alpha_j\|e_i\|_2$, where $\alpha_j = \max\{|x_j|\colon \|(x_1,\ldots,x_j,\ldots,x_n)\|_1 = 1\}$. In general, for $x \in \mathbb{C}^n$ and $1 \leqslant j \leqslant n$, if $C_{x,j} \in \mathscr{M}_n$ is defined by the operator $C_{x,j}(y) = y_j x$ then $\|C_{x,j}\|_{1,2} = \alpha_j\|x\|_2$.

Also if for $x \in \mathbb{C}^n$ we define $C_x \in \mathscr{M}_n$ by $C_x = \sum\limits_{j=1}^{n} C_{x,j}$, then clearly $\|C_x\|_{1,2} = \alpha\|x\|_2$, where $\alpha = \max\Big\{\Big|\sum\limits_{j=1}^{n} y_j\Big|\colon \|(y_1,\ldots,y_j,\ldots,y_n)\|_1 = 1\Big\}$.

Now we give a partial solution to Problem (i) and useful direction toward solving Problem (iii):

Proposition 2.1. *Let $\|\cdot\|$ be an algebra norm on $\mathscr{M}_n$. Then $\|\cdot\|$ is a g-ind norm on $\{A \in \mathscr{M}_n\colon \|A\| = \|A^{-1}\| = 1\}$.*

Proof. Put $\|x\|_1 = \max\{\|C_{Ax}\|\colon \|A\| = 1\}$, $\lambda^{-1} = \max\Big\{\Big|\sum\limits_{i=1}^{n} x_i\Big|\colon \|x\|_1 = 1\Big\}$ and $\|x\|_2 = \lambda\|C_x\|$. Then we have $\|C_y\|_{1,2} = \max\{\|C_y x\|_2\colon \|x\|_1 = 1\} = \max\Big\{\Big|\sum\limits_{i=1}^{n} x_i\Big|\|y\|_2\colon \|x\|_1 = 1\Big\} = \|y\|_2\lambda^{-1} = \|C_y\|$.

It follows that for each $y \in \mathbb{C}^n$ there is some $x \in \mathbb{C}^n$ such that $\|C_y x\|_2 = \|C_y\|\,\|x\|_1 = \|C_y\|\max\{\|C_{Dx}\|\colon \|D\| = 1\}$.

Now let A be invertible and $\|A^{-1}\| = \|A\| = 1$ and $z = A^{-1}C_y x$. Then $\lambda^{-1}\|Bz\|_2 = \lambda^{-1}\|BA^{-1}C_y x\|_2 = \lambda^{-1}\|Dx\|_2 = \|C_{Dx}\| \leqslant \|C_y\|^{-1}\|C_y x\|_2 = \|C_y\|^{-1}\|Az\|_2$.

Now choose y so that $\|C_y\| = 1$. Then $\|C_{Bz}\| \leqslant \|C_{Az}\|$ for all $B \in \mathscr{M}_n$. This implies that $\|C_{Az}\|$ is an upper bound for the set $\{\|C_{Bz}\|\colon \|B\| = 1\}$ and indeed $\|C_{Az}\| = \max\{\|C_{Bz}\|\colon \|B\| = 1\} = \|z\|_1$. It follows that $\|A\| = 1 = \|C_{A(z/\|z\|_1)}\| = \max\{\|C_{Au}\|\colon \|u\|_1 = 1\} = \max\{\|Au\|_2\colon \|u\|_1 = 1\} = \|A\|_{1,2}$. $\square$

Let us now answer Question (ii).

Proposition 2.2. *An induced norm $\|\cdot\|_{1,2}$ is unitarily invariant if and only if so are $\|\cdot\|_1$ and $\|\cdot\|_2$.*

P r o o f. Let U, V be unital operators and A be an arbitrary operator on $\mathbb{C}^n$. Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are unitarily invariant. Then

$$\|UAV\|_{1,2} = \max_{x\neq 0} \frac{\|UAVx\|_2}{\|x\|_1} = \max_{x\neq 0} \frac{\|AVx\|_2}{\|x\|_1} = \max_{y\neq 0} \frac{\|Ay\|_2}{\|V^{-1}y\|_1}$$
$$= \max_{y\neq 0} \frac{\|Ay\|_2}{\|y\|_1} = \|A_{1,2}\|.$$

Conversely, if $\|\cdot\|_{1,2}$ is unitarily invariant, then $\|Ux\|_1 = \max\{\|AUx\|_2\colon \|A\|_{1,2} \leqslant 1\} = \max\{\|Bx\|_2\colon \|U^{-1}B\|_{1,2} \leqslant 1\} = \max\{\|Bx\|_2\colon \|B\|_{1,2} \leqslant 1\} = \|x\|_1$ and $\|Ux\|_2 = \alpha^{-1}\|C_{Ux}\| = \alpha^{-1}\|UC_x\| = \alpha^{-1}\|C_x\| = \|x\|_2$. □

Modifying the proof of Theorem 5.6.18 of [3], we obtain a similar useful result for g-ind norms:

Theorem 2.3. *Let $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_3$ and $\|\cdot\|_4$ be four given norms on $\mathbb{C}^n$ and*

$$R_{i,j} = \max\left\{\frac{\|x\|_i}{\|x\|_j}\colon\ x \neq 0\right\}, \quad 1 \leqslant i,j \leqslant 4.$$

Then

$$\max\left\{\frac{\|A\|_{1,2}}{\|A\|_{3,4}}\colon\ A \neq 0\right\} = R_{2,4}R_{3,1}.$$

In particular, $\max\{\|A\|_{1,1}/\|A\|_{2,2}\colon\ A \neq 0\} = \max\{\|A\|_{2,2}/\|A\|_{1,1}\colon\ A \neq 0\} = R_{1,2}R_{2,1}$.

P r o o f. Let A be a matrix and $x \neq 0$. Then $\|Ax\|_2/\|x\|_1 = \|Ax\|_2/\|Ax\|_4 \cdot \|Ax\|_4/\|x\|_3 \cdot \|x\|_3/\|x\|_1$. Hence $\|A\|_{1,2} \leqslant R_{2,4}\|A\|_{3,4}R_{3,1}$. Thus $\|A\|_{1,2}/\|A\|_{3,4} \leqslant R_{2,4}R_{3,1}$.

There are vectors y, z in $\mathbb{C}^n$ such that $\|y\|_2 = \|z\|_2 = 1$, $\|y\|_2 = R_{2,4}\|y\|_4$ and $\|z\|_3 = R_{3,1}\|z\|_1$. By Proposition 1.5, there exists a vector $z_0 \in \mathbb{C}^n$ such that $|z_0^*x| \leqslant \|x\|_3$ and $z_0^*z = \|z\|_3$.

Put $A_0 = yz_0$. Then $\|A_0z\|_2/\|z\|_1 = \|yz_0^*z\|_2/\|z\|_1 = \|y\|_2\|z\|_3/\|z\|_1 = \|y\|_2R_{3,1}$. Hence $\|A_0\|_{1,2} \geqslant \|y\|_2/\|y\|_4 \cdot R_{3,1}\|y\|_4 = R_{2,4} \cdot R_{3,1}\|y\|_4$. On the other hand, $\|A_0x\|_4/\|x\|_3 = \|yz_0^*x\|_4/\|x\|_3 = \|y\|_4|z_0^*x|/\|x\|_3 \leqslant \|y\|_4$. Thus $\|A_0\|_{3,4} \leqslant \|y\|_4$. Hence $\|A_0\|_{1,2}/\|A_0\|_{3,4} \geqslant R_{2,4}R_{3,1}\|y\|_4/\|y\|_4 = R_{2,4}R_{3,1}$. □

Corollary 2.4.

(i) $\|\cdot\|_{1,2} \leqslant \|\cdot\|_{3,2}$ *if and only if* $\|\cdot\|_1 \geqslant \|\cdot\|_3$,

(ii) $\|\cdot\|_{1,2} \leqslant \|\cdot\|_{1,4}$ *if and only if* $\|\cdot\|_2 \leqslant \|\cdot\|_4$.

Proof. (i) $\|\cdot\|_{1,2} \leqslant \|\cdot\|_{3,2}$ if and only if $\max\{\|A\|_{1,2}/\|A\|_{3,2}\colon A \neq 0\} = R_{2,2}R_{3,1} \leqslant 1$ and this happens if and only if $R_{3,1} \leqslant 1$ or equivalently $\|\cdot\|_3 \leqslant \|\cdot\|_1$. The proof of (ii) is similar. □

The following corollary completely answers Question (iv):

Corollary 2.5. $\|\cdot\|_{1,2} = \|\cdot\|_{3,4}$ *if and only if there exists* $\gamma > 0$ *such that* $\|\cdot\|_1 = \gamma\|\cdot\|_3$ *and* $\|\cdot\|_2 = \gamma\|\cdot\|_4$.

Proof. If $\|A\|_{1,2} = \|A\|_{3,4}$, then $R_{4,2}R_{1,3} = \max\{\|A\|_{3,4}/\|A\|_{1,2}\colon A \neq 0\} = 1 = \max\{\|A\|_{1,2}/\|A\|_{3,4}\colon A \neq 0\} = R_{2,4}R_{3,1}$. Hence $\max\{\|x\|_2/\|x\|_4\colon x \neq 0\} = R_{2,4} = 1/R_{3,1} = \min\{\|x\|_1/\|x\|_3\colon x \neq 0\} \leqslant \max\{\|x\|_1/\|x\|_3\colon x \neq 0\} = R_{1,3} = 1/R_{4,2} = \min\{\|x\|_2/\|x\|_4\colon x \neq 0\}$. Thus there exists a number γ such that $\|x\|_2/\|x\|_4 = \gamma = \|x\|_1/\|x\|_3$. □

Remark 2.6. It is known that each induced norm $\|\cdot\|_{1,1}$ is minimal in the sense that for any matrix norm $\|\cdot\|$, the inequality $\|\cdot\| \leqslant \|\cdot\|_{1,1}$ implies that $\|\cdot\| = \|\cdot\|_{1,1}$. But this is not true for g-ind norms in general. For instance, put $\|\cdot\|_\alpha = \ell_\infty(\cdot)$, $\|\cdot\|_\beta = 2\ell_2(\cdot)$ and $\|\cdot\|_\gamma = \ell_2(\cdot)$. Then $\|\cdot\|_{\gamma,\beta} \leqslant \|\cdot\|_{\alpha,\beta}$ but $\|\cdot\|_{\gamma,\beta} \neq \|\cdot\|_{\alpha,\beta}$.

The following theorem is one of our main theorems and provides a complete solution for Problem (v):

Theorem 2.7. *Let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be two norms on* $\mathbb{C}^n$. *Then* $\|\cdot\|_{1,2}$ *is an algebra norm on* $\mathscr{M}_n$ *if and only if* $\|\cdot\|_1 \leqslant \|\cdot\|_2$.

Proof. For each A and B in $\mathscr{M}_n$ we have

$$\|ABx\|_2 \leqslant \|A\|_{1,2}\|Bx\|_1 \leqslant \|A\|_{1,2}\|Bx\|_2 \leqslant \|A\|_{1,2}\|B\|_{1,2}\|x\|_1.$$

Hence $\|AB\|_{1,2} \leqslant \|A\|_{1,2}\|B\|_{1,2}$.

Conversely, let $\|\cdot\|_{1,2}$ be an algebra norm. Then for each $A, B \in \mathscr{M}_n$ we have $\|AB\|_2 \leqslant \|A\|_{1,2}\|B\|_{1,2}\|x\|_1$. Let B be an arbitrary member of $\mathscr{M}_n$. For $Bx \neq 0$, take M to be the linear span of $\{Bx\}$ and define $f\colon (M, \|\cdot\|_1) \to \mathbb{C}$ by $f(cBx) = c\|Bx\|_1/\|Bx\|_2$. By the Hahn-Banach Theorem, there is an $F\colon (\mathbb{C}^n, \|\cdot\|_1) \to \mathbb{C}$ with $F|_M = f$ and $\|F\| = \|f\| = \max\{|f(cBx)|\colon \|cBx\|_1 = 1\} = \max\{|c|\|Bx\|_1/\|Bx\|_2\colon |c|\,\|Bx\|_1 = 1\} = 1/\|Bx\|_2$. Define $A\colon (\mathbb{C}^n, \|\cdot\|_1) \to (\mathbb{C}^n, \|\cdot\|_2)$ by $Ay = F(y)Bx$. Then $\|A\|_{1,2} = \max\{\|Ay\|_2\colon \|y\|_1 = 1\} = \max\{|F(y)|\,\|Bx\|_2\colon \|y\|_1 = 1\} = 1$, and

$\|ABx\|_2 = |F(Bx)|\,\|Bx\|_2 = |f(Bx)|\,\|Bx\|_2 = (\|Bx\|_1/\|Bx\|_2)\|Bx\|_2 = \|Bx\|_1$. Thus for all B,

$$\|Bx\|_1 = \|ABx\|_2 \leqslant \|A\|_{1,2}\|B\|_{1,2}\|x\|_1 = \|B\|_{1,2}\|x\|_1,$$

or

$$\|Bx\|_1 \leqslant \|B\|_{1,2}\|x\|_1.$$

Now take N to be the linear span of $\{x\}$ and define $g\colon (N,\|\cdot\|_1) \to \mathbb{C}$ by $g(cx) = c\|x\|_1/\|x\|_2$. By the Hahn-Banach Theorem, there is a $G\colon (\mathbb{C}^n,\|\cdot\|_1) \to \mathbb{C}$ with $G|_N = g$ and $\|G\| = \|g\| = \max\{|g(cx)|\colon \|cx\|_1\} = \max\{|c|\,\|x\|_1/\|x\|_2\colon |c|\|x\|_1 = 1\} = 1/\|x\|_2$. Define $B\colon (\mathbb{C}^n,\|\cdot\|_1) \to (\mathbb{C}^n,\|\cdot\|_2)$ by $By = G(y)x$. Then $\|B\|_{1,2} = \max\{\|By\|_2\colon \|y\|_1 = 1\} = \max\{|G(y)|\,\|x\|_2\colon \|y\|_1 = 1\} = \|x\|_2\|G\| = 1$, and $\|Bx\|_1 = |G(x)|\,\|x\|_1 = |g(x)|\,\|x\|_1 = (\|x\|_1/\|x\|_2)\|x\|_1 = \|x\|_1^2/\|x\|_2$.

So

$$\frac{\|x\|_1^2}{\|x\|_2} = \|Bx\|_1 \leqslant \|B\|_{1,2}\|x\|_1 = \|x\|_1.$$

Thus $\|\cdot\|_1 \leqslant \|\cdot\|_2$. □

Proposition 2.8. *Suppose that $\|\cdot\|_{1,2}$ is a g-ind norm and $\lambda > 0$. Then the scaled norm $\lambda\|\cdot\|_{1,2}$ is a g-ind algebra norm if and only if $\lambda \geqslant R_{1,2}$.*

Proof. Evidently, $\lambda\|\cdot\|_{1,2} = \|\cdot\|_{\|\cdot\|_1,\lambda\|\cdot\|_2}$. If $\|\cdot\|_{3,4} = \lambda\|\cdot\|_{1,2} = \|\cdot\|_{\|\cdot\|_1,\lambda\|\cdot\|_2}$ then Corollary 2.5 implies that there exists $\alpha > 0$ such that $\|\cdot\|_3 = \alpha\|\cdot\|_1$ and $\|\cdot\|_4 = \alpha\lambda\|\cdot\|_2$. Now Theorem 2.7 implies that $\lambda\|\cdot\|_{1,2} = \|\cdot\|_{3,4}$ is an algebra norm if and only if $\alpha\|\cdot\|_1 \leqslant \alpha\lambda\|\cdot\|_2$ or equivalently $R_{1,2} \leqslant \lambda$. □

Proposition 2.9. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{C}^n$ and $0 \neq \alpha,\beta \in \mathbb{C}$. Define $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on $\mathbb{C}^n$ by $\|x\|_\alpha = \|\alpha x\|_1$ and $\|x\|_\beta = \|\beta x\|_2$, respectively. Then $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two norms on $\mathbb{C}^n$ and $\|\cdot\|_{\alpha,\beta} = |\beta/\alpha|\,\|\cdot\|_{1,2}$.*

Proof. We have $\|A\|_{\alpha,\beta} = \max\{\|Ax\|_\beta\colon \|x\|_\alpha = 1\} = \max\{\|\beta Ax\|_2\colon \|\alpha x\|_1 = 1\} = |\beta/\alpha|\max\{\|Ay\|_2\colon \|y\|_1 = 1\} = |\beta/\alpha|\,\|A\|_{1,2}$. □

The preceding proposition leads us to give the following definition:

Definition 2.10. Let $(\|\cdot\|_1,\|\cdot\|_2)$ and $(\|\cdot\|_3,\|\cdot\|_4)$ be two pairs of norms on $\mathbb{C}^n$. We say that $(\|\cdot\|_1,\|\cdot\|_2)$ is generalized induced congruent (gi-congeruent) to $(\|\cdot\|_3, \|\cdot\|_4)$ and we write $(\|\cdot\|_1,\|\cdot\|_2) \equiv_{\rm gi} (\|\cdot\|_3,\|\cdot\|_4)$ if $\|\cdot\|_{1,2} = \gamma\|\cdot\|_{3,4}$ for some $0 < \gamma \in \mathbb{R}$.

Clearly $\equiv_{\rm gi}$ is an equivalence relation. We denote by $[(\|\cdot\|_1,\|\cdot\|_2)]_{\rm gi}$ the equivalence class of $(\|\cdot\|_1,\|\cdot\|_2)$. Proposition 2.9 shows that for each $0 < \alpha,\beta \in \mathbb{R}$ we have $(\alpha\|\cdot\|_1,\beta\|\cdot\|_2) \equiv_{\rm gi} (\|\cdot\|_1,\|\cdot\|_2)$. Indeed, we have the following result:

Theorem 2.11. *For each pair* $(\|\cdot\|_1, \|\cdot\|_2)$ *of norms on* $\mathbb{C}^n$ *we have*

$$[(\|\cdot\|_1, \|\cdot\|_2)]_{\mathrm{gi}} = \{(\alpha\|\cdot\|_1, \beta\|\cdot\|_2)\colon\ 0 < \alpha, \beta \in \mathbb{R}\}.$$

We can extend the above method to find some other norms on $\mathscr{M}_n$ which are not necessarily gi-congruent to a given pair $(\|\cdot\|_1, \|\cdot\|_2)$:

Proposition 2.12. *Let* $(\|\cdot\|_1, \|\cdot\|_2)$ *be a pair of norms on* $\mathbb{C}^n$ *and* $K, L \in \mathscr{M}_n$ *be two invertible matrices. Define* $\|\cdot\|_K$ *and* $\|\cdot\|_L$ *on* $\mathbb{C}^n$ *by* $\|x\|_K = \|Kx\|_1$ *and* $\|x\|_L = \|Lx\|_2$*. Then* $\|\cdot\|_K$ *and* $\|\cdot\|_L$ *are norms on* $\mathbb{C}^n$ *and* $\|A\|_{K,L} = \|LAK^{-1}\|_{1,2}$.

Proof. Clear. See also Lemma 3.1 of [4]. □

Remark 2.13. Note that the case $K = \alpha I$ and $L = \beta I$ gives Proposition 2.9.

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