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SAMUEL COMPACTIFICATION AND UNIFORM COREFLECTION OF NEARNESS σ-FRAMES

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Abstract. We introduce the structure of a nearness on a σ-frame and construct the coreflection of the category **NσFrm** of nearness σ-frames to the category **KRegσFrm** of compact regular σ-frames. This description of the Samuel compactification of a nearness σ-frame is in analogy to the construction by Baboolal and Ori for nearness frames in [1] and that of Walters for uniform σ-frames in [11]. We also construct the uniform coreflection of a nearness σ-frame, that is, the coreflection of the category of **NσFrm** to the category **UσFrm** of uniform σ-frames.

Keywords: σ-frame, nearness, Samuel compactification

MSC 2000: 06D22, 18B35, 54D35

1. Background

A *σ-frame* L is a bounded lattice with *top* and *bottom* (denoted by 1 and 0 respectively) which is countably complete satisfying the distributive law

$$x \wedge \bigvee S = \bigvee_{s \in S} (x \wedge s)$$

for all $x \in L$ and any countable $S \subseteq L$. For σ-frames L and M, a *σ-frame homomorphism* is a map $h\colon L \to M$ which preserves top and bottom, finite meets and all countable joins. The resulting category is denoted as σ**Frm**. The papers [5] and [9] provide further details on the category σ**Frm**.

A *frame* is a σ-frame which is closed under arbitrary joins (a complete lattice) satisfying the above distributive law but for arbitrary subsets $S \subseteq L$. Frame homomorphisms are σ-frame maps that preserve all joins. The text [8] provides a rich study of the category **Frm** of frames and frame homomorphisms. For any topological space X, the lattice of open sets $\mathfrak{O}X$ is a frame with the join of an arbitrary collection of open sets being the union and with the meet, the interior of the intersection. Another example of a σ-frame is the lattice of cozero sets in any topological space.

In any σ-frame (frame) L if $x, y \in L$ we say that y is *rather below* x (written as $y \prec x$) if there is $t \in L$ such that $y \wedge t = 0$ and $t \vee x = 1$. The σ-frame L is called a *regular* σ-frame if for each $x \in L$ there is a countable $T \subseteq \{y \in L\colon\ y \prec x\}$ such that $x = \bigvee T$. Consequently, we have the full coreflective subcategory **RegσFrm** of regular σ-frames. Regular frames are those in which each element can be expressed as a join of elements rather below it. **RegFrm** is the corresponding category of regular frames.

In the σ-frame (frame) L an element $c \in L$ is a *compact* element if $c \leqslant \bigvee X$ for any countable (arbitrary) $X \subseteq L$. We then have $c \leqslant \bigvee F$ for some finite $F \subseteq X$. We will denote finite subsets by the symbol $\Subset$. L is called a *compact* σ-frame (frame) provided that the unit in L is compact. **KRegσFrm** is the full subcategory of compact regular σ-frames which is shown to be coreflective in **RegσFrm** in [5] with the coreflection map given by the join $\kappa_L\colon\ \mathfrak{K}L \longrightarrow L$ for any $L \in$ **RegσFrm**. $\mathfrak{K}L$ is the σ-frame of all countably generated *regular* ideals of L where an ideal $I \subseteq L$ is *regular* if for each $a \in I$ there is $b \in I$ such that $a \prec b$.

L is a *normal* σ-frame (frame) if for each pair $a, b \in L$ with $a \vee b = 1$, there is $u, v \in L$ with $u \wedge v = 0$ such that $a \vee u = b \vee v = 1$. It is shown in [5] that every regular σ-frame is normal so that, in this special case, $\prec$ interpolates, i.e. if $a \prec c$ in L then $a \prec b \prec c$ for some $b \in L$. In contrast with regular σ-frames, regular frames need not be normal.

A *cover* on the σ-frame (frame) L is any countable (arbitrary) subset $A \subseteq L$ such that $\bigvee A = 1$. $\operatorname{cov} L$ will denote the collection of all covers on the σ-frame (frame) L and for $A, B \in \operatorname{cov} L$ we say that A *refines* B (written as $A \leqslant B$) if for each $a \in A$, $a \leqslant b$ for some $b \in B$. The *meet* of A and B is the set $A \wedge B = \{a \wedge b\colon\ a \in A$ and $b \in B\}$. For any element $x \in L$ the set $Ax = \bigvee\{a \in A\colon\ a \wedge x \neq 0\}$ is the *star* of x with respect to the cover A.

For any subcollection μ of covers in a σ-frame (frame) L and $a, b \in L$ we say that a is *μ-strongly below* b, $a \lhd_\mu b$ (or simply $a \lhd b$ for brevity) provided that $Aa \leqslant b$ for some μ-cover A. If for each element a in a σ-frame L,

$$a = \bigvee T \quad \text{for some countable } T \subseteq \{b \in L\colon\ b \lhd a\},$$

then μ is *admissible.* A *nearness* on the σ-frame L is any admissible filter $\mu \subseteq \operatorname{cov} L$. The couple (L, μ) is then a *nearness* σ-frame. The members of μ are called *uniform* or *nearness* covers. A map $h\colon\ (L, \mu) \longrightarrow (M, \nu)$ between nearness σ-frames (L, μ) and (M, ν) is a *uniform* or *nearness homomorphism* if h is a σ-frame homomorphism on the underlying σ-frames preserving uniform covers, i.e. $h(A) \in \nu$ whenever

$A \in \mu$. We then have the category **NσFrm** of nearness σ-frames and uniform homomorphisms.

A nearness on a frame L is any filter of covers μ in which each element in L can be expressed as a join of elements μ-strongly below it. The structure of a nearness on a frame and the category **NFrm** of nearness frames and uniform homomorphisms has been broadly studied in [7], [4] and [1].

2. Structured σ-frames

The ensuing results are in analogy to that of structured frames (see [4]). In this and subsequent sections L will denote a σ-frame unless otherwise stated.

The following asserts that regularity is a particularly important criterion as it is in the category **RegσFrm** that nearnesses live.

Lemma 1. *L has a nearness $\Leftrightarrow$ L is regular.*

Proof. Suppose that μ is a nearness on L. Let $a \in L$. Then $a = \bigvee T$ for some countable $T \subseteq \{b \in L\colon\ b \lhd a\}$. However, if $t \in T$ then $At \leqslant a$ for some μ-cover A. Then $s = \bigvee\{y \in A\colon\ y \wedge t = 0\}$ separates t and a. Thus $t \prec a$ and T is then a countable subset of $\{b \in L\colon\ b \prec a\}$. By admissibility, $a = \bigvee T$. So L is regular.

Now, if L is regular and $a \in L$ let μ be the filter generated by all countable covers on L. By regularity, $a = \bigvee S$ for some countable $S \subseteq \{x \in L : x \prec a\}$. If $b \prec a$, then there is $s \in L$ which separates b and a. Then $A = \{s, a\} \in \mu$ with $Ax = a$. So, $b \lhd a$ and thus S is a countable subset of $\{x \in L\colon\ x \lhd a\}$ with $a = \bigvee S$ rendering μ admissible and hence a nearness on L. $\square$

Consequently, the filter in L generated by all finite covers is a nearness on L. Moreover, $\operatorname{cov} L$ is a nearness on L which we call the *fine* nearness and any filter on L containing all finite covers is a nearness on L. A nearness σ-frame (L, μ) is called *fine* if $\mu = \operatorname{cov} L$.

If A is a cover on L then the *star of* A is defined as the set $A^* = AA = \{Aa\colon\ a \in A\}$ which is also a cover of L as $A \leqslant A^*$. We say that A *star refines* B (written as $A \leqslant^* B$) if $A^* \leqslant B$. A filter ν is a *preuniformity* on L if for each $A \in \nu$ there is $B \in \nu$ such that $B \leqslant^* A$. A *uniformity* on L is a nearness μ such that for each $A \in \mu$ there exists $B \in \mu$ with $B \leqslant^* A$, i.e. every μ-cover has a μ-star refinement. So, a uniformity on L is then a preuniformity with the additional admissibility criterion. We then have the category **UσFrm** of uniform σ-frames and uniform homomorphisms. For a more comprehensive treatment of **UσFrm** see [11], [12] and [13].

Lemma 2. *If L is compact and regular then L has a unique nearness, namely* $\operatorname{cov} L$*, which is a uniformity.*

P r o o f. We already have that $\operatorname{cov} L$ is a nearness. For uniqueness, let ν be any nearness on L. We show that ν contains all finite covers and hence all covers of L. Let $A = \{a_1, a_2, \dots, a_n\}$ be any finite cover on L. By admissibility, $a_i = \bigvee_m \{x_{im} \in L \colon x_{im} \lhd_\nu a_i\}$ for each $1 \leqslant i \leqslant n$. Then $\bigvee_i \bigvee_m \{x_{im} \in L \colon x_{im} \lhd_\nu a_i\} = 1$. By compactness, $\bigvee_i \bigvee_j \{x_{ij} \in L \colon x_{ij} \lhd_\nu a_i\} = 1$ for some $\{x_{ij}\} \Subset \{x_{im}\}$. Then for each i there exists $B_i \in \nu$ such that $B_i x_{ij} \leqslant a_i$. Then $B = \bigwedge B_i \in \nu$ and $B x_{ij} \leqslant a_i$ for each i. Then for each $b \in B$, some $b \wedge x_{ij} \neq 0$ so that $b \leqslant B x_{ij} \leqslant a_i$. Thus $B \leqslant A$ and so $A \in \nu$.

The proof that $\operatorname{cov} L$ is a uniformity essentially follows the proof in [4]. We need only to show that any finite cover on L has a finite star refinement. Let $A = \{a_1, a_2, \dots, a_n\}$ be any finite cover on L. By regularity, $a_i = \bigvee_m \{x_{im} \in L \colon x_{im} \prec a_i\}$ for each $1 \leqslant i \leqslant n$. Then $\bigvee_i \bigvee_m \{x_{im} \in L \colon x_{im} \prec a_i\} = 1$. Again, by compactness, $\bigvee_i \bigvee_j \{x_{ij} \in L \colon x_{ij} \prec a_i\} = 1$ for some $\{x_{ij}\} \Subset \{x_{im}\}$. Then for each i there exists $t_i \in L$ such that $x_{ij} \wedge t_i = 0$ and $t_i \vee a_i = 1$. Since $\{x_{ij}\}$ is a cover on L, $\wedge t_i = 0$. Let $B = \bigwedge_i \{a_i, t_i\}$. Then B is a cover on L and each element in B can be expressed for $E \subseteq \{1, 2, \dots, n\}$ as

$$a_E = \bigwedge_i \{a_i \colon i \in E\} \wedge \bigwedge_j \{t_j \colon j \notin E\}.$$

Then $a_E \leqslant a_i$ for each i and $B \leqslant A$. Hence, if one obtains a finite cover $C_i \leqslant^* \{a_i, t_i\}$ for each i, then

$$C = \bigwedge_i C_i \leqslant^* B \leqslant A.$$

It then suffices to show that each two-cover $\{a, b\}$ has a finite star-refinement. Since L is normal according to [6], there exist $u \prec a$ and $v \prec b$ such that $u \vee v = 1$. Then there exist $s, t \in L$ such that

$$u \wedge s = 0, \quad s \vee a = 1, \quad v \wedge t = 0, \quad t \vee b = 1.$$

Let $D = \{a, s\} \wedge \{b, t\} \wedge \{u, v\} = \{a \wedge b \wedge u,\ a \wedge b \wedge v,\ a \wedge t \wedge u,\ b \wedge s \wedge v\}$. Then $D \in \operatorname{cov} L$ and $D(s \wedge b \wedge v) \leqslant b$, $D(a \wedge b \wedge u) \leqslant a$, $D(a \wedge b \wedge v) \leqslant b$ and $D(a \wedge t \wedge u) \leqslant a$. Thus $D \leqslant^* \{a, b\}$. □

Thus for any compact regular σ-frame, $\operatorname{cov} L$ is its unique nearness which in fact is a uniformity. Also, by the above Lemma, every compact nearness σ-frame is fine.

A nearness μ is *strong* if for each $A \in \mu$ there exists $B \in \mu$ such that for each $b \in B$, $b \lhd a$ for some $a \in A$. We call the nearness μ *almost uniform* if μ is strong and $\lhd$ interpolates.

Lemma 3. *If μ is a uniformity on L, then μ is almost uniform.*

Proof. Let $A \in \mu$. Find $B \in \mu$ such that $B \leqslant^* A$. Then for each $b \in B$, $Bb \leqslant a$ for some $a \in A$. Thus $b \lhd a$ and so μ is strong.

Now let $x \lhd y$ in L. Then $Cx \leqslant y$ for some $C \in \mu$. Again as μ is a uniformity we can find $D \in \mu$ such that $D \leqslant^* C$. If $d \in D$ and $d \wedge Dx \neq 0$, then

$$0 \neq d \wedge Dx = d \wedge \bigvee\{t \in D\colon\ t \wedge x \neq 0\} = \bigvee\{d \wedge t\colon\ t \in D,\ t \wedge x \neq 0\}.$$

Thus $d \wedge \tilde{d} \neq 0$ for some $\tilde{d} \in D$ such that $\tilde{d} \wedge x \neq 0$. Then $d \leqslant D\tilde{d}$ with $\tilde{d} \wedge x \neq 0$.

Since $D \leqslant^* C$, we have $D\tilde{d} \leqslant c$ for some $c \in C$. If $c \wedge x = 0$, then $x \wedge \tilde{d} \leqslant x \wedge D\tilde{d} \leqslant c \wedge x$. Thus $\tilde{d} \wedge x = 0$, a contradiction. So we have that $c \wedge x \neq 0$. Then $d \leqslant D\tilde{d} \leqslant c \leqslant Cx$ and hence, $\bigvee\{d \in D\colon\ d \wedge Dx \neq 0\} = D(Dx) \leqslant Cx$. Since $x \lhd Dx$ we have $x \lhd Dx \lhd D(Dx) \leqslant Cx \leqslant y$. Thus $x \lhd Dx \lhd y$ and so $\lhd$ interpolates. Hence, μ is almost uniform. □

Lemma 4. *For any nearness σ-frame (L, μ) with $h\colon\ L \longrightarrow M$ any onto σ-frame homomorphism, $\nu = \{B \in \operatorname{cov} M\colon\ h(A) \leqslant B$ for some $A \in \mu\}$ is a nearness on M. Furthermore, ν is strong or a uniformity whenever μ is strong or a uniformity, respectively.*

Proof. If $A, B \in \nu$ then $h(C) \leqslant A$ and $h(D) \leqslant B$ for some $C, D \in \mu$. Then $C \wedge D \in \mu$ and as h is a σ-frame map we have $h(C \wedge D) = h(C) \wedge h(D) \leqslant A \wedge B$. Thus $A \wedge B \in \nu$. Clearly, if $S \in \nu$ and $S \leqslant T$ then $h(E) \leqslant S \leqslant T$ for some $E \in \mu$. Thus $T \in \nu$. Hence, ν is a filter of M-covers.

For admissibility, let $y \in M$. Since h is onto, $y = h(x)$ for some $x \in L$. By the admissibility of μ, $x = \bigvee T$ for some countable $T \subseteq \{t \in L\colon\ t \lhd_\mu x\}$. If $t \in T$ then $At \leqslant x$ for some $A \in \mu$. Since h is a σ-frame homomorphism we have

$$\bigvee h(A) = h\Big(\bigvee A\Big) = h(1_L) = 1_M.$$

Thus $h(A) \in \nu$. Further, if $h(a) \wedge h(t) \neq 0_M$ for $a \in A$, then $h(a \wedge t) \neq 0_M$ and hence $a \wedge t \neq 0_L$. Thus $a \leqslant At$. Then $h(A)h(t) \leqslant h(At) \leqslant h(x) = y$. Thus $t \lhd_\mu x$ implies that $h(t) \lhd_\nu y$. Let $S = \{h(t)\colon\ t \in T\}$. Then S is a countable subset of $\{s \in M\colon\ s \lhd_\nu y\}$ and since $x = \bigvee T$, we conclude that $y = \bigvee S$ showing the admissibility of ν.

Now suppose that μ is strong. Let $B \in \nu$ and find $C \in \mu$ such that $h(C) \leqslant B$. Since μ is strong there exists $A \in \mu$ such that for each $a \in A$, $a \lhd_\mu c$ for some $c \in C$. Then $h(A) \in \nu$ and $h(c) \lhd_\nu h(a)$. But $h(a) \leqslant b$ for some $b \in B$. Thus $h(c) \lhd_\nu b$. Thus $h(A) \in \nu$ is such that for each $h(a) \in h(A)$, $h(a) \lhd_\nu b$ for some $b \in B$ showing that ν is strong.

Now let $(L, \mu) \in \mathbf{U\sigma Frm}$ and $B \in \nu$. Then $h(C) \leqslant B$ as above. Since μ is a uniformity, $A \leqslant^* C$ for some $A \in \mu$. Then for each $a \in A$, $Aa \leqslant c$ for some $c \in C$. Then

$$\begin{aligned}
h(A)h(a) &= \bigvee_{y \in A} \{h(y)\colon\ h(y) \wedge h(a) \neq 0_M\} \\
&= \bigvee_{y \in A} \{h(y)\colon\ h(y \wedge a) \neq 0_M\} \quad \text{(since } h \text{ is a } \sigma\text{-frame map)} \\
&\leqslant \bigvee_{y \in A} \{h(y)\colon\ y \wedge a \neq 0_L\} \quad \text{(since } h(y \wedge a) \neq 0_M \text{ implies } y \wedge a \neq 0_L) \\
&= h\Big(\bigvee \{y \in A\colon\ y \wedge a \neq 0_L\}\Big) \quad \text{(since } h \text{ is a } \sigma\text{-frame map)} \\
&= h(Aa) \leqslant h(c).
\end{aligned}$$

Thus $h(A) \leqslant^* h(C) \leqslant B$. Thus $h(A) \in \nu$ is such that $h(A) \leqslant^* B$. Hence ν is a uniformity. □

For a nearness μ on the σ-frame (frame) L, $A \in \mu$ is *normal* if there is a sequence of uniform covers $(A_n) \subseteq \mu$ such that $A = A_1$ and $A_{n+1} \leqslant^* A_n$ for each n. We denote by μ_N the normal covers of μ for any $(L, \mu) \in \mathbf{N\sigma Frm}$ (or $\mathbf{NFrm}$). Clearly μ_N is a preuniformity on L. If $x \lhd_{\mu_N} a$ in L, we say that x is *uniformly (strongly) normally below* a and in keeping with [1], we write this as $x \blacktriangleleft a$. The following results for the relation $\blacktriangleleft$ are in analogy with those in [1] and [11] respectively.

Theorem 1. *Let* $(L, \mu) \in \mathbf{N\sigma Frm}$.

1. $\blacktriangleleft$ *is a sublattice of* $L \times L$.
2. *For any* $a, b, x, y \in L$ *with* $a \leqslant x \blacktriangleleft y \leqslant b$, $a \blacktriangleleft b$.
3. *In* L, $x \blacktriangleleft y \Rightarrow x \lhd_\mu y \Rightarrow x \prec y$.
4. $\blacktriangleleft$ *interpolates.*
5. $\blacktriangleleft$ *is preserved by uniform* σ*-frame homomorphisms.*

P r o o f. 1. If $a \blacktriangleleft b$ and $c \blacktriangleleft d$ in (L, μ), then $Aa \leqslant b$ and $Cc \leqslant d$ for some $A, C \in \mu_N$. As μ_N is a preuniformity, $A \wedge C \in \mu_N$ and $(A \wedge C)(a \vee c) \leqslant b \vee d$. Also $(A \wedge C)(a \wedge c) \leqslant b \wedge d$. Thus $a \vee c \blacktriangleleft b \vee d$ and $a \wedge c \blacktriangleleft b \wedge d$. As $B0 \leqslant 0$ and $B1 \leqslant 1$ for any $B \in \mu_N$, thus $0 \blacktriangleleft 0$ and $1 \blacktriangleleft 1$.

2. If $a \leqslant x \blacktriangleleft y \leqslant b$ in L, then $Ax \leqslant y$ for some $A \in \mu_N$ and $Aa \leqslant Ax \leqslant y \leqslant b$. So, $a \blacktriangleleft b$.

3. Obvious.

4. If $x \blacktriangleleft y$, then $Ax \leqslant y$ for some $A \in \mu_n$. As μ_N is a preuniformity, we have $B \leqslant^* A$ for some $B \in \mu_N$. Then $B(Bx) \leqslant Ax \leqslant y$ and $x \blacktriangleleft Ax \blacktriangleleft y$. Thus $\blacktriangleleft$ interpolates.

5. Let $h\colon (L,\mu) \longrightarrow (M,\nu)$ be any uniform σ-frame homomorphism in $\mathbf{N\sigma Frm}$ with $x \blacktriangleleft y$ in (L,μ). Then $Ax \leqslant y$ for some μ_N-cover A. Then $A = A_1$ and $A_{n+1} \leqslant^* A_n$ for each n for some sequence $\{A_n\} \subseteq \mu$. Since h is uniform, $h(A) = h(A_1)$ and $h(A_{n+1}) \leqslant^* h(A_n)$ for each n. So, $\{h(A_n)\} \subseteq \nu_N$. Thus $h(A) \in \nu_N$ and $Ax \leqslant y$ implies that $h(A)h(x) \leqslant h(Ax) \leqslant h(y)$. Thus $h(x) \blacktriangleleft h(y)$. □

Lemma 5. *For $a \blacktriangleleft b \blacktriangleleft c$ in $(L,\mu) \in$ $\mathbf{N\sigma Frm}$ there exists $s, t \in L$ such that s separates $a \prec b$, $t \vee c = 1$ and $t \blacktriangleleft s$.*

Proof. Let $a \blacktriangleleft b$ in L. Then there exists $A \in \mu_N$ such that $Aa \leqslant b$. Since μ_N is a preuniformity, $B \leqslant^* A$ for some $B \in \mu_N$. But $B(Ba) \leqslant Aa$. Thus $a \blacktriangleleft Ba \blacktriangleleft Aa \leqslant b \blacktriangleleft c$. Let $s = \bigvee\{b \in B : b \wedge a = 0\}$. Then $a \wedge s = 0$ and $s \vee Ba = 1$. So, $a \prec Ba$. Since $Ba \leqslant b$, also $a \prec b$. Set $t = \bigvee\{b \in B : b \wedge Ba = 0\}$. Then $t \vee Ba = 1$ and as $Ba \blacktriangleleft c$, we have $t \vee c = 1$. Now $Bt = \bigvee\{b \in B\colon b \wedge t \neq 0\} \leqslant s$ as, if $b \in B$ with $b \wedge t \neq 0$, then $b \wedge b_m \neq 0$ for some $b_m \in B$ with $b_m \wedge Ba = 0$. Then $b_m \wedge x = 0$ for each $x \in B$ with $x \wedge a \neq 0$. In particular, if $b \wedge a \neq 0$, then $b \wedge b_m = 0$, which contradicts $b \wedge b_m \neq 0$. Thus $b \wedge a = 0$ and so $b \leqslant s$. Hence, $Bt \leqslant s$ and thus $t \blacktriangleleft s$. □

3. The Samuel compactification

In this section we present the compact regular coreflection of a nearness σ-frame as an adaptation of the corresponding results in [1] and [11]. An ideal I in any nearness σ-frame (L,μ) is

1. *uniformly regular* if for each $x \in I$, $x \lhd y$ for some $y \in I$,
2. *uniformly normally regular* if for each $x \in I$, $x \blacktriangleleft y$ for some $y \in I$ and
3. *countably generated* if there is a sequence (y_n) in I such that for each $x \in I$, $x \leqslant y_m$ for some m.

It should be noted that if μ is a uniformity on L then there is no distinction between $\blacktriangleleft$ and $\lhd$. So in uniform sigma-frames there is no distinction between the uniformly regular ideals and the uniformly normally regular ones. However, in $\mathbf{N\sigma Frm}$ this need not be the case. Every uniformly normally regular ideal is regular but the converse need not be true. Let $\mathfrak{NR}_\sigma L$ be the set of all countably generated uniformly normally regular ideals of (L,μ). Clearly any $J \in \mathfrak{NR}_\sigma L$ may be generated by a sequence $a_1 \blacktriangleleft a_2 \ldots$ and any ideal generated by any such sequence belongs to $\mathfrak{NR}_\sigma L$. Using the same method as that in [6] together with [11] we show that $\mathfrak{NR}_\sigma L$ is a compact regular σ-frame.

Theorem 2. *$\mathfrak{NR}_\sigma L$ is a compact regular σ-frame.*

Proof. Suppose that $I, J \in \mathfrak{NR}_\sigma L$. If $x \in I$ and $x \in J$, then $x \blacktriangleleft s$ and $x \blacktriangleleft t$ for some $s \in I$ and $t \in J$ as I and J are normally regular. Then $x \blacktriangleleft s \wedge t \in I \cap J$. Thus $I \cap J \in \mathfrak{NR}_\sigma L$. Again by the properties of $\blacktriangleleft$, $I \vee J \in \mathfrak{NR}_\sigma L$. As any updirected join of normally regular ideals is again normally regular, $\mathfrak{NR}_\sigma L$ is closed under finite $\wedge$ and (countable) $\vee$. Since $\blacktriangleleft \Rightarrow \prec$, we have $\mathfrak{NR}_\sigma L \subseteq \mathfrak{K}L$, where $\mathfrak{K}L$ is the compact regular coreflection of the σ-frame L (see [5]). Thus $\mathfrak{NR}_\sigma L$ is compact.

For regularity, consider any $J \in \mathfrak{NR}_\sigma L$ with the generating sequence $a_1, a_2, \ldots$. By repeated interpolation of $\blacktriangleleft$, for each n let J_n be the ideal generated by a sequence

$$a_n = a_{n_0} \blacktriangleleft a_{n_1} \blacktriangleleft a_{n_2} \blacktriangleleft \ldots \blacktriangleleft a_{n+1}.$$

Then $J_n \in \mathfrak{NR}_\sigma L$ and $J = \bigvee J_n$. Also for each n, $a_n \blacktriangleleft a_{n+1} \blacktriangleleft a_{n+2}$ and by Lemma 5 we can find x_n, y_n such that $a_n \wedge x_n = 0$, $x_n \vee a_{n+1} = 1$, $y_n \vee a_{n+2} = 1$ and $y_n \blacktriangleleft x_n$. Let I_n be the ideal generated by the sequence

$$y_{n_0} = y_n \blacktriangleleft y_{n_1} \blacktriangleleft y_{n_2} \blacktriangleleft \ldots \blacktriangleleft x_n.$$

Then $I_n \in \mathfrak{NR}_\sigma L$ and $I_n \cap J_{n+1} = \{0\}$ (as $a_n \wedge x_n = 0$) and $I_n \vee J_{n+2} = L$ (as $a_{n+2} \vee y_n = 1$). Thus $J_{n+1} \prec J_n$ in $\mathfrak{NR}_\sigma L$. Hence $\mathfrak{NR}_\sigma L$ is regular. □

As a compact regular σ-frame has a unique nearness (the fine nearness), category **KRegσFrm** is a full subcategory of **NσFrm**.

Lemma 6. *For $(L,\mu) \in$ **NσFrm**, $\varrho_L\colon \mathfrak{NR}_\sigma L \longrightarrow (L,\mu)$ given by join is a uniform σ-frame homomorphism.*

Proof. ϱ_L is a σ-frame homomorphism which is the restriction of κ_L to the countably generated uniformly normally regular ideals. For uniformity, take any finite cover $\{J_1, J_2, \ldots, J_n\}$ of $\mathfrak{NR}_\sigma L$. Then there is $a_i \in J_i$ such that $a_1 \vee a_2 \vee \ldots \vee a_n = 1$. Let $c_i = \varrho_L(J_i)$. By the uniformly normal regularity of J_i, $a_i \blacktriangleleft c_i$. Then $B_i a_i \leqslant c_i$ for some $B_i \in \mu_N$ for each $i = 1, 2, \ldots, n$. Thus $B = \bigwedge B_i \in \mu$ and $Ba_i \leqslant c_i$ for each i. As $\bigvee\limits_{i=1}^{n} a_i = 1$, for each $t \in B$ we have $t \wedge a_i \neq 0$ for some i. Thus $t \leqslant Ba_i \leqslant c_i$. Hence, $B \leqslant \{c_1, c_2, \ldots, c_n\} = C$. Thus $C \in \mu$, i.e. $\varrho_L(\{J_1, J_2, \ldots, J_n\}) \in \mu$ and so ϱ_L is uniform. □

Lemma 7. *If $M \in$ **KRegσFrm**, then $\varrho_M\colon \mathfrak{N}\mathfrak{R}_\sigma M \longrightarrow M$ is an isomorphism.*

Proof. The proof is immediate since in **KRegσFrm**, $\blacktriangleleft\ = \lhd = \prec$ so that $\mathfrak{N}\mathfrak{R}_\sigma M = \mathfrak{K}M$ and $\varrho_M = \kappa_L\colon \mathfrak{K}M \longrightarrow M$ (see [5]) is the coreflection. Hence, if M is a compact regular σ-frame, then ϱ_M is an isomorphism. $\square$

Theorem 3. *$\mathfrak{N}\mathfrak{R}_\sigma L$ is the compact regular coreflection of the nearness σ-frame (L,μ) with coreflection $\varrho_L\colon \mathfrak{N}\mathfrak{R}_\sigma L \longrightarrow (L,\mu)$ and coreflection functor $\mathfrak{N}\mathfrak{R}_\sigma$.*

Proof. Let $h\colon (M,\nu) \longrightarrow (L,\mu)$ be any uniform σ-frame morphism with M compact. Then $\mathfrak{N}\mathfrak{R}_\sigma h$ is the map taking each $I \in \mathfrak{N}\mathfrak{R}_\sigma M$ to the ideal generated by $h(I)$, $\langle h(I)\rangle$. By Theorem 1, h preserves $\blacktriangleleft$ and so $\mathfrak{N}\mathfrak{R}_\sigma h$ is a well-defined σ-frame homomorphism to $\mathfrak{N}\mathfrak{R}_\sigma L$. We then have the following diagram:

$$\begin{array}{ccc}
\mathfrak{N}\mathfrak{R}_\sigma L & \xleftarrow{\ \mathfrak{N}\mathfrak{R}_\sigma h\ } & \mathfrak{N}\mathfrak{R}_\sigma M \\
\big\downarrow & & \simeq \big\downarrow \varrho_M \\
(L,\mu) & \xleftarrow[\ h\ ]{} & (M,\nu)
\end{array}$$

As M is compact regular, by the previous result, ϱ_M is an isomorphism. Put $\bar h = \mathfrak{N}\mathfrak{R}_\sigma h \varrho_M^{-1}$. We then have that $\varrho_L \bar h = \varrho_L \mathfrak{N}\mathfrak{R}_\sigma h \varrho_M^{-1} = h$. Since ϱ_L is dense and monic, the uniqueness of $\bar h$ follows. $\square$

The above establishes $\mathfrak{N}\mathfrak{R}_\sigma L$ as the Samuel compactification of a nearness σ-frame via its countably generated uniformly normally regular ideals.

4. The uniform coreflection

For details on nearness frames see [1], [4], or [7]. A *uniformity* on a frame is a nearness in which each uniform cover has a uniform star refinement, the structural archetype for the development of the theory of uniform σ-frames (see [11], [12] and [13]). **UFrm** is the category of uniform frames and uniform frame maps discussed in the paper [6].

For the nearness frame (L,μ), the normal uniform cover μ_N is a preuniformity on L. Let $k\colon L \longrightarrow L$ be the interior operator given by

$$k(a) = \bigvee\{x \in L\colon\ x \blacktriangleleft a\}$$

where as before $x \blacktriangleleft a$ means that there is $A \in \mu_N$ such that $Ax \leqslant a$. Then $\mathcal{U}L = \operatorname{Fix} k$ is a subframe of L (see [6]), and $\mathcal{U}\mu = \{k(A)\colon\ A \in \mu_N\}$ is a uniformity on $\mathcal{U}L$. Then $(\mathcal{U}L, \mathcal{U}\mu)$ is the uniform coreflection of the nearness frame (L, μ) with the coreflection map given by the inclusion $j\colon \mathcal{U}L \longrightarrow L$ (see [1]).

Now, let (M, ν) be any uniform frame. As in [11], the *cozero part* of (M, ν) is the set

$$\mathrm{Coz}_u M = \{a \in M\colon\ a = h((0,1]) \text{ for some } h\colon \mathfrak{D}[0,1] \to (M,\nu) \in \mathbf{UFrm}\}.$$

The members of $\mathrm{Coz}_u M$ are called *uniformly cozero elements*. Also let

$$\mathrm{Coz}_u \nu = \{(a_n) = A \in \mathrm{cov}\, M\colon\ a_n \in \mathrm{Coz}_u M\ \forall n\},$$

i.e. $\mathrm{Coz}_u \nu$ is the collection of all countable uniform covers consisting of uniformly cozero elements. By [11], $(\mathrm{Coz}_u M, \mathrm{Coz}_u \nu)$ is a uniform sigma-frame. Given any nearness sigma-frame (L, μ) we show that $(\mathrm{Coz}_u \mathcal{U}(\mathcal{H}L), \mathrm{Coz}_u \mathcal{U}(\mathcal{H}\mu))$ is the uniform coreflection of (L, μ), with $(\mathcal{H}L, \mathcal{H}\mu)$ the nearness frame of all sigma ideals of L and $(\mathcal{U}(\mathcal{H}L), \mathcal{U}(\mathcal{H}\mu))$ its uniform coreflection. It should be noted that an ideal I in L is a σ-ideal in case I is closed under countable joins.

Let L be any σ-frame. Consider the frame envelope of L, $\mathcal{H}L$, the Lindelöf frame of all σ-ideals of L. Then $\downarrow\colon L \longrightarrow \mathcal{H}L$ taking each $a \in L$ to the principal ideal generated by a, $\downarrow a = \{y \in L\colon\ y \leqslant a\}$, is the universal homomorphism from σ-frames to frames (see [3]).

Lemma 8. *For each countable collection* (a_n) *in* L, $\bigvee\limits_{\mathcal{H}L} \downarrow a_n = \downarrow \bigvee a_n$.

Proof. Indeed, $\downarrow \vee a_n$ is a σ-ideal. So $\downarrow \bigvee a_n \in \mathcal{H}L$. Certainly, $\downarrow \bigvee a_n \subseteq \bigvee\limits_{\mathcal{H}L} \downarrow a_n$. Thus, $\downarrow \bigvee a_n$ is an upper bound for $\downarrow a_n$ for each n. Moreover, it is the least one for, if $J \in \mathcal{H}L$ is such that $\downarrow a_n \subseteq J$ for each n, then $a_n \in J$ for each n. Since J is a σ-ideal, we have $\bigvee a_n \in J$. Thus $\downarrow \bigvee a_n \subseteq J$. Hence, $\downarrow \bigvee a_n = \bigvee\limits_{\mathcal{H}L} \downarrow a_n$. □

Now, let (L, μ) be any nearness σ-frame and let $\mathcal{H}\mu$ be the filter on L generated by $\downarrow A = \{\downarrow a\colon\ a \in A\}$ where $A \in \mu$. Then by the above Lemma for each $A = (a_n) \in \mu$,

$$\bigvee_n \downarrow a_n = \downarrow \bigvee_n a_n = \downarrow 1_L = L = 1_{\mathcal{H}L}.$$

Thus $\downarrow A \in \mathrm{cov}\, \mathcal{H}L$ for each $A \in \mu$. We then have the following result for any nearness σ-frame (L, μ).

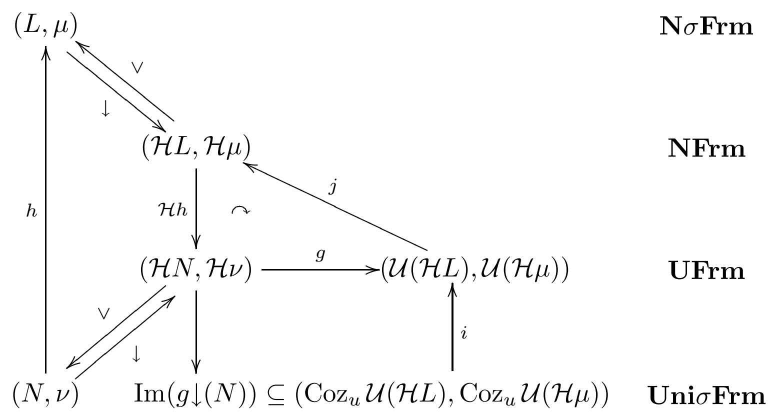
Lemma 9. *$\mathcal{H}\mu$ is a nearness on the frame $\mathcal{H}L$.*

Proof. We need only to show admissibility. So let $J \in \mathcal{H}L$. Then $J = \bigvee\{\downarrow a\colon a \in J\}$. But for each $a \in J$, by the admissibility of μ, $a = \bigvee\{b_n\colon b_n \lhd_\mu a\}$. Let $I_n = \downarrow b_n$ for each n with $b_n \lhd_\mu a$. Then $I_n \in \mathcal{H}L$ for each n. Also, $Bb_n \leqslant a$ for some $B \in \mu$ whenever $b_n \lhd_\mu a$. Then $\downarrow B \in \mathcal{H}\mu$ and $(\downarrow B)(\downarrow b_n) = \downarrow(Bb_n) \subseteq \downarrow a \subseteq J$. So, $(\downarrow B)I_n \subseteq J$ and thus $I_n \lhd_{\mathcal{H}\mu} J$ for each n. Moreover, $J = \bigvee\{I_n\colon I_n \lhd_{\mathcal{H}\mu} J\}$. Hence $\mathcal{H}\mu$ is a nearness on $\mathcal{H}L$. □

We now conclude this section with our final result.

Theorem 4. $(\mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}L), \mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}\mu))$ *is the uniform coreflection of the nearness σ-frame (L,μ).*

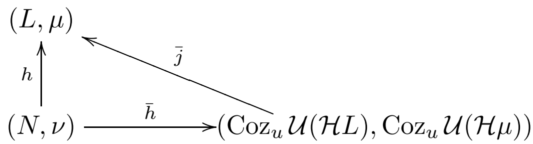
Proof. Let (N,ν) be any uniform σ-frame with uniform homomorphism $h\colon (N,\nu) \longrightarrow (L,\mu)$. We then have the following diagram:



The map $\downarrow\colon (N,\nu) \longrightarrow (\mathcal{H}N, \mathcal{H}\nu)$ is a σ-frame homomorphism. Also $\mathcal{H}h$ is a σ-frame homomorphism, where

$$\mathcal{H}h\Big(\bigvee_{\mathcal{H}N} \downarrow a_n\Big) = \Big\langle h\Big(\bigvee \downarrow a_n\Big)\Big\rangle = \Big\langle \bigvee h(\downarrow a_n)\Big\rangle = \bigvee \mathcal{H}h(\downarrow a_n)$$

and $\langle h(\bigvee \downarrow a_n)\rangle$ is the ideal generated by $h(\bigvee \downarrow a_n)$. We claim that $\bar{j} = \bigvee \circ j \circ i$ is the coreflection map, where $\vee\colon (\mathcal{H}L, \mathcal{H}\mu) \longrightarrow (L,\mu)$, j (the inclusion) is the uniform coreflection map of the nearness frame $(\mathcal{H}L, \mathcal{H}\mu)$ and i is the inclusion. We need to find $\bar{h}\colon (N,\nu) \longrightarrow (\mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}L), \mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}\mu))$ such that the triangle below commutes, i.e. $\bar{j}\bar{h} = h$.



Since j is the uniform coreflection map and $\mathcal{H}h\colon (\mathcal{H}n, \mathcal{H}\nu) \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ with $(\mathcal{H}N, \mathcal{H}\nu)$ is uniform there is a unique uniform homomorphism $g\colon \mathcal{H}N \longrightarrow M$ such that $jg = \mathcal{H}h$. But $(\mathcal{H}N, \mathcal{H}\nu)$ is a Lindelöf uniform frame so that the Lindelöf elements are precisely the uniformly cozero elements (see [11]) which are precisely the principle ideals $\downarrow x$ for each $x \in N$ (see [3]). Since g is uniform and uniform homomorphisms preserve uniform cozero elements, g maps cozero elements to cozero elements. Thus $\mathrm{Im}(g{\downarrow}N) \subseteq (\mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}L), \mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}\mu))$. Let $\bar{h} = g{\downarrow}$. Then the desired triangle above commutes. Since $\mathcal{H}h(\downarrow x) = \langle h(\downarrow x)\rangle = \downarrow h(x)$, we have

$$\bigvee \mathcal{H}h{\downarrow}(x) = \bigvee(\mathcal{H}h(\downarrow x)) = \bigvee \downarrow h(x) = h(x).$$

Thus $\bigvee \mathcal{H}h{\downarrow} = h$. Then

$$\bar{j}\bar{h} = \bigvee jig{\downarrow} = \bigvee(jg){\downarrow} = \bigvee \mathcal{H}h{\downarrow} = h.$$

It remains to show that $\bar{h}$ is unique. Suppose that $h'\colon (N, \nu) \longrightarrow (\mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}L),$ $\mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}\mu))$ with $\bar{j}h' = h$. But for any $I \in \mathrm{Coz}_u\,\mathcal{U}(\mathcal{H}L)$, if $\bar{j}(I) = 0$, then $\bigvee ji(I) = 0$. Thus $\bigvee I = 0$ and hence $I = \{0\}$. Thus $\bar{j}$ is dense and hence monic. Since $\bar{j}h' = h = \bar{j}\bar{h}$, $h = h'$. Hence $\bar{h}$ is unique with the property that $\bar{j}\bar{h} = h$, which completes the proof. □

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