Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 1, 53–65

Persistent URL: http://dml.cz/dmlcz/128154

Czechoslovak Mathematical Journal, 57 (132) (2007), 53–65

INTERTWINING NUMBERS; THE n-ROWED SHAPES

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(Received August 30, 2006)

Abstract. A fairly old problem in modular representation theory is to determine the vanishing behavior of the Hom groups and higher Ext groups of Weyl modules and to compute the dimension of the $\mathbb{Z}/(p)$-vector space $\operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu)$ for any partitions λ, μ of r, which is the intertwining number. K. Akin, D. A. Buchsbaum, and D. Flores solved this problem in the cases of partitions of length two and three. In this paper, we describe the vanishing behavior of the groups $\operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu)$ and provide a new formula for the intertwining number for any n-rowed partition.

Keywords: representation theory, intertwining number, Weyl module, Ext group, partition

MSC 2000: 20G15, 20C20D, 13D02

1. Introduction

Two historical developments of the representation theory of the general linear group GL_n began with the geometrical Borel-Weil-Bott theory and the combinatorial analysis of Schur, Young and Weyl. Over a field of characteristic zero, the geometric theory realizes the irreducible representations of GL_n with highest weight λ as the sections of a line bundle over the flag variety. By contrast, the Schur-Weyl construction produces these representations inside the tensor powers of the standard representation F, by a process of symmetrization and anti-symmetrization defined by λ considered as a Young diagram (see [13] for an accessible reference). Characteristic-free approaches to the representation theory of the general linear group and the symmetric groups are of current interest because of their own advantages, and the extension and modification of the representation theory over a field of characteristic zero to arbitrary commutative rings [5], [9], [14], [18], [20]. Such a characteristic-free

This work was supported by BK21 Project.

approach admits a natural generalization to Schur modules and Weyl modules associated to skew diagrams as well as more general diagrams of squares in the plane, and involves constructions which are "universal" in the sense that they do not depend upon the base rings [5], [9], [19], [21]. Using this approach one may gain clear knowledge of characteristic-independent features of the representation category, and some insight into modular representations by a base change from $\mathbb{Z}$ to a field of positive characteristic. One of the fairly old questions in modular representation is to determine the vanishing behavior of Hom groups and higher Ext groups of Weyl modules [2], [8], [10], [11], [12]. They could be used to compute some homological invariants of Weyl modules for GL_n [2], [4], [17]. This problem was solved in the cases of partitions of length two or three by K. Akin and D. A. Buchsbaum, and D. Flores de Chela [7], [8], [12].

In this paper we describe the vanishing behavior of $\mathrm{Hom}_{\bar{A}_r}\,(\overline{K}_\lambda, \overline{K}_\mu)$ as a group and then use our constructions to provide a new formula for the intertwining number, i.e., the dimension of the $\mathbb{Z}/(p)$-vector space $\mathrm{Hom}_{\bar{A}_r}\,(\overline{K}_\lambda, \overline{K}_\mu)$ for any n-rowed partition. From this formula we obtain the results of [7], [8], [12] as special cases. In this case, we also show that the Ext group is cyclic and describe the highest invariant factor. This result provides a curious link between the representation theory of GL_n and the natural tensor complexes used in the construction of minimal resolutions associated to generic determinantal ideals [1], [2], [6].

We now proceed to explain the contents of this paper. The second section covers some of the basic notation and basic facts. In the third section we describe the intertwining number in terms of the boundary map of the A_r-projective resolution of the Weyl module K_λ. The fourth section treats the problem of the intertwining number for any n-rowed shapes.

2. Preliminaries

In this section we will review some of the basic facts and give some of the basic notation that will be used throughout. For proofs and details, we refer to papers [2], [8] of the references.

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition, d is a positive integer less than or equal to λ_n, and $\mu = (\lambda_1, \ldots, \lambda_k + d, \ldots, \lambda_{k+j} - d, \ldots, \lambda_n)$ is the partition obtained from λ by taking d boxes away from some row of λ and attaching them to some higher row (assuming that we still obtain a partition). Let F be a free $\mathbb{Z}$-module and let $K_\lambda F$ and $K_\mu F$ be the Weyl modules of shape λ and μ over $\mathbb{Z}$, respectively. Then $\overline{K_\lambda F} \cong \mathbb{Z}/(p) \otimes_{\mathbb{Z}} K_\lambda F$ and $\overline{K_\mu F} \cong \mathbb{Z}/(p) \otimes_{\mathbb{Z}} K_\mu F$ are the Weyl modules of shape λ and μ over $\mathbb{Z}/(p)$ where p is a prime number. Also, for brevity we shall write K_λ and $\overline{K}_\lambda$ instead of $K_\lambda F$ and $\overline{K_\lambda F}$, respectively. One of the important problems in modular representation theory is to determine the vanishing behavior of the groups $\operatorname{Ext}^i_{\mathrm{GL}\,\overline{F}}(\overline{K}_\lambda, \overline{K}_\mu)$ and to compute their dimensions, as vector spaces over $\mathbb{Z}/(p)$, when they do not vanish. The dimensions of $\operatorname{Ext}^i_{\mathrm{GL}\,\overline{F}}(\overline{K}_\lambda, \overline{K}_\mu)$, as the $\mathbb{Z}/(p)$-vector spaces, are called *the intertwining numbers of $\overline{K}_\lambda$ and $\overline{K}_\mu$*, denoted by $\varepsilon^i_p(\lambda, d, k, k+j)$. These are the numbers that we want to compute.

The attempts to determine these intertwining numbers have been varied. One approach suggested in [2] will be elaborated. By the universal coefficient theorem,

$$\begin{aligned} 0 \to \mathbb{Z}/(p) \otimes \operatorname{Ext}^i_{\mathrm{GLF}}(K_\lambda, K_\mu) &\to \operatorname{Ext}^i_{\mathrm{GL}\,\overline{F}}(\overline{K}_\lambda, \overline{K}_\mu) \\ &\to \operatorname{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/(p), \operatorname{Ext}^{i+1}_{\mathrm{GLF}}(K_\lambda, K_\mu)) \to 0 \end{aligned}$$

is exact. Thus we see that it suffices to calculate the integral Ext groups, since the modular ones are simply the p-torsion part of one integral Ext plus the reduction modulo p of another. Now, we want a resolution

$$\mathbb{X}\colon \ \ldots \to X_2 \to X_1 \to X_0 \to K_\lambda \to 0$$

of K_λ such that $\operatorname{Ext}^i_{\mathrm{GLF}}(K_\lambda, K_\mu) = H^i(\operatorname{Hom}_{\mathrm{GLF}}(\mathbb{X}, K_\mu))$. In order to really make sense of this, we want to replace GLF by a suitable ring, we want the X_i to be projective modules over that ring, and we want the groups $\operatorname{Hom}_{\mathrm{GLF}}(\mathbb{X}, K_\mu)$ to be somewhat computable. This involves introducing the Schur algebra and weight modules as discussed in [2], [15].

It is easy to see from [2] that if we let $E = \operatorname{End}(F)$, then the Schur algebra A_r is (as a module) isomorphic to $D_r E$, where D signifies a divided power. Thus if the partition λ has weight $r = |\lambda| = \sum_{i=1}^{n} \lambda_i$, then K_λ and K_μ are A_r-modules, and we may replace the computation of $\operatorname{Ext}^i_{\mathrm{GLF}}(K_\lambda, K_\mu)$ by that of $\operatorname{Ext}^i_{A_r}(K_\lambda, K_\mu)$. Consequently, the resolution $\mathbb{X}$ that we are looking for should include all of the X_i projective modules over A_r. And now we know that

$$\operatorname{Hom}_{A_r}(\mathbb{X}, K_\mu) = \oplus \operatorname{Hom}_{A_r}(D_{a_1}F \otimes \ldots \otimes D_{a_n}F, K_\mu)$$

is the free abelian group generated by those standard tableaux of shape μ that have content $(1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$. Thus we can write down the maps in $\operatorname{Hom}_{A_r}(\mathbb{X}, K_\mu)$ as integral matrices, and the cohomology groups are determined by the invariant factors of those matrices.

First of all, we must know the maps in the projective resolution $\mathbb{X}$. This problem of exhibiting such resolutions with their maps is still very much an ongoing one. Fortunately, we can study the intertwining numbers for $i = 0$, that is, the dimension of $\operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu)$, denoted by $\varepsilon_p(\lambda, d, k, k+j)$, because we know a presentation of K_λ, i.e., $D_\lambda \to K_\lambda \to 0$. Also, a fairly straightforward argument shows that it is enough to consider the case where we take d boxes from the last row and attach them to the first, i.e., $\varepsilon_p(\lambda, d, 1, n)$. For two-rowed shapes and three-rowed shapes, we have the following results.

Theorem 2.1 ([3], [4], [7]). *Let $\lambda = (\lambda_1, \lambda_2)$ be a two-rowed partition with $|\lambda| = r$, let d be an arbitrary positive integer $\leqslant \lambda_2$, and let $\mu = (\lambda_1 + d, \lambda_2 - d)$ be a partition. If $s_1 = \lambda_1 - \lambda_2$, then*

(1) $\operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu) \cong \mathbb{Z}/(\delta)$, *which is cyclic, and*

(2) $\operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu) = \begin{cases} 0 & \text{if } p \nmid \delta, \\ \mathbb{Z}/(p) & \text{if } p \mid \delta, \end{cases}$

i.e., $\varepsilon_p(\lambda, d, 1, 2) = \begin{cases} 0 & \text{if } p \nmid \delta, \\ 1 & \text{if } p \mid \delta, \end{cases}$

where

$$\delta = \frac{s_1 + d + 1}{\gcd(s_1 + d + 1,\ \operatorname{lcm}\{1, 2, \ldots, d\})}.$$

Theorem 2.2 ([8], [12]). *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a three-rowed partition with $|\lambda| = r$, let d be an arbitrary positive integer $\leqslant \lambda_3$, and let $\mu = (\lambda_1 + d, \lambda_2, \lambda_3 - d)$ be a partition. If $s_1 = \lambda_1 - \lambda_2$ and $s_2 = \lambda_2 - \lambda_3$, then*

(1) $\operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu) \cong \mathbb{Z}/(\delta)$, *which is cyclic, and*

(2) $\operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu) = \begin{cases} 0 & \text{if } p \nmid \delta, \\ \mathbb{Z}/(p) & \text{if } p \mid \delta, \end{cases}$

i.e., $\varepsilon_p(\lambda, d, 1, 3) = \begin{cases} 0 & \text{if } p \nmid \delta, \\ 1 & \text{if } p \mid \delta, \end{cases}$

where

$$\delta = \frac{\alpha}{\gamma(\alpha, s_1 + d + 1, d)} \prod_{k=1}^{[\frac{1}{2}(d-1)]} \frac{\gamma(\alpha, s_1 + d - k + 1, k)}{\gamma(\alpha, s_1 + d - k + 1, d - k)},$$

$\gamma(u, v, w) = \gcd(u, v, \operatorname{lcm}\{1, 2, \ldots, w\})$, *and* $\alpha = s_1 + s_2 + d + 2$.

3. Intertwining number and Ext groups

In this section we describe the relationship between the intertwining number and the Ext group of the Weyl modules.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition with $|\lambda| = r$, let d be an arbitrary positive integer $\leqslant \lambda_n$, and let $\mu = (\lambda_1 + d, \lambda_2, \ldots, \lambda_n - d)$ be a partition. By taking $i = 0$, we have an exact sequence

$$\begin{gathered} 0 \to \mathbb{Z}/(p) \otimes \operatorname{Hom}_{A_r}(K_\lambda, K_\mu) \to \operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu) \\ \to \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/(p), \operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu)) \to 0. \end{gathered}$$

Since $\operatorname{Hom}_{A_r}(K_\lambda, K_\mu) = 0$ for $\lambda \neq \mu$, we have

$$\operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu) \cong \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/(p), \operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu)),$$

which is the p-torsion subgroup of $\operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu)$.

To find $\operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu)$, we start the A_r-projective resolution

$$\mathbb{X}\colon \ \ldots \longrightarrow X_2 \longrightarrow X_1 \xrightarrow{\square_\lambda} X_0 \longrightarrow K_\lambda \longrightarrow 0$$

where each X_i is a direct sum of divided powers.

Since $\operatorname{Hom}_{A_r}(K_\lambda, K_\mu) = 0$, we conclude that

$$\begin{gathered} 0 \longrightarrow \operatorname{Hom}_{A_r}(X_0, K_\mu) \xrightarrow{u=\square_\lambda^*} \operatorname{Hom}_{A_r}(X_1, K_\mu) \\ \xrightarrow{v} \operatorname{Hom}_{A_r}(X_2, K_\mu) \longrightarrow \ldots \end{gathered}$$

is a complex whose first cohomology is

$$\operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu) = \operatorname{Ker} v / \operatorname{Im} u,$$

and then

$$\begin{aligned} \operatorname{Coker} u &= \operatorname{Hom}_{A_r}(X_1, K_\mu)/\operatorname{Im} u \\ &\cong (\operatorname{Ker} v \oplus \operatorname{Im} v)/\operatorname{Im} u \\ &\cong (\operatorname{Ker} v/\operatorname{Im} u) \oplus \operatorname{Im} v \\ &= \operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu) \oplus \operatorname{Im} v. \end{aligned}$$

On the other hand, if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$ are the invariant factors of the map u, we have

$$\operatorname{Coker} u \cong \mathbb{Z}/(\varepsilon_1) \oplus \ldots \oplus \mathbb{Z}/(\varepsilon_t) \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}.$$

Hence

$$\operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu) \cong \mathbb{Z}/(\varepsilon_1) \oplus \ldots \oplus \mathbb{Z}/(\varepsilon_t),$$

since it is a torsion group.

Now, recall that if we write the map u as an integral matrix, the invariant factors of u may be computed as follows:

Let u be a $t \times t'$ integral matrix, $t' \geqslant t$, and let δ_i, $i = 1, 2, \ldots, t$, be the gcd of the i-minors of u. Then $\varepsilon_1 = \delta_1$, $\varepsilon_i = \delta_i/\delta_{i-1}$, $i = 2, 3, \ldots, t$ [16]. Thus we have the following results.

Theorem 3.1. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-rowed partition with $|\lambda| = r$, and let $\mu = (\lambda_1 + d, \lambda_2, \ldots, \lambda_n - d)$ be a partition, where d is an arbitrary positive integer $\leqslant \lambda_n$. If*

$$\ldots \longrightarrow X_1 \xrightarrow{\square_\lambda} X_0 \longrightarrow K_\lambda \longrightarrow 0$$

is an A_r-projective resolution, and if

$$u\colon \operatorname{Hom}_{A_r}(X_0, K_\mu) \to \operatorname{Hom}_{A_r}(X_1, K_\mu)$$

is a boundary map, then

(1) *$\operatorname{Ext}^1_{A_r}(K_\lambda, K_\mu) \cong \mathbb{Z}/(\varepsilon_1) \oplus \ldots \oplus \mathbb{Z}/(\varepsilon_t)$, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$ are the invariant factors of the map u, and*

(2) *the intertwining number $\varepsilon_p(\lambda, d, 1, n)$ is the dimension of $\operatorname{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/(p), \mathbb{Z}/(\varepsilon_1) \oplus \mathbb{Z}/(\varepsilon_2) \oplus \ldots \oplus \mathbb{Z}/(\varepsilon_t))$ as the $\mathbb{Z}/(p)$-vector space.*

4. The n-rowed cases

In this section we state and prove the main results of this paper for any n-rowed partition with $d = 1$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-rowed partition with $|\lambda| = r$, and let $\mu = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_n - 1)$ be a partition. Assume that $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$, and let $s_k = \lambda_k - \lambda_{k+1}$ for $k = 1, 2, \ldots, n-1$. We start with a complex

$$0 \longrightarrow \operatorname{Hom}_{A_r}(X_0, K_\mu) \xrightarrow{u=\square^*} \operatorname{Hom}_{A_r}(X_1, K_\mu) \longrightarrow \ldots$$

where $X_0 = D_{\lambda_1} \otimes D_{\lambda_2} \otimes \ldots \otimes D_{\lambda_n}$ and

$$X_1 = \sum_{k=1}^{n-1} \biggl(\sum_{l \geqslant 1} D_{\lambda_1} \otimes \ldots \otimes D_{\lambda_k + l} \otimes D_{\lambda_{k+1} - l} \otimes \ldots \otimes D_{\lambda_n} \biggr).$$

In order to find a matrix representation of the map u we need $\mathbb{Z}$-bases for $\mathrm{Hom}_{A_r}(X_0, K_\mu)$ and $\mathrm{Hom}_{A_r}(X_1, K_\mu)$. As already observed in the preliminary, we know that $\mathrm{Hom}_{A_r}(X_0, K_\mu)$ is generated by the standard tableaux of shape μ and content $(1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n})$. So we have a basis for $\mathrm{Hom}_{A_r}(X_0, K_\mu)$ consisting of $n-1$ standard tableaux: for $i = 1, 2, \ldots, n-1$,

$$\begin{aligned}\alpha_i = {} & 1^{(\lambda_1)}(i+1)^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes i^{(\lambda_i)} \otimes (i+1)^{(\lambda_{i+1}-1)}(i+2)^{(1)} \\ & \otimes (i+2)^{(\lambda_{i+2}-1)}(i+3)^{(1)} \otimes \ldots \otimes (n-1)^{(\lambda_{n-1}-1)} n^{(1)} \otimes n^{(\lambda_n-1)}.\end{aligned}$$

Similarly the group $\mathrm{Hom}_{A_r}(X_1, K_\mu)$, which is equal to

$$\sum_{k=1}^{n-1} \mathrm{Hom}_{A_r}(D_{(\lambda_1,\ldots,\lambda_k+1,\lambda_{k+1}-1,\ldots,\lambda_n)}, K_\mu)$$

since it is zero for $l > 1$, is $(n-1)$-dimensional because only one standard tableau β_k forms a basis for

$$\mathrm{Hom}_{A_r}(D_{(\lambda_1,\ldots,\lambda_k+1,\lambda_{k+1}-1,\ldots,\lambda_n)}, K_\mu)$$

for each $k = 1, 2, \ldots, n-1$, where β_k is a standard tableau of shape μ and content $(1^{\lambda_1}, \ldots, k^{\lambda_k+1}, (k+1)^{\lambda_{k+1}-1}, \ldots, n^{\lambda_n})$: for $1 \leqslant k \leqslant n-1$,

$$\begin{aligned}\beta_k = {} & 1^{(\lambda_1)}k^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes (k+1)^{(\lambda_{k+1}-1)}(k+2)^{(1)} \\ & \otimes (k+2)^{(\lambda_{k+2}-1)}(k+3)^{(1)} \otimes \ldots \otimes (n-1)^{(\lambda_{n-1}-1)} n^{(1)} \otimes n^{(\lambda_n-1)}.\end{aligned}$$

Thus our matrix for the map u is an $(n-1) \times (n-1)$ matrix.

To compute the matrix of u with respect to the bases

$$\{\alpha_i \colon 1 \leqslant i \leqslant n-1\} \text{ and } \{\beta_k \colon 1 \leqslant k \leqslant n-1\},$$

we have to compute $u(\alpha_i)$ for $i = 1, 2, \ldots, n-1$. But

$$u(\alpha_i) = \Box_\lambda^*(\alpha_i) = \alpha_i \circ \Box_\lambda \colon X_1 \xrightarrow{\Box_\lambda} D_\lambda \xrightarrow{\alpha_i} K_\mu$$

is a composition, and we can compute it by looking at

$$\alpha_i \circ \Box_\lambda(1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k+1)} \otimes (k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)})$$

for $k = 1, 2, \ldots, n-1$.

Theorem 4.1. *The matrix of the map u $(= \Box_\lambda^*)$ is a square matrix of order $n-1$ defined by*

$$\begin{bmatrix} s_1+2 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 1 & \dots & 0 & 0 \\ -1 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-2} & 0 & 0 & 0 & \dots & 1 & 0 \\ (-1)^{n-1} & 0 & 0 & 0 & \dots & 1 & s_{n-1}+1 \\ (-1)^n & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

P r o o f. We will compute the matrix representation of the boundary map u:

$$\alpha_i \circ \Box_\lambda (1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k+1)} \otimes (k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)})$$

for $i, k = 1, 2, \ldots, n-1$.

Case 1. $k=1$: For $i=1$,

$$\begin{aligned}
&\alpha_1 \circ \Box_\lambda (1^{(\lambda_1+1)} \otimes 2^{(\lambda_2-1)} \otimes 3^{(\lambda_3)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&\quad = \alpha_1 (1^{(\lambda_1)} \otimes 1^{(1)} 2^{(\lambda_2-1)} \otimes 3^{(\lambda_3)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&\quad = 1^{(\lambda_1)} 1^{(1)} \otimes 2^{(\lambda_2-1)} 3^{(1)} \otimes 3^{(\lambda_3-1)} 4^{(1)} \otimes \ldots \otimes n^{(\lambda_n-1)} \\
&\qquad + 1^{(\lambda_1)} 2^{(1)} \otimes 1^{(1)} 2^{(\lambda_2-2)} 3^{(1)} \otimes 3^{(\lambda_3-1)} 4^{(1)} \otimes \ldots \otimes n^{(\lambda_n-1)} \\
&\quad \overset{\text{st}}{\equiv} \left[\binom{\lambda_1+1}{1} - \binom{\lambda_2-1}{1} \right] 1^{(\lambda_1+1)} \otimes 2^{(\lambda_2-1)} 3^{(1)} \otimes 3^{(\lambda_3-1)} 4^{(1)} \otimes \ldots \otimes n^{(\lambda_n-1)} \\
&\quad = (\lambda_1 - \lambda_2 + 2)\beta_1 = (s_1+2)\beta_1.
\end{aligned}$$

Here the symbol $\overset{\text{st}}{\equiv}$ means that the straightening law is used. We will continue to use this symbol in the sequel. For $i \geqslant 2$,

$$\begin{aligned}
&\alpha_i \circ \Box_\lambda (1^{(\lambda_1+1)} \otimes 2^{(\lambda_2-1)} \otimes 3^{(\lambda_3)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&= \alpha_i (1^{(\lambda_1)} \otimes 1^{(1)} 2^{(\lambda_2-1)} \otimes 3^{(\lambda_3)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&= 1^{(\lambda_1)} (i+1)^{(1)} \otimes 1^{(1)} 2^{(\lambda_2-1)} \otimes 3^{(\lambda_3)} \otimes \ldots \otimes i^{(\lambda_i)} \\
&\quad \otimes (i+1)^{(\lambda_{i+1}-1)} (i+2)^{(1)} \otimes \ldots \otimes (n-1)^{(\lambda_{n-1}-1)} n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\overset{\text{st}}{\equiv} (-1) 1^{(\lambda_1+1)} \otimes 2^{(\lambda_2-1)} (i+1)^{(1)} \otimes 3^{(\lambda_3)} \otimes \ldots \otimes i^{(\lambda_i)} \\
&\quad \otimes (i+1)^{(\lambda_{i+1}-1)} (i+2)^{(1)} \otimes \ldots \otimes (n-1)^{(\lambda_{n-1}-1)} n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\overset{\text{st}}{\equiv} 1^{(\lambda_1+1)} \otimes 2^{(\lambda_2-1)} 3^{(1)} \otimes 3^{(\lambda_3-1)} (i+1)^{(1)} \otimes \ldots \otimes i^{(\lambda_i)} \\
&\quad \otimes (i+1)^{(\lambda_{i+1}-1)} (i+2)^{(1)} \otimes \ldots \otimes (n-1)^{(\lambda_{n-1}-1)} n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\overset{\text{st}}{\equiv} \ldots \overset{\text{st}}{\equiv} (-1)^{i+1} 1^{(\lambda_1+1)} \otimes 2^{(\lambda_2-1)} 3^{(1)} \otimes \ldots \otimes i^{(\lambda_i-1)} (i+1)^{(1)} \\
&\quad \otimes (i+1)^{(\lambda_{i+1}-1)} (i+2)^{(1)} \otimes \ldots \otimes (n-1)^{(\lambda_{n-1}-1)} n^{(1)} \otimes n^{(\lambda_n-1)} = (-1)^{i+1}\beta_1.
\end{aligned}$$

Case 2. $k \geqslant 2$:
For $i \leqslant k-2$,

$$
\begin{aligned}
\alpha_i \circ \Box_\lambda & (1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k+1)} \otimes (k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&= \alpha_i(1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&= 1^{(\lambda_1)}(i+1)^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes i^{(\lambda_i)} \otimes (i+1)^{(\lambda_{i+1}-1)}(i+2)^{(1)} \otimes \ldots \\
&\quad \otimes (k-1)^{(\lambda_{k-1}-1)}k^{(1)} \otimes k^{(\lambda_k-1)}k^{(1)} \otimes (k+1)^{(\lambda_{k+1}-1)}(k+2)^{(1)} \otimes \ldots \\
&\quad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\quad + 1^{(\lambda_1)}(i+1)^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes i^{(\lambda_i)} \otimes (i+1)^{(\lambda_{i+1}-1)}(i+2)^{(1)} \otimes \ldots \\
&\quad \otimes (k-1)^{(\lambda_{k-1}-1)}k^{(1)} \otimes k^{(\lambda_k-1)}(k+1)^{(1)} \\
&\quad \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-2)}(k+2)^{(1)} \otimes \ldots \\
&\quad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\overset{\text{st}}{\equiv} \left[\binom{\lambda_k}{1} - \binom{\lambda_{k+1}-1}{1}\right] 1^{(\lambda_1)}(i+1)^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \\
&\quad \otimes i^{(\lambda_i)} \otimes (i+1)^{(\lambda_{i+1}-1)}(i+2)^{(1)} \otimes \ldots \\
&\quad \otimes (k-1)^{(\lambda_{k-1}-1)}k^{(1)} \otimes k^{(\lambda_k)} \otimes (k+1)^{(\lambda_{k+1}-1)}(k+2)^{(1)} \otimes \ldots \\
&\quad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\overset{\text{st}}{\equiv} 0.
\end{aligned}
$$

For $i = k-1$,

$$
\begin{aligned}
\alpha_{k-1} \circ \ \Box_\lambda & \ (1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k+1)} \otimes (k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&= \alpha_{k-1}(1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&= 1^{(\lambda_1)}k^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes (k-1)^{(\lambda_{k-1})} \otimes k^{(\lambda_k-1)}k^{(1)} \\
&\quad \otimes (k+1)^{(\lambda_{k+1}-1)}(k+2)^{(1)} \otimes \ldots \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\quad + 1^{(\lambda_1)}k^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes (k-1)^{(\lambda_{k-1})} \otimes k^{(\lambda_k-1)}(k+1)^{(1)} \\
&\quad \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-2)}(k+2)^{(1)} \otimes \ldots \\
&\quad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\overset{\text{st}}{\equiv} \left[\binom{\lambda_k}{1} - \binom{\lambda_{k+1}-1}{1}\right] 1^{(\lambda_1)}k^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \\
&\quad \otimes (k-1)^{(\lambda_{k-1})} \otimes k^{(\lambda_k)} \otimes (k+1)^{(\lambda_{k+1}-1)}(k+2)^{(1)} \otimes \ldots \\
&\quad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&= (\lambda_k - \lambda_{k+1} + 1)\beta_k = (s_k+1)\beta_k \\
&= \begin{cases} 1\beta_k & \text{if } k = 2, 3, \ldots, n-2, \\ (s_{n-1}+1)\beta_k & \text{if } k = n-1. \end{cases}
\end{aligned}
$$

For $i = k$,

$$\begin{aligned}
&\alpha_k \circ \Box_\lambda\ (1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k+1)} \otimes (k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&\quad = \alpha_k(1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&\quad = 1^{(\lambda_1)}k^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes (k+1)^{(\lambda_{k+1}-1)}(k+2)^{(1)} \otimes \ldots \\
&\qquad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\qquad + 1^{(\lambda_1)}(k+1)^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-2)}(k+2)^{(1)} \otimes \ldots \\
&\qquad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \\
&\quad \overset{\mathrm{st}}{\equiv} 1^{(\lambda_1)}k^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes (k+1)^{(\lambda_{k+1}-1)}(k+2)^{(1)} \otimes \ldots \\
&\qquad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} = 1\beta_k,
\end{aligned}$$

since the above second term is straightened into zero.

For $i \geqslant k+1$,

$$\begin{aligned}
&\alpha_i \circ \Box_\lambda\ (1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k+1)} \otimes (k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&\quad = \alpha_i(1^{(\lambda_1)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \otimes n^{(\lambda_n)}) \\
&\quad = 1^{(\lambda_1)}(i+1)^{(1)} \otimes 2^{(\lambda_2)} \otimes \ldots \otimes k^{(\lambda_k)} \otimes k^{(1)}(k+1)^{(\lambda_{k+1}-1)} \otimes \ldots \\
&\qquad \otimes i^{(\lambda_i)} \otimes (i+1)^{(\lambda_{i+1}-1)}(i+2)^{(1)} \otimes \ldots \\
&\qquad \otimes (n-1)^{(\lambda_{n-1}-1)}n^{(1)} \otimes n^{(\lambda_n-1)} \overset{\mathrm{st}}{\equiv} 0.
\end{aligned}$$

Hence we get the matrix of u as required. $\Box$

Notice that our matrix representation of the boundary map u just depends on s_1 and s_{n-1}. Since $d = 1$, we denote this matrix by $M(s_1, s_{n-1}, 1)$.

Theorem 4.2. *The r-minors, for $r = 1, 2, \ldots, n-2$, of the matrix $M(s_1, s_{n-1}, 1)$ are coprimes.*

P r o o f. We shall show that there exists an r-minor Δ_r of the matrix $M(s_1, s_{n-1}, 1)$ such that $\Delta_r = 1$ for each $r = 1, 2, \ldots, n-2$. By taking the rows $2, 3, \ldots, r+1$ and the columns $1, 2, \ldots, r$ from $M(s_1, s_{n-1}, 1)$, we get an r-minor Δ_r of $M(s_1, s_{n-1}, 1)$, which is of the form

$$\Delta_r = \begin{vmatrix} -1 & 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 1 & \ldots & 0 \\ -1 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{r+1} & 0 & 0 & 0 & \ldots & 1 \\ (-1)^{r+2} & 0 & 0 & 0 & \ldots & 0 \end{vmatrix} = \begin{vmatrix} * & \triangle' \\ (-1)^{r+2} & 0 \ \ldots \ 0 \end{vmatrix}.$$

Here $\Delta' = 1$, since Δ' is the determinant of an upper triangular matrix with 1's in the main diagonal. Thus

$$\Delta_r = (-1)^{r+2}(-1)^{r+1}\Delta' = (-1)^{2r+3} = -1.$$

By altering the order of rows(or columns), we have an r-minor of $M(s_1, s_{n-1}, 1)$ which is equal to one. This comletes the proof of the theorem. $\square$

From this theorem we immediately obtain the following two results.

Corollary 4.3. *The invariant factors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-2}$ of the matrix $M(s_1, s_{n-1}, 1)$ are all equal to one.*

Corollary 4.4. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-rowed partition with $|\lambda| = r$ and $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$, and let $\mu = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_n - 1)$ be a partition. Then*
(1) $\mathrm{Ext}^1_{A_r}(K_\lambda, K_\mu) \cong \mathbb{Z}/(\delta)$, *which is cyclic, and*
(2) $\mathrm{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu) = \begin{cases} 0 & \text{if } p \nmid \delta, \\ \mathbb{Z}/(p) & \text{if } p \mid \delta, \end{cases}$

i.e., $\varepsilon_p(\lambda, 1, 1, n) = \begin{cases} 0 & \text{if } p \nmid \delta, \\ 1 & \text{if } p \mid \delta, \end{cases}$ *where δ is the highest invariant factor of the matrix $M(s_1, s_{n-1}, 1)$.*

Finally, we get the important result, i.e., a formula for δ in this case.

Theorem 4.5. *Let $\delta = \delta(n, 1)$ be the highest invariant factor of the matrix $M(s_1, s_{n-1}, 1)$ when $d = 1$. Then $\delta = s_1 + s_{n-1} + n = \lambda_1 - \lambda_n + n$. Moreover, $\delta = \alpha_n$ where $\alpha_n = s_1 + s_2 + \ldots + s_{n-1} + d + (n-1)$.*

Proof. We know that δ is the gcd of the maximal minors of the matrix $M(s_1, s_{n-1}, 1)$. Since $M(s_1, s_{n-1}, 1)$ is a square matrix,

$$\delta = \det M(s_1, s_{n-1}, 1).$$

Thus we have

$$\delta = \begin{vmatrix} s_1+2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 1 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-1)^{n-2} & 0 & 0 & \ldots & 1 & 1 & 0 \\ (-1)^{n-1} & 0 & 0 & \ldots & 0 & 1 & s_n+1 \\ (-1)^n & 0 & 0 & \ldots & 0 & 0 & 1 \end{vmatrix}.$$

Add $(-1)^{n-1}$ times the last column to the first column, and then expand along the last row. Then

$$\delta = (-1)^{(n-1)+(n-1)} \begin{vmatrix} s_1+2 & 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & 1 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-2} & 0 & 0 & \dots & 1 & 1 \\ (-1)^{n-1}(s_{n-1}+2) & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

Add $(-1)^{n-2}(s_{n-1}+2)$ times the last column to the first column, and then expand along the last row. Then

$$\delta = (-1)^{(n-2)+(n-2)} \begin{vmatrix} s_1+2 & 1 & 0 & \dots & 0 \\ -1 & 1 & 1 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n-2}(s_{n-1}+3) & 0 & 0 & \dots & 1 \end{vmatrix}.$$

By repeating the above process, we arrive at

$$\begin{aligned}\delta &= \begin{vmatrix} s_1+2 & 1 & 0 \\ -1 & 1 & 1 \\ (s_{n-1}+n-3) & 0 & 1 \end{vmatrix} = \begin{vmatrix} s_1+2 & 1 \\ -(s_{n-1}+n-2) & 1 \end{vmatrix} \\ &= s_1+2+s_{n-1}+n-2 = s_1+s_{n-1}+n = \lambda_1-\lambda_n+n.\end{aligned}$$

□

Corollary 4.6. *If $\lambda = (r-a, 1^a)$, $0 \leqslant a \leqslant r-1$, is a hook partition with $|\lambda| = r$ and $\mu = (r-a+1, 1^{a-1})$, then*

$$\operatorname{Hom}_{\bar{A}_r}(\overline{K}_\lambda, \overline{K}_\mu) = \begin{cases} 0 & \text{if } p \nmid r, \\ \mathbb{Z}/(p) & \text{if } p \mid r. \end{cases}$$

Proof. This is the case $d = 1$. By Theorem 4.5,

$$\delta = \lambda_1 - \lambda_n + n = (r-a) - 1 + (a+1) = r.$$

From Corollary 4.4, we have our result. □

Remark. M. Teresa and F. Oliveira-Martins [22] proved the above hook partition case by using somewhat complicated ways, which is different from our attempts.

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