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ON LEFT C-$\mathcal{U}$-LIBERAL SEMIGROUPS

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Abstract. In this paper the equivalence $\widetilde{\mathcal{Q}}^U$ on a semigroup S in terms of a set U of idempotents in S is defined. A semigroup S is called a $\mathcal{U}$-liberal semigroup with U as the set of projections and denoted by $S(U)$ if every $\widetilde{\mathcal{Q}}^U$-class in it contains an element in U. A class of $\mathcal{U}$-liberal semigroups is characterized and some special cases are considered.

Keywords: equivalence $\widetilde{\mathcal{Q}}^U$, left C-$\mathcal{U}$-liberal semigroup, left semi-spined product, band-formal construction, left C-liberal semigroup

MSC 2000: 20M10

1. Introduction

It is well known that Green's equivalences on semigroups have played a fundamental role in the study of regular semigroups. In terms of various generalized Green's equivalences on semigroups (such as $*$-*Green's equivalences* defined by Fountain [5] and $**$-*Green's equivalences* defined by Tang [23] and Du and He [1]), some classes of generalized regular semigroups (such as *abundant semigroups* and *weakly abundant semigroups*) have been defined and studied. In general, in the procession of discussing regular and generalized regular semigroups, all idempotents in semigroups are involved.

Recently, some authors found that the set of some idempotents in a semigroup, such as the C-set of a $\mathcal{P}$-regular semigroup (see [10]–[13], [24]–[26]) and the set of projections of a U-semiabundant semigroup (see [6], [7], [12], [16]–[18]), is perhaps very important to the description for the whole semigroup and, sometimes, is more dominant than the set of all idempotents.

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In this paper, we shall go forward along the path charted by Lawson [17], [18]. The semigroups considered in this paper will be all equipped with a plentiful supply of idempotents. For all undefined terminology and notation the reader is referred to Fountain [5], Howie [15] and Lawson [17].

2. Preliminaries

The aim of this section is to introduce some basic concepts and give some characterizations for C-$\mathcal{U}$-liberal semigroups.

Throughout this section, we always assume that S is a semigroup and U is a non-empty subset of the set $E(S)$ of all idempotents in S. The set of idempotents in a subset A of S is denoted by $E(A)$. The set of all regular elements in S is denoted by $\mathrm{Reg}(S)$. By $S=[Y;S_\alpha]$ we mean that S is a semilattice of the semigroups S_α $(\alpha\in Y)$. In particular, if S is a band, then $[Y;S_\alpha]$ is the greatest semilattice decomposition of S. We use 1_S to denote the identity in the monoid S^1. The lattices of all binary relations, equivalences, left congruences, right congruences and congruences on S are denoted by $\mathcal{B}(S)$, $\mathcal{E}(S)$, $\mathcal{LC}(S)$, $\mathcal{RC}(S)$ and $\mathcal{C}(S)$, respectively. For any $\varrho\in\mathcal{B}(S)$, if it is necessary, ϱ is written specifically as $\varrho(S)$.

The elements in the set $\mathrm{Reg}_U(S)=\{a\in S\colon (\exists\, e,f\in U)\ e\ \mathcal{L}\ a\ \mathcal{R}\ f\}$ are called *U-regular elements* (see [17]). It is obvious that $U\subseteq \mathrm{Reg}_U(S)$. Moreover, we can routinely show that $a\in\mathrm{Reg}_U(S)$ if and only if $a\in\mathrm{Reg}(S)$ and the set $V_U(a)=\{a'\in V(a)\colon aa',a'a\in U\}$ of *the U-inverses of a* is non-empty.

For any $a\in S$, let

$$U_a^l=\{e\in U\colon ea=a\},\quad U_a^r=\{e\in U\colon ae=a\},\quad U_a=U_a^l\cap U_a^r.$$

It is evident that

$$\widetilde{\mathcal{Q}}^U=\{(a,b)\in S\times S\colon U_a=U_b\}\in\mathcal{E}(S).$$

Lawson [17] defined the equivalences $\widetilde{\mathcal{L}}^U$, $\widetilde{\mathcal{R}}^U$ and $\widetilde{\mathcal{H}}^U$ on S by

$$\widetilde{\mathcal{L}}^U=\{(a,b)\in S\times S\colon U_a^r=U_b^r\},\ \ \widetilde{\mathcal{R}}^U=\{(a,b)\in S\times S\colon U_a^l=U_b^l\},\ \ \widetilde{\mathcal{H}}^U=\widetilde{\mathcal{L}}^U\cap\widetilde{\mathcal{R}}^U.$$

He also indicated that in general $\widetilde{\mathcal{L}}\notin\mathcal{RC}(S)$ and $\widetilde{\mathcal{R}}^U\notin\mathcal{LC}(S)$. The semigroup S is said to *satisfy condition* (CR) if $\widetilde{\mathcal{L}}\in\mathcal{RC}(S)$. Furthermore, S is said to *satisfy condition* (C) if it satisfies condition (CR) and its dual condition (CL).

The $\widetilde{\mathcal{Q}}^U$, $\widetilde{\mathcal{L}}^U$, $\widetilde{\mathcal{R}}^U$ and $\widetilde{\mathcal{H}}^U$-classes in S containing the element a are denoted by $\widetilde{Q}_a^U$, $\widetilde{L}_a^U$, $\widetilde{R}_a^U$ and $\widetilde{H}_a^U$, respectively. The following basic result will be used frequently.

Lemma 2.1. *Let a and b be two arbitrary elements in S. The following statements hold:*

(1) *if $(a,b)\in\mathcal{L}$, then $(a,b)\in\widetilde{\mathcal{L}}^U$; conversely, if $(a,b)\in\widetilde{\mathcal{L}}^U|_{\mathrm{Reg}_U(S)}$, then $(a,b)\in\mathcal{L}$;*

(2) *if $(a,b)\in\mathcal{R}$, then $(a,b)\in\widetilde{\mathcal{R}}^U$; conversely, if $(a,b)\in\widetilde{\mathcal{R}}^U|_{\mathrm{Reg}_U(S)}$, then $(a,b)\in\mathcal{R}$;*

(3) *if $\widetilde{H}^U_a\cap U\neq\emptyset$, then it is a singleton contained in U_a;*

(4) *$\widetilde{H}^U_a\subseteq\widetilde{Q}^U_a$;*

(5) *$\widetilde{Q}^U_a\cap U\neq\emptyset$ if and only if U_a has the minimum element with respect to the natural partial order $\leqslant$ on $E(S)$ and, in this case, $\widetilde{Q}^U_a\cap U$ is a singleton contained in U_a.*

Proof. The first part of the statements (1) and (3) can be found in Lawson [17]. The statement (2) is the result dual to (1). We now check the left ones.

(1) We only need to establish the second part. Suppose that $(a,b)\in\widetilde{\mathcal{L}}^U|_{\mathrm{Reg}_U(S)}$. For any $a'\in V_U(a)$, since $a'a\in U^r_a$, we have $a'a\in U^r_b$ and hence $b=ba'a\in L(a)$, where $L(a)$ is the principle left ideal of S generated by a. Dually, also $a\in L(b)$. Thus $(a,b)\in\mathcal{L}$ as required.

(4) If $(a,b)\in\widetilde{\mathcal{H}}^U$, then $U^l_a=U^l_b$ and $U^r_a=U^r_b$. So $U_a=U_b$ and hence $(a,b)\in\widetilde{\mathcal{Q}}^U$.

(5) It is a routine matter to show that, for any $e\in U$, $U_e=U_a$ if and only if e is the minimum element in U_a. By the statements (3) and (4), we can see that this statement is true. $\square$

For any $a\in S$, if they exist, the unique element in $\widetilde{Q}^U_a\cap U$ is denoted by a°_U, while the unique element in $\widetilde{H}^U_a\cap U$ is denoted by $a^\diamond_U$.

The pair (S,U) is called *a U-semiabundant semigroup* if every $\widetilde{\mathcal{L}}^U$ and $\widetilde{\mathcal{R}}^U$-class in it meet with U (see [17, 18]). A U-semiabundant semigroup (S,U) is called *an Ehresmann semigroup* if it satisfies condition (C) and U is a semilattice (see [6], [7]). We call the Ehresmann semigroups with central projections *C-Ehresmann semigroups*.

Definition 2.2. The pair (S,U) is called a $\mathcal{U}$-liberal semigroup if every $\widetilde{\mathcal{Q}}^U$-class in S contains an element from U. A $\mathcal{U}$-liberal semigroup (S,U) is called an orthomonoid if U is a subsemigroup of S such that

$$(\forall\, a,b\in S)\quad (ab)^\circ_U\ \mathcal{D}\ a^\circ_U b^\circ_U.$$

Orthomonoids with central projections are called C-$\mathcal{U}$-liberal semigroups.

Remark 2.3. It is easy to check that U is central in S if and only if it satisfies the condition

$$\text{(C.1)}\qquad (\forall\, u\in U)\quad uS=Su.$$

If it is this case, U is a semilattice when it is a subsemigroup of S. Thus $S(U)$ is a C-$\mathcal{U}$-liberal semigroup if and only if it is an orthomonoid satisfying (C.1).

Definition 2.4. The pair (S,U) is called a $\mathcal{U}$-semi-rpp semigroup if every $\widetilde{\mathcal{L}}^U$-class in it contains an element from U. A $\mathcal{U}$-semi-rpp semigroup (S,U) is said to be strongly if

$$(\forall\, a \in S) \quad |\widetilde{L}^U_a \cap U_a| = 1.$$

In this case, the unique element in $\widetilde{L}^U_a \cap U_a$ $(a \in S)$ is denoted by a^+_U. A $\mathcal{U}$-semi-rpp semigroup (S,U) is called a C-$\mathcal{U}$-semi-rpp semigroup if it satisfies conditions (CR), (C.1) and U is a subsemigroup.

$\mathcal{U}$-semi-lpp semigroups, strongly $\mathcal{U}$-semi-lpp semigroups and C-$\mathcal{U}$-semi-lpp semigroups are defined dually in terms of the relation $\widetilde{\mathcal{R}}^U$. If (S,U) is a strongly $\mathcal{U}$-semi-lpp semigroup, then the unique element in $\widetilde{R}^U_a \cap U_a$ $(a \in S)$ is denoted by a^*_U.

Definition 2.5. The pair (S,U) is called a $\mathcal{U}$-semi-superabundant semigroup if every $\widetilde{\mathcal{H}}^U$-class in it contains an element from U. A $\mathcal{U}$-semi-superabundant semigroup (S,U) is called a C-$\mathcal{U}$-semi-superabundant semigroup if it satisfies conditions (C), (C.1) and U is a subsemigroup.

In case that (S,U) is a U-semiabundant semigroup, a $\mathcal{U}$-liberal semigroup, a $\mathcal{U}$-semi-rpp semigroup, a $\mathcal{U}$-semi-lpp semigroup or a $\mathcal{U}$-semi-superabundant semigroup, we denote (S,U) by $S(U)$ and call U *the set of projections*. By virtue of Lemma 2.1 (3)–(4), we get

Corollary 2.6. *If $S(U)$ is a $\mathcal{U}$-semi-superabundant semigroup, then it is a $\mathcal{U}$-liberal semigroup such that $\widetilde{\mathcal{Q}}^U = \widetilde{\mathcal{H}}^U$ and $a^\circ_U = a^\diamond_U$ for any $a \in S$.*

Lemma 2.7. *Let T be a semigroup and E a non-empty subset of $E(T)$ contained in the center of T. Then $\widetilde{\mathcal{Q}}^E = \widetilde{\mathcal{L}}^E = \widetilde{\mathcal{R}}^E = \widetilde{\mathcal{H}}^E$. Moreover, the following statements are equivalent:*

(1) *$T(E)$ is a $\mathcal{U}$-liberal semigroup;*
(2) *$T(E)$ is a $\mathcal{U}$-semi-rpp semigroup;*
(3) *$T(E)$ is a strongly $\mathcal{U}$-semi-rpp semigroup;*
(4) *$T(E)$ is a $\mathcal{U}$-semi-lpp semigroup;*
(5) *$T(E)$ is a strongly $\mathcal{U}$-semi-lpp semigroup;*
(6) *$T(E)$ is a $\mathcal{U}$-semi-abundant semigroup;*
(7) *$T(E)$ is a $\mathcal{U}$-semi-superabundant semigroup.*

Furthermore, if the statements (1) *(and hence* (3), (5) *and* (6)*) holds, then, for any $a \in T$, $a^\circ_U = a^*_U = a^+_U = a^\diamond_U$.*

Proof. This result holds in view of $E^l_a = E_a = E^r_a$ for any $a \in T$. □

If $T = [Y; T_\alpha]$ is a semilattice of the monoids T_α and $E = \{1_{T_\alpha} \colon \alpha \in Y\}$ is a subsemigroup of T, then, by Petrich [22], Exercise IV.2 (iv), T is a strong semilattice of T_α $(\alpha \in Y)$ with respect to the homomorphism transitive system defined by, for any $\alpha, \beta \in E$ with $\beta \leqslant \alpha$,

$$\varphi_{\alpha,\beta}\colon\ T_\alpha \longrightarrow T_\beta,\ x \longmapsto x1_{T_\beta}.$$

It is evident that, in this case, E is isomorphic to Y and is central in T. Sometimes, for any $\alpha \in Y$, we set $\alpha = 1_{T_\alpha}$. Fountain, Gomes and Gould [6] called T *an E-semilattice of monoids.*

Theorem 2.8. *Let T be a semigroup and $E \subseteq E(S)$. The following statements are equivalent:*
(1) *$T(E)$ is a C-$\mathcal{U}$-liberal semigroup;*
(2) *$T(E)$ is a $\mathcal{U}$-liberal semigroup satisfying the identity $a^\circ_U b^\circ_U = (ab)^\circ_U$ and E is a semilattice;*
(3) *$T(E)$ is an E-semilattice of monoids;*
(4) *$T(E)$ is a $\mathcal{U}$-liberal semigroup, E is a central subsemigroup of T and $\widetilde{\mathcal{Q}}^E \in \mathcal{C}(T)$;*
(5) *$T(E)$ is a $\mathcal{U}$-liberal semigroup, E is a central subsemigroup of T and $\widetilde{\mathcal{Q}}^E \in \mathcal{RC}(T)$;*
(6) *$T(E)$ is a C-$\mathcal{U}$-semi-rpp semigroup;*
(7) *$T(E)$ is a $\mathcal{U}$-liberal semigroup, E is a central subsemigroup of T and $\widetilde{\mathcal{Q}}^E \in \mathcal{LC}(T)$;*
(8) *$T(E)$ is a C-$\mathcal{U}$-semi-lpp semigroup;*
(9) *$T(E)$ is a C-$\mathcal{U}$-semi-superabundant semigroup;*
(10) *$T(E)$ is a C-Ehresmann semigroup.*

Proof. Fountain, Gomes and Gould [6] proved that the statements (3) and (10) are equivalent. Clearly, the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), (9) $\Rightarrow$ (10) and (3) $\Rightarrow$ (1) are true. Furthermore, by virtue of Lemma 2.7 and Remark 2.3, we can see that the statements (5) and (7) are equivalent to (6) and (7), respectively.

(1) $\Rightarrow$ (2). Assume that $T(E)$ is a C-$\mathcal{U}$-liberal semigroup. Then E is a semilattice and a subsemigroup of T. It follows by the definition of C-$\mathcal{U}$-liberal semigroups that $a^\circ_E b^\circ_U \mathcal{D} (ab)^\circ_E$ holds for any $a, b \in T$. Suppose that $c \in S$ is such that $a^\circ_E b^\circ_E \mathcal{L}\ c\ \mathcal{R}\ (ab)^\circ_E$. Then $c \in \mathrm{Reg}_E(S)$ and there exists $c' \in V_E(c)$ such that $c'c = a^\circ_E b^\circ_E$ and $cc' = (ab)^\circ_E$. By noting that $ca^\circ_E b^\circ_E = c = (ab)^\circ_E c$ and E is central in T, we have

$$\begin{aligned}(ab)^\circ_E = cc' = ca^\circ_E b^\circ_E c' = cc'a^\circ_E b^\circ_E c' = (ab)^\circ_E a^\circ_E b^\circ_E = (ab)^\circ_E c'c \\ = c'(ab)^\circ_E c = c'c = a^\circ_E b^\circ_E.\end{aligned}$$

(5) $\Rightarrow$ (7). Assume that the statement (5) holds. Then $\widetilde{\mathcal{Q}}^E \in \mathcal{LC}(S)$ follows by

$$(\forall\, (a,b) \in \widetilde{\mathcal{Q}}^E)(\forall\, c \in S) \quad ca\ \widetilde{\mathcal{Q}}^E\ c^\circ_E a = ac^\circ_E\ \widetilde{\mathcal{Q}}^E\ bc^\circ_E = c^\circ_E b\ \widetilde{\mathcal{Q}}^E\ cb.$$

(7) $\Rightarrow$ (9). If the statement (7) holds, then so does (8). By Remark 2.3 and Lemma 2.7, we claim that $T(E)$ is a $\mathcal{U}$-semi-superabundant semigroup satisfying (CL) and E is a central subsemigroup of T. It is dual that the statements (5) and (6) hold also. Thereby, $T(E)$ satisfies (CR), and hence it is a C-semi-superabundant semigroup. Here is the end of the proof.

Hereafter, by "a C-Ehresmann semigroup $T = [Y; T_\alpha]$" we mean that $T(E)$ is a C-Ehresmann semigroup which is the E-semilattice of the monoids T_α $(\alpha \in Y)$.

3. Left C-$\mathcal{U}$-liberal semigroups

Definition 3.1. An orthomonoid semigroup $S(U)$ is called a left C-U-liberal semigroup if

$$(\forall\, u \in U) \quad uS \subseteq Su. \tag{C.2}$$

Lemma 3.2. *Let S be a semigroup and $U \subseteq E(S)$. The following statements are equivalent:*

(1) *S satisfies condition* (C.2)*;*
(2) *for any $a \in S$ and $e \in U$, $ea = eae$;*
(3) *for any $a \in S$, $U^l_a = U_a$;*
(4) *for any $a \in \mathrm{Reg}_U(S)$, $aS \subseteq Sa$;*
(5) *for any $e \in U$, the mapping $\xi\colon\ x \mapsto ex$ of S^1 onto eS^1 is a semigroup homomorphism.*

Moreover, if S satisfies condition (C.2)*, then the following statements hold:*

(a) *if $U = E(S)$, then U is a subsemigroup of S;*
(b) *if U is a subsemigroup of S, then it is a left regular band.*

P r o o f. (1) $\Rightarrow$ (2). Assume that S satisfies condition (C.2). Then for any $e \in U$ and $a \in S$, there exists $u \in S$ such that $ea = ue$. Thus $ea = ue = uee = eae$.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (4). Assume that the statement (3) holds. Then, for any $b \in S$, $a \in \mathrm{Reg}_U(S)$ and $a' \in V_U(a)$ we have $ab = aa'ab = aba'a \in Sa$. Thereby, the statement (4) also holds.

(4) $\Rightarrow$ (5). If the statement (4) holds, then the statement (1) and hence the statement (2) also hold. Now, the statement (5) is immediate.

(5) $\Rightarrow$ (1). Assume that the statement (5) holds and $e \in U$. Then for any $b = ea \in eS$ we have

$$b = ea = ea \cdot 1_{S^1} = ea \cdot e1_{S^1} = eae \in Se.$$

This implies that the statement (1) holds also.

We now assume that S satisfies condition (C.2). If $U = E(S)$, then, for any $e, f \in U$, it follows by the statement (2) that $ef = eff = efef$, and hence $E(S)$ forms a subsemigroup of S. Thus, the statement (a) holds. If U is a subsemigroup, then it is a band satisfying the identity $ea = eae$. This implies that U is a left regular band. This completes the proof. $\square$

Remark 3.3. If S is the direct product of a left zero band I and a monoid T, then we call S *a left monoid*. In this case $S(I \times \{1_T\})$ is a left C-$\mathcal{U}$-liberal semigroup. Hereafter, we will identify the left monoid S with the left C-$\mathcal{U}$-liberal semigroup $S(U)$.

Lemma 3.4. *Let $S(U)$ be a $\mathcal{U}$-liberal semigroup satisfying condition* (C.2) *and U a subsemigroup of S. The following statements are equivalent:*
(1) *$S(U)$ is a left C-$\mathcal{U}$-liberal semigroup;*
(2) *for any $a, b \in S$, $(ab)^\circ_U \ \mathcal{L}(U)\ a^\circ_U b^\circ_U$;*
(3) *$\mathcal{L}^\circ_U = \{(a,b) \in S \times S\colon\ a^\circ_U \ \mathcal{L}(U)\ b^\circ_U\} \in \mathcal{C}(S)$.*

P r o o f. (1) $\Rightarrow$ (2). Assume that S is a left C-$\mathcal{U}$-liberal semigroup. Then, for any $a, b \in S$, we have $(ab)^\circ_U \ \mathcal{D}\ a^\circ_U b^\circ_U$. Consequently, there are $c, d \in S$ such that, in S,

$$(ab)^\circ_U \ \mathcal{R}\ c \ \mathcal{L}\ a^\circ_U b^\circ_U \ \mathcal{R}\ d \ \mathcal{L}\ (ab)^\circ_U.$$

Since $(ab)^\circ_U \in U^l_c$ and $a^\circ_U b^\circ_U \in U^l_d$, by Lemma 3.2 we have $(ab)^\circ_U \in U_c$ and $a^\circ_U b^\circ_U \in U_d$, and whence $c(ab)^\circ_U = c$ and $da^\circ_U b^\circ_U = d$. This implies that

$$L_{a^\circ_U b^\circ_U} = L_c \leqslant_l L_{(ab)^\circ_U} = L_d \leqslant_l L_{a^\circ_U b^\circ_U},$$

so that $(ab)^\circ_U \ \mathcal{L}(S)\ a^\circ_U b^\circ_U$. Since U is a subsemigroup of S, by Lemma 3.2, we can see that U is a left regular band. It follows that $\mathcal{L}(S)|_U = \mathcal{L}(U)$, yields $(ab)^\circ_U \ \mathcal{L}(U)\ a^\circ_U b^\circ_U$.

(2) $\Rightarrow$ (3). If the statement (2) holds, then $\mathcal{L}^\circ_U \in \mathcal{E}(S)$. Since U is a left regular band, we have $\mathcal{L}(U) \in \mathcal{C}(U)$. Thus, for any $(a,b), (c,d) \in \mathcal{L}^\circ_U$, we have

$$(ac)^\circ_U \ \mathcal{L}(U)\ a^\circ_U c^\circ_U \ \mathcal{L}(U)\ b^\circ_U d^\circ_U \ \mathcal{L}(U)\ (bd)^\circ_U.$$

It follows that the relation $\mathcal{L}^\circ_U$ on S is a congruence.

(3) $\Rightarrow$ (1). Assume that the statement (3) holds. For any $a, b \in S$, since $(a, a^\circ_U), (b, b^\circ_U) \in \mathcal{L}^\circ_U$, we have $(ab, a^\circ_U b^\circ_U) \in \mathcal{L}^\circ_U$, and whence $(ab)^\circ_U \ \mathcal{L}(U)\ a^\circ_U b^\circ_U$. By noting that $\mathcal{L}(U) \subseteq \mathcal{D}(S)$, we conclude that the statement (1) holds. $\square$

If X is a subdirect product of sets Y and Z we denote the first and the second projections of X onto Y and Z by $\mathcal{P}_Y$ and $\mathcal{P}_Z$, respectively. The set of all transformations on a set X is denoted by $\mathcal{T}(X)$ which also stands for the semigroup of all transformations on X. For any $\tau, \sigma \in \mathcal{T}(X)$ and $x \in X$, the image of x under τ is denoted by $x\tau$ and the product of τ and σ in $\mathcal{T}(X)$ is denoted by $\tau\sigma$. Let $\mathcal{T}^*(X)$ be the dual semigroup of the semigroup $\mathcal{T}(X)$. The product of τ and σ in $\mathcal{T}^*(X)$ is denoted by $\tau * \sigma$. Then $\tau * \sigma = \sigma\tau$. Let $I = [Y; I_\alpha]$ and $T = [Y; T_\alpha]$ be two semigroups. Define $S_\alpha = I_\alpha \times T_\alpha$ for any $\alpha \in Y$ and let $S = \bigcup_{\alpha\in Y} S_\alpha$. If

$$\eta\colon\ S \longrightarrow \mathcal{T}^*(I),\ (i,a) \longrightarrow (i,a)^{\#}$$

is a mapping satisfying the following statements for any $(i,a) \in S_\alpha$ and $(j,b) \in S_\beta$

(C.3) (i) $j(i,a)^{\#} \in I_{\alpha\beta}$, in particular, if $\alpha \leqslant \beta$, then $j(i,a)^{\#} = ij$;

(ii) $(i,a)^{\#} * (j,b)^{\#} = (j(i,a)^{\#}, ab)^{\#}$,

then S forms a semigroup with respect to the binary operation

$$(i,a)\cdot(j,b) = (j(i,a)^{\#}, ab).$$

Zhu, Guo and Shum [28] called this semigroup *the left semi-spined product of I and T with respect to Y and η* or simply *a left semi-spined product of I and T*, and denoted it by $I \times_{Y,\eta} T$. In this case, η is called *a structural homomorphism.*

Theorem 3.5. *Let S be a semigroup. The following statements are equivalent:*

(1) *$S(U)$ is a left C-$\mathcal{U}$-liberal semigroup for some $U \subseteq E(S)$;*

(2) *S is a semilattice Y of left monoids $S_\alpha = I_\alpha \times T_\alpha$ ($\alpha \in Y$) and $U = \{(i, 1_{T_\alpha})\colon i \in I_\alpha, \alpha \in Y\}$ is a subsemigroup of S;*

(3) *S is a semilattice Y of left monoids $S_\alpha = I_\alpha \times T_\alpha$ ($\alpha \in Y$) such that, for any $\beta \leqslant \alpha$ in Y,*

$$(\exists\, i_\alpha \in I_\alpha,\ i_\beta \in I_\beta) \quad ((i_\beta, 1_{T_\beta})(1_\alpha, 1_{T_\alpha}))\mathcal{P}_{T_\beta} = 1_{T_\beta};$$

(4) *S is a left semi-spined product of a left regular band $I = [Y; I_\alpha]$ and a C-Ehresmann semigroup $T = [Y; T_\alpha]$ with respect to Y and some structural homomorphism.*

Proof. (1) $\Rightarrow$ (2). Let $S(U)$ be a left C-$\mathcal{U}$-liberal semigroup. Then, by Lemma 3.2 (b), we can see that U is a left regular band. Let $U = [Y; I_\alpha]$ where I_α ($\alpha \in Y$) are left zero bands. By Lemma 3.4, we have

$$S = [Y; S_\alpha = \{x \in S\colon\ x_U^\circ \in I_\alpha\}].$$

For any $e \in I_\alpha$ ($\alpha \in Y$), the subset $S_e = \{x \in S\colon\ x_U^\circ = e\}$ obviously forms a monoid with e as its identity. If also $f \in I_\alpha$, then, for any $a \in S_e$, we have $f \in U^l_{fa}$ and $(f, (fa)_U^\circ) \in \mathcal{L}$. By Lemma 2.1 and Lemma 3.4, we conclude that $(fa)_U^\circ = f$, whence $fa \in S_f$. Therefore

$$\xi_{e,f}\colon\ S_e \longrightarrow S_f,\ x \longmapsto fx$$

is a mapping. Since $\xi_{f,e} = \xi_{e,f}^{-1}$ and, for any $a, b \in S_e$,

$$f(ab) = faefb = fa \cdot fb,$$

the mapping $\xi_{e,f}$ is a semigroup isomorphism. For any $\alpha \in Y$, we choose $e_\alpha \in I_\alpha$ and let $T_\alpha = S_{e_\alpha}$. It is a routine matter to check that the mapping

$$\xi_\alpha \colon\ S_\alpha \longrightarrow I_\alpha \times T_\alpha,\ x \longmapsto (x_U^\circ, x\xi_{x_U^\circ, e_\alpha}) = (x_U^\circ, e_\alpha x)$$

is also an isomorphism. Thus there is an isomorphism from S onto a semilattice Y of the left monoids $I_\alpha \times T_\alpha$ such that the image of U is exactly $\{(i, 1_{T_\alpha})\colon\ i \in I_\alpha, \alpha \in Y\}$.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (4). Assume that the statement (3) holds and let $U = \bigcup\limits_{\alpha \in Y} U_\alpha$ where $U_\alpha = I_\alpha \times \{1_{T_\alpha}\}$ $(\alpha \in Y)$. Then, for any $\beta \leqslant \alpha$ in Y, $(i, 1_{T_\alpha}) \in U_\alpha$ and $(j, 1_{T_\beta}) \in U_\beta$, we have

$$\begin{aligned}(j, 1_{T_\beta})(i, 1_{T_\alpha}) &= (j, 1_{T_\beta})(i_\beta, 1_{T_\beta})(i_\alpha, 1_{T_\alpha})(i, 1_{T_\alpha})\\ &= (j, 1_{T_\beta})(i_\beta, 1_{T_\beta})(i_\alpha, 1_{T_\alpha}) = (j, 1_{T_\beta}).\end{aligned}$$

Therefore, for any $\alpha, \beta \in Y$, $(i, 1_{T_\alpha}) \in U_\alpha$ and $(j, 1_{T_\beta}) \in U_\beta$, we have further

$$\begin{aligned}(j, 1_{T_\beta})(i, 1_{T_\alpha})\mathcal{P}_{T_{\alpha,\beta}} &= ((((j, 1_{T_\beta})(i, 1_{T_\alpha}))\mathcal{P}_{I_{\alpha\beta}}, 1_{T_{\alpha\beta}})(j, 1_{T_\beta})(i, 1_{T_\alpha}))\mathcal{P}_{T_{\alpha\beta}}\\ &= ((((j, 1_{T_\beta})(i, 1_{T_\alpha}))\mathcal{P}_{I_{\alpha\beta}}, 1_{T_{\alpha\beta}})(i, 1_{T_\alpha}))\mathcal{P}_{T_{\alpha\beta}}\\ &= (((j, 1_{T_\beta})(i, 1_{T_\alpha}))\mathcal{P}_{I_{\alpha\beta}}, 1_{T_{\alpha\beta}})\mathcal{P}_{T_{\alpha\beta}}\\ &= 1_{T_{\alpha\beta}}.\end{aligned}$$

Thus U is a subsemigroup of S. It is obvious that $U = [Y; U_\alpha]$ is a left regular band. Moreover, the set $I = \bigcup\limits_{\alpha \in Y} I_\alpha$ forms a left regular band with respect to the operation

$$(\forall\, i \in I_\alpha, j \in I_\beta) \quad i \circ j = ((i, 1_{T_\alpha})(j, 1_{T_\beta}))\mathcal{P}_I.$$

Denote $T = \bigcup\limits_{\alpha \in Y} T_\alpha$. For any $a \in T_\alpha$ and $b \in T_\beta$, suppose that $(i, a) \in I_\alpha \times T_\alpha$, $(j, b) \in I_\beta \times T_\beta$ and $k \in I_{\alpha\beta}$ are such that $((k, 1_{T_{\alpha\beta}})(i, a))\mathcal{P}_T = a'$ and $((k, 1_{T_{\alpha\beta}})(j, b))\mathcal{P}_T = b'$. Then

$$(\forall\, k' \in I_{\alpha\beta}) \quad (k', 1_{T_{\alpha\beta}})(i, a) = (k', 1_{T_{\alpha\beta}})(k, 1_{T_{\alpha\beta}})(i, a) = (k', a').$$

Thus a' is independent of k. Similarly, b' is independent of k too. Furthermore, for any $i' \in I_\alpha$ and $j' \in I_\beta$, we have

$$\begin{aligned}(i',a)(j',b) &= (((i',a)(j',b))\mathcal{P}_I, 1_{T_{\alpha\beta}})(i',a)(j',b)\\ &= (((i',a)(j',b))\mathcal{P}_I, a')(j',b)\\ &= (((i',a)(j',b))\mathcal{P}_I, a'(k,1_{T_{\alpha\beta}})(j',b)\\ &= (((i',a)(j',b))\mathcal{P}_I, a')(k,b')\\ &= (((i',a)(j',b))\mathcal{P}_I, a'b').\end{aligned}$$

So $((i,a)(j,b))\mathcal{P}_T$ is independent of both i and j. Consequently, we can define an operation $\bullet$ on T as below: for any $a \in T_\alpha$ and $b \in T_\beta$,

$$a \bullet b = c \Longleftrightarrow (\exists\, i \in I_\alpha, j \in I_\beta) \quad ((i,a)(j,b))\mathcal{P}_T = c.$$

By noticing that $\mathcal{P}_T$ is a homomorphism of S onto the groupoid $(T,\bullet)$, we conclude that $(T,\bullet)$ is a semigroup. Moreover, it is evident that $(T,\bullet)$ is a semilattice of monoids T_α $(\alpha \in Y)$. Since U is a subsemigroup of S, we can easily show that $E = \{1_{T_\alpha}\colon\ \alpha \in Y\}$ is a subsemigroup of T, and hence T is the E-semilattice of T_α $(\alpha \in Y)$. Thus T is a C-Ehresmann semigroup.

We now define a mapping η from the set S into $\mathcal{T}^*(I)$ by

$$(\forall\,(i,a) \in S_\alpha)\,(\forall\, j \in I_\beta) \quad j(i,a)^{\#} = ((i,a)(j,1_{T_\beta}))\mathcal{P}_I.$$

For any $(i,a) \in S_\alpha$ and $j \in I_\beta$ $(\alpha,\beta \in Y)$, we can easily see that $j(i,a)^{\#} \in I_{\alpha\beta}$. In particular, if $\alpha \leqslant \beta$, then $j(i,a)^{\#} = i = ij$ in view of

$$(i,a)(j,1_{T_\beta}) = (i,a)(i,1_{T_\alpha})(j,1_{T_\beta}) = (i,a)(i,1_{T_\beta}) = (i,a).$$

Moreover, for any $b \in T_\beta$ we have

$$\begin{aligned}(i,a)(j,b) &= (i,a)(j,1_{T_\beta})(j,b)\\ &= (j(i,a)^{\#}, a \bullet 1_{T_\beta})(j,b)\\ &= (j(i,a)^{\#}, a \bullet 1_{T_\beta})(j(i,a)^{\#}, 1_{T_{\alpha\beta}})(j,b)\\ &= (j(i,a)^{\#}, a \bullet 1_{T_\beta})(j(i,a)^{\#}, 1_{T_{\alpha\beta}})(j(i,a)^{\#}, 1_{T_{\alpha\beta}})(j,b)\\ &= (j(i,a)^{\#}, a \bullet 1_{T_\beta})(j(i,a)^{\#}, 1_{T_{\alpha\beta}})(((j(i,a)^{\#}, 1_{T_{\alpha\beta}})(j,b))\mathcal{P}_I, 1_{T_{\alpha\beta}} \bullet b)\\ &= (j(i,a)^{\#}, a \bullet 1_{T_\beta})(j(i,a)^{\#}, 1_{T_{\alpha\beta}} \bullet b)\\ &= (j(i,a)^{\#}, a \bullet b).\end{aligned}$$

Hence, for any $k \in I_\gamma$ $(\gamma \in Y)$,

$$\begin{aligned} k(j(i,a)^{\#}, a \bullet b)^{\#} &= k((i,a)(j,b))^{\#} \\ &= (((i,a)(j,b))(k, 1_{T_\gamma}))\mathcal{P}_I \\ &= ((i,a)((j,b)(k,1_{T_\gamma})))\mathcal{P}_I \\ &= ((i,a)(k(b,j)^{\#}, b \bullet 1_{T_\gamma}))\mathcal{P}_I \\ &= (k(b,j)^{\#}(a,i)^{\#}, a \bullet b \bullet 1_{T_\gamma})\mathcal{P}_I \\ &= k((i,a)^{\#} * (j,b)^{\#}). \end{aligned}$$

Thus η is a structural homomorphism such that $S = I \times_{Y,\eta} T$.

(4) $\Rightarrow$ (1). Let S be the left semi-spined product of a left regular band $I = [Y; I_\alpha]$ and a C-Ehresmann semigroup $T = [Y; T_\alpha]$ with respect to the semilattice Y and a structural homomorphism η, and let $U = \{(i, 1_{T_\alpha})\colon\ i \in I_\alpha, \alpha \in Y\}$. Then $U = [Y; I_\alpha \times \{1_{T_\alpha}\}]$ is a left regular band and $S = [Y; I_\alpha \times T_\alpha]$. Suppose that (i,a) and (j,b) are two arbitrary elements of $I_\alpha \times T_\alpha$ and $I_\beta \times T_\beta$, respectively, and suppose that $(k, 1_{T_\gamma})$ is an arbitrary element in U. Then

$$\begin{aligned} (k,1_{T_\gamma})(i,a)(k,1_{T_\gamma}) &= (i(k,1_{T_\gamma})^{\#}, 1_{T_\gamma} \cdot a)(k, 1_{T_\gamma}) \\ &= (i(k,1_{T_\gamma})^{\#}, 1_{T_\gamma} \cdot a \cdot 1_{T_\gamma}) \\ &= (i(k,1_{T_\gamma})^{\#}, 1_{T_\gamma} \cdot a) \\ &= (k,1_{T_\gamma})(i,a). \end{aligned}$$

It follows by Lemma 3.2 that S satisfies condition (C.2). It is evident that $(i, 1_{T_\alpha}) \in U_{(i,a)}$. If also $(k, 1_{T_\gamma}) \in U_{(i,a)}$, then of course $\alpha \leqslant \gamma$ and hence $(k,1_{T_\gamma})(i,1_{T_\alpha}), (i,1_{T_\alpha})(k,1_{T_\gamma}) \in (I_\alpha \times T_\alpha) \cap U_{(i,a)}$. Since $(i, 1_{T_\alpha})$ is the unique element in $(I_\alpha \times T_\alpha) \cap U_{(i,a)}$, we have

$$(k,1_{T_\gamma})(i,1_{T_\alpha}) = (i,1_{T_\alpha}) = (i,1_{T_\alpha})(k,1_{T_\gamma}).$$

Thus $(i,1_{T_\alpha}) = (i,a)^\circ_U$ whence $S(U)$ is a $\mathcal{U}$-liberal semigroup. Since $(i,a)^\circ_U(j,b)^\circ_U$ and $((i,a)(j,b))^\circ_U$ are elements in $I_{\alpha\beta} \times \{1_{T_{\alpha\beta}}\}$, we have

$$(i,a)^\circ_U(j,b)^\circ_U \ \mathcal{L}(U)\ ((i,a)(j,b))^\circ_U$$

and hence, by Lemma 3.4, $S(U)$ is a left C-$\mathcal{U}$-liberal semigroup. Here is the end of the proof.

For a transformation ψ on a set X and $i \in X$, we use $\langle\psi\rangle$ to denote that ψ is a constant mapping on X with value $\langle\psi\rangle$, and use $\langle i\rangle$ to denote the constant mapping on X with value i. Since it is similar to the well-known construction for bands, the following characterization for left C-$\mathcal{U}$-liberal semigroups is called *the band-formal construction.*

Theorem 3.6. *Let $T = [Y; T_\alpha]$ be a C-Ehresmann semigroup and, for any $\alpha \in Y$, let I_α be a non-empty set. Denote $S = \bigcup_{\alpha \in Y} S_\alpha$ where $S_\alpha = I_\alpha \times T_\alpha$ and let $U = \{(i, 1_{T_\alpha})\colon i \in I_\alpha, \alpha \in Y\}$. For any $\gamma \leqslant \alpha$ in Y, define a mapping*

$$\Psi_{\alpha,\gamma}\colon S_\alpha \longrightarrow \mathcal{T}^*(I_\gamma),\ (i,a) \longmapsto \psi_{\alpha,\gamma}^{(i,a)}$$

such that the following statements hold for any $\alpha, \beta \in Y$:

(C.4) (i) *for any $(i,a) \in S_\alpha$, $\psi_{\alpha,\alpha}^{(i,a)} = \langle i \rangle$;*

(ii) *if $(i,a) \in S_\alpha$ and $(j,b) \in S_\beta$, then $\psi_{\alpha,\alpha\beta}^{(i,a)} * \psi_{\beta,\alpha\beta}^{(j,b)} = \big\langle \psi_{\alpha,\alpha\beta}^{(i,a)} * \psi_{\beta,\alpha\beta}^{(j,b)} \big\rangle$; moreover,*

(iii) *for any $\delta \leqslant \alpha\beta$ in Y, $\psi_{\alpha\beta,\delta}^{(k,ab)} = \psi_{\alpha,\delta}^{(i,a)} * \psi_{\beta,\delta}^{(j,b)}$ where $k = \big\langle \psi_{\alpha,\alpha\beta}^{(a,i)} * \psi_{\beta,\alpha\beta}^{(b,j)} \big\rangle$.*

Then $S(U)$ is a left C-$\mathcal{U}$-liberal semigroup with respect to the multiplication defined by

$$(*) \qquad (\forall\, (i,a) \in S_\alpha,\ (j,b) \in S_\beta) \quad (i,a)(j,b) = \big(\big\langle \psi_{\alpha,\alpha\beta}^{(i,a)} * \psi_{\beta,\alpha\beta}^{(j,b)} \big\rangle, ab\big).$$

Conversely, every left C-$\mathcal{U}$-liberal semigroup can be obtained in this way.

Proof. *The first part.* By conditions (C.4) (i) and (ii), we can see that the multiplication $(*)$ on S is well-defined. Let $(i,a) \in S_\alpha$, $(j,b) \in S_\beta$ and $(k,c) \in S_\gamma$ where $\alpha, \beta, \gamma \in Y$. It is obvious that the condition (C.4) (iii) can be translated into

$$(\forall\, \delta \leqslant \alpha\beta \text{ in } Y) \quad \psi_{\alpha\beta,\delta}^{(i,a)(j,b)} = \psi_{\alpha,\delta}^{(i,a)} * \psi_{\beta,\delta}^{(j,b)}.$$

Therefore

$$\begin{aligned} ((i,a)(j,b))(k,c) &= \big(\big\langle \psi_{\alpha\beta,\alpha\beta\gamma}^{(i,a)(j,b)} * \psi_{\gamma,\alpha\beta\gamma}^{(k,c)} \big\rangle, abc\big) \\ &= \big(\big\langle \psi_{\alpha,\alpha\beta\gamma}^{(i,a)} * \psi_{\beta,\alpha\beta\gamma}^{(j,b)} * \psi_{\gamma,\alpha\beta\gamma}^{(k,c)} \big\rangle, abc\big) \\ &= \big(\big\langle \psi_{\alpha,\alpha\beta\gamma}^{(i,a)} * \psi_{\beta\gamma,\alpha\beta\gamma}^{(j,b)(k,c)} \big\rangle, abc\big) \\ &= (i,a)((j,b)(k,c)). \end{aligned}$$

So S is a semigroup. It is a routine matter to check that U is a subsemigroup of S and S is a semilattice Y of the left monoids S_α. By Theorem 3.5, we claim that $S(U)$ is a left C-$\mathcal{U}$-liberal semigroup.

The converse part. Let $S(U)$ be a left C-$\mathcal{U}$-liberal semigroup. Then we are reasonable to assume that S is a semilattice Y of left monoids $S_\alpha = I_\alpha \times T_\alpha$ and $U = \{(i, 1_{T_\alpha})\colon i \in I_\alpha, \alpha \in Y\}$ is a subsemigroup of S. For any $\beta \leqslant \alpha$ in Y and $(i,a) \in S_\alpha$, define $\psi_{\alpha,\beta}^{(i,a)} \in \mathcal{T}^*(I_\beta)$ by

$$(\forall\, j \in I_\beta) \quad j\psi_{\alpha,\beta}^{(i,a)} = ((i,a)(j, 1_{T_\beta}))\mathcal{P}_{I_\beta}.$$

Then (C.4) (i) holds obviously. Let $\alpha, \beta \in Y$, $(i,a) \in S_\alpha$ and $(j,b) \in S_\beta$. Then

$$\begin{aligned}
(\forall\, h \in I_{\alpha\beta}) \quad ((i,a)(j,b))\mathcal{P}_I &= ((i,a)(j,b)(h, 1_{T_{\alpha\beta}}))\mathcal{P}_I \\
&= \big((i,a)\big(h\psi^{(j,b)}_{\beta,\alpha\beta}, b \cdot 1_{T_{\alpha\beta}}\big)\big)\mathcal{P}_I \\
&= \big((i,a)\big(h\psi^{(j,b)}_{\beta,\alpha\beta}, 1_{T_{\alpha\beta}}\big)(h, b \cdot 1_{T_{\alpha\beta}})\big)\mathcal{P}_I \\
&= \big(\big(h\psi^{(b,j)}_{\beta,\alpha\beta}\psi^{(a,i)}_{\alpha,\alpha\beta}, a \cdot 1_{T_{\alpha\beta}}\big)(h, b \cdot 1_{T_{\alpha\beta}})\big)\mathcal{P}_I \\
&= \big(h\psi^{(j,b)}_{\beta,\alpha\beta}\psi^{(i,a)}_{\alpha,\alpha\beta}, ab\big)\mathcal{P}_I \\
&= h\big(\psi^{(i,a)}_{\alpha,\alpha\beta} * \psi^{(j,b)}_{\beta,\alpha\beta}\big).
\end{aligned}$$

So $\psi^{(i,a)}_{\alpha,\alpha\beta} * \psi^{(j,b)}_{\beta,\alpha\beta}$ is a constant transformation on $I_{\alpha\beta}$ with value $k = ((i,a)(j,b))\mathcal{P}_{I_{\alpha\beta}}$. Hence (C.4) (ii) also holds. If $\delta \leqslant \alpha\beta$ in Y, then for any $h \in I_\delta$, we have

$$\begin{aligned}
h\psi^{(k,ab)}_{\alpha\beta,\delta} &= ((k,ab)(h, 1_{T_\delta}))\mathcal{P}_I \\
&= ((i,a)(j,b)(h, 1_{T_\delta}))\mathcal{P}_I \\
&= \big((i,a)\big(h\psi^{(j,b)}_{\beta,\delta}, b \cdot 1_{T_\delta}\big)\big)\mathcal{P}_I \\
&= \big((i,a)\big(h\psi^{(j,b)}_{\beta,\delta}, 1_{T_\delta}\big)(h, b \cdot 1_{T_\delta})\big)\mathcal{P}_I \\
&= h\big(\psi^{(i,a)}_{\alpha,\delta} * \psi^{(j,b)}_{\beta,\delta}\big).
\end{aligned}$$

So (C.4) (iii) holds. Consequently, S can be constructed as in the first part since

$$(i,a)(j,b) = \big(\langle \psi^{(i,a)}_{\alpha,\alpha\beta} * \psi^{(j,b)}_{\beta,\alpha\beta} \rangle, ab\big).$$

□

Definition 3.7. Let S be a semigroup and U a subsemigroup of S contained in $E(S)$ such that (C.2) holds. If $S(U)$ is a strong $\mathcal{U}$-semi-rpp semigroup satisfying (CR), then it is called a left C-$\mathcal{U}$-semi-rpp semigroup. If $S(U)$ is a $\mathcal{U}$-semi-superabundant semigroup satisfying (C), then it is called a left C-$\mathcal{U}$-semi-superabundant semigroup.

Theorem 3.8. *Let S be a semigroup and $U \subseteq E(S)$. The following statements are equivalent:*

(1) *$S(U)$ is a left C-$\mathcal{U}$-liberal semigroup;*
(2) *$S(U)$ is a $\mathcal{U}$-semi-abundant semigroup satisfying condition* (C)*, U is a subsemigroup of S and $\widetilde{\mathcal{R}}^U = \widetilde{\mathcal{H}}^U$;*
(3) *$S(U)$ is a left C-$\mathcal{U}$-semi-superabundant semigroup;*
(4) *$S(U)$ is a left C-$\mathcal{U}$-semi-rpp semigroup.*

Proof. (1) $\Rightarrow$ (2). Assume that $S(U)$ is a left C-$\mathcal{U}$-liberal semigroup. By Theorem 3.5, it is reasonable to suppose that S is a semilattice Y of left monoids $S_\alpha = I_\alpha \times T_\alpha$ and $U = \{(i, 1_{T_\alpha})\colon i \in I_\alpha, \alpha \in Y\}$ is a subsemigroup of S. Then U is a left regular band with structural semilattice decomposition $[Y; U_\alpha = \{(i, 1_{T_\alpha})\colon i \in I_\alpha\}]$. Let $I = \bigcup_{\alpha\in Y} I_\alpha$ and $T = \bigcup_{\alpha\in Y} T_\alpha$. It is a routine matter to verify that for any $(i,a) \in S_\alpha$ and $e \in U$,

$$(i,a)\ \widetilde{\mathcal{L}}^U\ e \Longleftrightarrow e \in U_\alpha, \quad (i,a)\ \widetilde{\mathcal{R}}^U\ e \Longleftrightarrow e\mathcal{P}_I = i.$$

So S is a $\mathcal{U}$-semi-abundant semigroup, in which

$$\widetilde{\mathcal{L}}^U = \{(x,y) \in S \times S\colon (\exists\, \alpha \in Y)\, x, y \in S_\alpha\},$$
$$\widetilde{\mathcal{R}}^U = \{(x,y) \in S \times S\colon x\mathcal{P}_I = y\mathcal{P}_I\}.$$

By noting that $\widetilde{\mathcal{R}}^U \subseteq \widetilde{\mathcal{L}}^U$, we have $\widetilde{\mathcal{R}}^U = \widetilde{\mathcal{H}}^U$. The relation $\widetilde{\mathcal{L}}^U$ on S is obviously a semilattice congruence. By the definition of S and $\widetilde{\mathcal{R}}^U$, we can easily establish that $\widetilde{\mathcal{R}} \in \mathcal{LC}(S)$. Thus S satisfies condition (C).

(2) $\Rightarrow$ (3). Assume that $S(U)$ is a $\mathcal{U}$-semi-abundant semigroup satisfying condition (C) in which U is a subsemigroup and $\widetilde{\mathcal{R}}^U = \widetilde{\mathcal{H}}^U$. Then S is of course a $\mathcal{U}$-semi-superabundant semigroup satisfying condition (C) and every $\widetilde{\mathcal{R}}^U$-class in it contains a unique projection. For any $a \in S$ and $e \in U$, since $eea = ea$, we have $e(ea)^\diamond_U = (ea)^\diamond_U$ and hence $(ea)^\diamond_U\ \mathcal{R}\ (ea)^\diamond_U e$. Since U is a subsemigroup and $\mathcal{R} \subseteq \widetilde{\mathcal{R}}^U$, we have $(ea)^\diamond_U = (ea)^\diamond_U e$, whence $eae = ea$. It follows by Lemma 3.2 that S satisfies condition (C.2). So the statement (3) holds.

(3) $\Rightarrow$ (4). Assume that $S(U)$ is a left C-$\mathcal{U}$-semi-superabundant semigroup. Then $S(U)$ is of course a $\mathcal{U}$-semi-rpp semigroup satisfying conditions (CR) and (C.2), and U is a subsemigroup of S. Moreover, by Lemma 3.2 (b), we can see that U is a left regular band. For any $a \in S$, it is obvious that $a^\diamond_U \in U_a \cap \widetilde{L}^U_a$. If also $e \in U_a \cap \widetilde{L}^U_a$, then $(e, a^\diamond_U) \in \widetilde{\mathcal{L}}^U$. It follows by Lemma 2.1 (1) that $(e, a^\diamond_U) \in \mathcal{L}$. Since $a^\diamond_U$ is the minimum element in U_a, we have $a^\diamond_U \leqslant e$ and hence $e = ea^\diamond_U = a^\diamond_U$. So $a^\diamond_U = a^+_U$ whence $S(U)$ is a strongly $\mathcal{U}$-semi-rpp semigroup.

(4) $\Rightarrow$ (1). Assume that $S(U)$ is a left C-$\mathcal{U}$-semi-rpp semigroup. Then S satisfies condition (C.2), U is a left regular band and a subsemigroup of S. For any $a \in S$ and $e \in U_a$, since $S(U)$ satisfies condition (CR), we have $a = ae\ \widetilde{\mathcal{L}}^U\ a^+_U e$. By noting that also $a^+_U e \in U_a$, we have further $a^+_U e = a^+_U$ whence $ea^+_U \in U$ is $\mathcal{L}$-equivalent with a^+_U. It follows by Lemma 2.1 (1) that $ea^+_U\ \widetilde{\mathcal{L}}^U\ a$. Thus $ea^+_U \in U_a \cap \widetilde{L}^U_a$ so that $ea^+_U = a^+_U$. Therefore $a^+_U = a^\circ_U$, whence S is a $\mathcal{U}$-liberal semigroup. For any $a, b \in S$, since $S(U)$

satisfies (CR) again, we have $ab\ \widetilde{\mathcal{L}}^U\ a^+_U b$ and hence

$$\begin{aligned}(\forall\, e \in U) \quad (ab)^+_U e = (ab)^+_U &\iff abe = ab \\ &\iff a^+_U b a^+_U b^+_U e = a^+_U b b^+_U e = a^+_U b e = a^+_U b = a^+_U b a^+_U b^+_U \\ &\iff a b a^+_U b^+_U e = a b a^+_U b^+_U \\ &\iff (ab)^+_U a^+_U b^+_U e = (ab)^+_U a^+_U b^+_U.\end{aligned}$$

Thus $(ab)^+_U\ \widetilde{\mathcal{L}}^U\ (ab)^+_U a^+_U b^+_U$. By Lemma 2.1 (1), we claim that $(ab)^+_U\ \mathcal{L}\ (ab)^+_U a^+_U b^+_U$. Thereby

$$(ab)^+_U = (ab)^+_U a^+_U b^+_U.$$

Since a and b are arbitrary, by replacing a and b in the equation above by b and a^+_U, respectively, we have further

$$(ba^+_U)^+_U = (ba^+_U)^+_U b^+_U a^+_U = (ba^+_U)^+_U b^+_U (a^+_U)^+_U a^+_U = (ba^+_U)^+_U a^+_U.$$

Moreover, by virtue of $a^+_U b^+_U\ \mathcal{L}\ b^+_U a^+_U$ we conclude that $a^+_U b^+_U\ \widetilde{\mathcal{L}}^U\ b^+_U a^+_U$ and hence

$$\begin{aligned}(\forall\, e \in U) \quad abe = ab &\Longrightarrow a^+_U b a^+_U e = a^+_U b e = a^+_U b = a^+_U b a^+_U \\ &\Longrightarrow b a^+_U e = (ba^+_U)^+_U a^+_U b a^+_U e = (ba^+_U)^+_U a^+_U b a^+_U = b a^+_U \\ &\Longrightarrow b^+_U a^+_U e = b^+_U a^+_U \\ &\Longrightarrow a^+_U b^+_U e = a^+_U b^+_U \\ &\Longrightarrow (ab)^+_U e = (ab)^+_U a^+_U b^+_U e = (ab)^+_U a^+_U b^+_U = (ab)^+_U \\ &\Longrightarrow abe = ab.\end{aligned}$$

Thus $a^+_U b^+_U\ \widetilde{\mathcal{L}}^U\ ab\ \widetilde{\mathcal{L}}^U\ (ab)^+_U$, whence $a^+_U b^+_U\ \mathcal{L}\ (ab)^+_U$. So $S(U)$ is a left C-$\mathcal{U}$-liberal semigroup. □

4. Left C-liberal semigroups

Let S be a semigroup. For any $a \in S$ we denote $E(S)^l_a$, $E(S)^r_a$ and $E(S)_a$ by I^l_a, I^r_a and I_a, respectively, while the equivalences $\widetilde{\mathcal{Q}}^{E(S)}$, $\widetilde{\mathcal{L}}^{E(S)}$, $\widetilde{\mathcal{R}}^{E(S)}$ and $\widetilde{\mathcal{H}}^{E(S)}$ will be written simply as $\widetilde{\mathcal{Q}}$, $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{H}}$. In fact, $\widetilde{L}$, $\widetilde{R}$ and $\widetilde{H}$ were defined for the first time by El-Quallali [2].

Definition 4.1. Let S be a semigroup. If $S(E(S))$ is an Ehresmann semigroup, then it is called a full Ehresmann semigroup. Full Ehresmann semigroups with central idempotents are called C-full Ehresmann semigroups. If $S(E(S))$ is a $\mathcal{U}$-liberal semigroup, then S is called *a liberal semigroup* and, in this case, $a^\circ_{E(S)}$ $(a \in S)$ is denoted by a°. If $S(E(S))$ is an orthomonoid, then S is called a full orthomonoid. Full orthomonoids with central idempotents are called C-liberal semigroups. A full orthomonoid S is called a left C-liberal semigroup if

$$(\forall\, e \in E(S)) \quad eS \subseteq Se. \tag{C.5}$$

By *semi-rpp semigroups*, *strongly semi-rpp semigroups* and *left C-semi-rpp semigroups* we mean, respectively, the $\mathcal{U}$-semi-rpp semigroups, strongly $\mathcal{U}$-semi-rpp semigroups and left C-$\mathcal{U}$-semi-rpp semigroups with all idempotents as the projections. The concepts of *semi-lpp semigroup*, *semi-superabundant semigroup*, *left C-semi-superabundant semigroup* and so on are similarly defined, therefore we omit the detailed explanation. For an element a in a semigroup S, if they exist, $a^+_{E(S)}$, $a^*_{E(S)}$ and $a^\diamond_{E(S)}$ are denoted by a^+, a^* and $a^\diamond$, respectively.

Remark 4.2.

(1) The direct product of a left zero band and a unipotent semigroup is called *a left unipotent semigroup.* Left unipotent semigroups are left C-liberal semigroups.

(2) A semigroup satisfying condition (C.5) is not necessarily liberal. For example, let $S = \{e, f, a, 0\}$ be a semigroup with Cayley table

$\cdot$	e	f	a	0
e	e	e	a	0
f	f	f	a	0
a	a	a	0	0
0	0	0	0	0

Then S satisfies condition (C.5) but is not liberal.

(3) Non-trivial right zero semigroups are liberal but do not satisfy condition (C.5).

(4) A liberal semigroup satisfying condition (C.5) is not necessarily left C-liberal. For example, let $S = \{e, f, a, b, 0\}$ be a semigroup with Cayley table

$\cdot$	e	f	a	b	0
e	e	e	a	a	0
f	f	f	b	b	0
a	a	a	0	0	0
b	b	b	0	0	0
0	0	0	0	0	0

Then S is a liberal semigroup satisfying condition (C.5) but $a^\circ b^\circ = ef = e \neq 0 = (ab)^\circ$.

Lemma 4.3. *If S is a semilattice Y of left unipotent semigroups $I_\alpha \times T_\alpha$, then $E(S)$ is a subsemigroup of S.*

Proof. It is obvious that $E(S) = \{(i, 1_{T_\alpha})\colon i \in I_\alpha, \alpha \in Y\}$. For any $\beta \leqslant \alpha$ in Y, $i \in I_\alpha$ and $j \in I_\beta$, the equation $((j, 1_{T_\beta})(i, 1_{T_\alpha}))\mathcal{P}_{T_\beta} = 1_{T_\beta}$ holds in view of

$$\begin{aligned}(j, 1_{T_\beta})(i, 1_{T_\alpha}) \cdot (j, 1_{T_\beta})(i, 1_{T_\alpha}) &= (j, 1_{T_\beta})(i, 1_{T_\alpha})(j, 1_{T_\beta}) \cdot (i, 1_{T_\alpha}) \\ &= (j, 1_{T_\beta})(i, 1_{T_\alpha})(j, 1_{T_\alpha}) \\ &= (j, 1_{T_\beta})(i, 1_{T_\alpha}).\end{aligned}$$

□

Corollary 4.4. *Let T be a semigroup. The following statements are equivalent:*

(1) *T is a C-liberal semigroup;*

(2) *T is a liberal semigroup satisfying the identity $a^\circ b^\circ = (ab)^\circ$ and $E(T)$ is a semilattice and a subsemigroup of T;*

(3) *T is a liberal semigroup with central idempotents and $\widetilde{\mathcal{Q}} \in \mathcal{C}(T)$;*

(4) *T is a liberal semigroup with central idempotents and $\widetilde{\mathcal{Q}} \in \mathcal{RC}(T)$;*

(5) *T is a C-semi-rpp semigroup;*

(6) *T is a liberal semigroup with central idempotents and $\widetilde{\mathcal{Q}} \in \mathcal{LC}(T)$;*

(7) *T is a C-semi-lpp semigroup;*

(8) *T is a C-semi-superabundant semigroup;*

(9) *T is a C-full Ehresmann semigroup;*

(10) *T is a semilattice of unipotent semigroups;*

(11) *T is a strong semilattice of unipotent semigroups.*

Proof. It is obvious that, if $E(T)$ is central in T, then it is a subsemigroup of T and is a semilattice. By virtue of Theorem 2.8 and Lemma 4.3, the present result holds. □

Remark 4.5. A liberal semigroup with central idempotents is not necessarily a C-full Ehresmann semigroup. For example, if S is *a non-trivial null monoid* (i.e., a non-trivial null semigroup with an identity adjoined), then it is a liberal semigroup with central idempotents but not a C-full Ehresmann semigroup.

Corollary 4.6. *Let S be a semigroup. The following statements are equivalent:*

(1) *S is a left C-liberal semigroup;*

(2) *S is a semilattice Y of left unipotent semigroups;*

(3) *S is a left semi-spined product of a left regular band and a C-full Ehresmann semigroup;*

(4) *S is a semi-abundant semigroup satisfying* (C) *and $\widetilde{\mathcal{R}} = \widetilde{\mathcal{H}}$;*

(5) *S is a left C-semi-superabundant semigroup;*

(6) *S is a left C-semi-rpp semigroup.*

P r o o f. By virtue of Theorem 3.5, Theorem 3.8 and Lemma 4.3, we only need to prove that if the statement (4) holds then $E(S)$ is a subsemigroup of S. Assume that (4) holds. Then S is a semi-superabundant semigroup and each $\widetilde{\mathcal{R}}$-class in it contains a unique idempotent. For any $e, f \in E(S)$, since $eef = ef$, we have $e(ef)^\diamond = (ef)^\diamond$, whence $(ef)^\diamond e$ is an idempotent which is $\mathcal{R}$-equivalent to $(ef)^\diamond$. It follows by Lemma 2.1 (2) that $(ef)^\diamond e = (ef)^\diamond$. Since $(ef)^\diamond \mathrel{\widetilde{\mathcal{H}}} ef$, we have $efe = ef$. So $efef = eff = ef$, whence $E(S)$ is a subsemigroup of S. $\square$

5. Left C-ϱ-rpp semigroups

Let S be a semigroup. For any $a \in S$ and $\varrho \in \mathcal{LC}(S)$, define $a \cdot \varrho = \{(ax, ay)\colon (x, y) \in \varrho\}$. We say that S *is ϱ-left cancellative* if

$$(\forall\, a, b, c \in S) \quad (ab, ac) \in \varrho \Longrightarrow (b, c) \in \varrho.$$

It is a routine matter to show that

$$\mathcal{L}^\varrho = \{(a, b) \in S \times S\colon\ (\forall\, x, y \in S^1)\ (ax, ay) \in \varrho \Longleftrightarrow (bx, by) \in \varrho\} \in \mathcal{RC}(S).$$

Definition 5.1. The semigroup S is said to be ϱ-rpp if every $\mathcal{L}^\varrho$-class in S contains at least one idempotent. Moreover, S is said to be strongly ϱ-rpp if, for any $a \in S$, the set $I_a \cap L^\varrho_a$ is a singleton, where L^ϱ_a stands for the $\mathcal{L}^\varrho$-class in S containing a. In this case, the unique element in $I_a \cap L^\varrho_a$ is denoted by a^+_ϱ. [Strongly] ϱ-rpp semigroups with central idempotents are called [strongly] C-ϱ-rpp semigroups and strongly ϱ-rpp semigroups satisfying (C.5) are called left C-ϱ-rpp semigroups.

Remark 5.2. By a ϱ-rpp semigroup S we also mean that S is a semigroup which is ϱ-rpp for some $\varrho \in \mathcal{LC}(S)$. It is obvious that, if ϱ is the identical relation ε and $\mathcal{R}$-equivalence on S, respectively, then $\mathcal{L}^\varrho$ is exactly the relations $\mathcal{L}^*$ and $\mathcal{L}^{**}$ on S stated in McAlister [19] and Tang [23]. In the sequel, we will identify the concepts of ε-rpp and $\mathcal{R}$-rpp semigroups, respectively, with *rpp* and *wrpp semigroups*. Guo, Shum and Zhu [9] called a rpp semigroup S a *strongly rpp semigroup* if for any $a \in S$, the set $I^l_a \cap L^*_a$ is a singleton. Since $E(L^*_a) \subseteq I^r_a$, their definition for strongly rpp semigroups conincides with ours.

Theorem 5.3. *If S is a left C-ϱ-rpp semigroup, then it is a left C-liberal semigroup such that*

$$(\forall\, a \in S) \quad a^\circ = a^+_\varrho.$$

P r o o f. Let S be a left C-ϱ-rpp semigroup. Then $E(S)$ is a left regular band. Assume that $E(S) = [Y; I_\alpha]$. For any $e, f \in E(S)$, if $e \mathrel{\mathcal{L}} f$, then, by $\varrho \in \mathcal{LC}(S)$, we have

$$(\forall\, x, y \in S^1) \quad efx = ex \;\varrho\; ey = efy \Longleftrightarrow fx = fex \;\varrho\; fey = fy,$$

that is to say, $e \;\mathcal{L}^\varrho\; f$; conversely, if $e \;\mathcal{L}^\varrho\; f$ where $e \in I_\alpha$ and $f \in I_\beta$, then by $\varrho \in \mathcal{LC}(S)$ again we have

$$(\forall\, x, y \in S^1) \quad fex \;\varrho\; fey \Longleftrightarrow ex = eex \;\varrho\; eey = ey.$$

Therefore $fe \;\mathcal{L}^\varrho\; e \;\mathcal{L}^\varrho\; f$. Noting that $fe \leqslant f$, we conclude that $f \in L^\varrho_{fe} \cap I_{fe}$. So $f = (fe)^+_\varrho = fe$, whence $\beta \leqslant \alpha$. Dually, also $\alpha \leqslant \beta$. Thus $e \;\mathcal{L}\; f$. Till now, we have established that

$$\mathcal{L}^\varrho|_{E(S)} = \mathcal{L}(E(S)).$$

For any $a \in S$ and $e \in I_a$, since $a^+_\varrho e \in I_a$ is such that $a = ae \;\mathcal{L}^\varrho\; a^+_\varrho e$, we have $a^+_\varrho = a^+_\varrho e$. So ea^+_ϱ is in I_a and is $\mathcal{L}$-equivalent to a^+_ϱ, and hence is $\mathcal{L}^\varrho$-equivalent to a. Noting that $ea^+_\varrho \in I_a$ as well, we conclude that $a^+_\varrho = ea^+_\varrho$. Thus a^+_ϱ is the minimum element in I_a. Since a is arbitrary, we claim that S is liberal and such that $a^\circ = a^+_\varrho$.

For any $a, b \in S$, it follows by virtue of $a^+_\varrho b = a^+_\varrho b b^+_\varrho$ that

$$\begin{aligned}
(\forall\, x, y \in S^1) \quad (ab)^+_\varrho x \;\varrho\; (ab)^+_\varrho y &\Longleftrightarrow abx \;\varrho\; aby \\
&\Longleftrightarrow a^+_\varrho b b^+_\varrho x \;\varrho\; a^+_\varrho b b^+_\varrho y \\
&\Longleftrightarrow a^+_\varrho b a^+_\varrho b^+_\varrho x \;\varrho\; a^+_\varrho b a^+_\varrho b^+_\varrho y \\
&\Longleftrightarrow aba^+_\varrho b^+_\varrho x \;\varrho\; aba^+_\varrho b^+_\varrho y \\
&\Longleftrightarrow (ab)^+_\varrho a^+_\varrho b^+_\varrho x \;\varrho\; (ab)^+_\varrho a^+_\varrho b^+_\varrho y.
\end{aligned}$$

So $(ab)^+_\varrho \;\mathcal{L}^\varrho\; (ab)^+_\varrho a^+_\varrho b^+_\varrho$ and hence $(ab)^+_\varrho \;\mathcal{L}\; (ab)^+_\varrho a^+_\varrho b^+_\varrho$. Furthermore,

$$(ab)^+_\varrho = (ab)^+_\varrho (ab)^+_\varrho a^+_\varrho b^+_\varrho = (ab)^+_\varrho a^+_\varrho b^+_\varrho.$$

In particular, if we replace a and b in equation above by b and a^+_ϱ, respectively, then

$$(ba^+_\varrho)^+_\varrho a^+_\varrho = (ba^+_\varrho)^+_\varrho b^+_\varrho (a^+_\varrho)^+_\varrho a^+_\varrho = (ba^+_\varrho)^+_\varrho b^+_\varrho a^+_\varrho = (ba^+_\varrho)^+_\varrho.$$

Since $a^+_\varrho b^+_\varrho \;\mathcal{L}\; b^+_\varrho a^+_\varrho$, we have $a^+_\varrho b^+_\varrho \;\mathcal{L}^\varrho\; b^+_\varrho a^+_\varrho$. Thus, by virtue of Lemma 2.1 and $\varrho \in \mathcal{LC}(S)$, we conclude that

$$\begin{aligned}
(\forall\, x, y \in S^1) \quad abx \;\varrho\; aby &\Longrightarrow a^+_\varrho bx \;\varrho\; a^+_\varrho by \\
&\Longrightarrow a^+_\varrho ba^+_\varrho x \;\varrho\; a^+_\varrho ba^+_\varrho y \\
&\Longrightarrow ba^+_\varrho x = (ba^+_\varrho)^+_\varrho a^+_\varrho ba^+_\varrho x \;\varrho\; (ba^+_\varrho)^+_\varrho a^+_\varrho ba^+_\varrho y = ba^+_\varrho y \\
&\Longrightarrow b^+_\varrho a^+_\varrho x \;\varrho\; b^+_\varrho a^+_\varrho y \\
&\Longrightarrow a^+_\varrho b^+_\varrho x \;\varrho\; a^+_\varrho b^+_\varrho y \\
&\Longrightarrow (ab)^+_\varrho a^+_\varrho b^+_\varrho x \;\varrho\; (ab)^+_\varrho a^+_\varrho b^+_\varrho y \\
&\Longrightarrow (ab)^+_\varrho x \;\varrho\; (ab)^+_\varrho y \\
&\Longrightarrow abx \;\varrho\; aby,
\end{aligned}$$

that is to say, $a_\varrho^+ b_\varrho^+ \ \mathcal{L}^\varrho \ ab \ \mathcal{L}^\varrho \ (ab)_\varrho^+$. So $a_\varrho^+ b_\varrho^+ \ \mathcal{L} \ (ab)_\varrho^+$, whence S is left C-liberal. $\square$

Lemma 5.4. *Let S be a left C-ϱ-rpp semigroup, $E(S) = [Y; I_\alpha]$ and $e, f \in E(S)$. Then*
(1) *$S_e = \{a \in S \colon a_\varrho^+ = e\}$ is a $\varrho|_{S_e}$-left cancellative unipotent semigroup;*
(2) *if $fe = f$, then $f \cdot \varrho|_{S_e} \subseteq \varrho|_{S_f}$.*

Proof. (1) By Theorem 5.3, we can see that S_e is a unipotent semigroup with e as its identity. It is obvious that $\varrho|_{S_e} \in \mathcal{LC}(S_e)$. If $a, b, c \in S_e$ are such that $(ab, ac) \in \varrho|_{S_e}$, then $ab \ \varrho \ ac$ and hence $b = eb \ \varrho \ ec = c$. So S_e is $\varrho|_{S_e}$-left cancellative.

(2) Assume that $fe = f$. For any $a \in S_e$, it is obvious that $f \in I^l_{fa} = I_{fa}$. Moreover, since $(fa)_\varrho^+ \ \mathcal{L} \ f_\varrho^+ a_\varrho^+ = fe = f$, we have $f = (fa)_\varrho^+$ whence $fa \in S_f$. Noting that $\varrho \in \mathcal{LC}(S)$, we conclude that $f \cdot \varrho|_{S_e} \subseteq \varrho|_{S_f}$. $\square$

Corollary 5.5. *If $T = [Y; T_\alpha]$ is a C-full Ehresmann semigroup satisfying the condition*
(C.6) (i) *for any $\alpha \in Y$, T_α is ϱ_α-left cancellative for some $\varrho_\alpha \in \mathcal{LC}(T_\alpha)$, and*
(ii) *for any $\beta \leqslant \alpha$ in Y, $1_{T_\beta} \cdot \varrho_\alpha \subseteq \varrho_\beta$,*
then T is strongly C-ϱ-rpp where $\varrho = \bigcup\limits_{\alpha \in Y} \varrho_\alpha$.

Conversely, every strongly C-ϱ-rpp semigroup can be obtained in this way.

Proof. Let $T = [Y; T_\alpha]$ be a C-full Ehresmann semigroup satisfying condition (C.6). Then $E(S)$ is central in T. Now, $\varrho = \bigcup\limits_{\alpha \in Y} \varrho_\alpha \in \mathcal{LC}(T)$ in view of

$$\begin{aligned}(\forall\,(a,b) \in \varrho_\alpha, c \in T_\beta) \quad (ca, cb) &= (c1_{T_{\alpha\beta}} \cdot 1_{T_{\alpha\beta}} a, c1_{T_{\alpha\beta}} \cdot 1_{T_{\alpha\beta}} b)\\ &\in c1_{T_{\alpha\beta}} \cdot 1_{T_{\alpha\beta}} \cdot \varrho_\alpha\\ &\subseteq c1_{T_{\alpha\beta}} \cdot \varrho_{\alpha\beta}\\ &\subseteq \varrho_{\alpha\beta}.\end{aligned}$$

Let $a \in T_\alpha$ and $b \in T_\gamma$. If $\alpha = \gamma$, then by (C.6) (ii) we have

$$\begin{aligned}(\forall\, x, y \in S^1) \quad (ax, ay) \in \varrho &\Longleftrightarrow (\exists\, \beta \in Y)\ (ax, ay) \in \varrho_\beta\\ &\Longleftrightarrow (\exists\, \beta \in Y)\ ax, ay \in T_\beta,\ (1_{T_\beta} x, 1_{T_\beta} y) \in \varrho_\beta\\ &\Longleftrightarrow (\exists\, \beta \in Y)\ bx, by \in T_\beta,\ (1_{T_\beta} x, 1_{T_\beta} y) \in \varrho_\beta\\ &\Longleftrightarrow (\exists\, \beta \in Y)\ (bx, by) \in \varrho_\beta\\ &\Longleftrightarrow (bx, by) \in \varrho\end{aligned}$$

and so $a \ \mathcal{L}^\varrho \ b$. Conversely, if $a \ \mathcal{L}^\varrho \ b$, then it follows by $a1_{T_\alpha} \ \varrho \ a$ that $b1_{T_\alpha} \ \varrho \ b$. Therefore $\gamma \leqslant \alpha$. Dually, also $\alpha \leqslant \gamma$ whence $\alpha = \gamma$. Therefore $\mathcal{L}^\varrho$ is exactly the semilattice congruence on T induced by the semilattice decomposition $[Y; T_\alpha]$. Thus T is strongly C-ϱ-rpp.

Conversely, if T is a strongly C-ϱ-rpp semigroup, then, by Theorem 5.3, T is a C-full Ehresmann semigroup. Assume that T is a semilattice Y of unipotent monoids T_α. For any $\alpha \in Y$, define $\varrho_\alpha = \varrho|_{T_\alpha}$. Then obviously $\varrho_\alpha \in \mathcal{LC}(T_\alpha)$. By Lemma 5.4, ϱ_α satisfies (C.6). □

In what follows, by a strongly C-ϱ-rpp semigroup $T = [Y; T_\alpha; \varrho_\alpha]$ we mean that S is a strongly C-ϱ-rpp semigroup constructed as in Corollary 5.5.

Theorem 5.6. *Let $S = I \times_{Y,\eta} T$ be a left semi-spined product of a left regular band $I = [Y; I_\alpha]$ and a strongly C-ϱ-rpp semigroup $T = [Y; T_\alpha; \varrho_\alpha]$ which satisfies the following condition:*

(C.7) (i) *there is an equivalence δ on I contained in $\mathcal{L}(I)$ such that*

$$(\forall\,(i,a) \in S)(\forall\,(k,j) \in \delta|_{I_\beta}) \quad (j(i,a)^{\#}, k(i,a)^{\#}) \in \delta;$$

(ii) *for any $(i,a) \in T_\alpha \times I_\alpha$, $j \in I_\beta$ and $k \in I_\gamma$,*

$$(j(i,a)^{\#}, k(i,a)^{\#}) \in \delta \Longrightarrow (j(i,1_{T_\alpha})^{\#}, k(i,1_{T_\alpha})^{\#}) \in \delta.$$

Then S is a left C-ϱ-rpp semigroup where

$$\varrho = \{((i,a),(j,b)) \in S \times S\colon \ (\exists\,\alpha \in Y) \ \ (a,b) \in \varrho_\alpha, (i,j) \in \delta\}.$$

Conversely, every left C-ϱ-rpp semigroup can be constructed in this way.

Proof. Let S be a semigroup constructed as in the theorem. Then evidently $\varrho \in \mathcal{E}(S)$. For any $((i,a),(j,b)) \in \varrho|_{I_\alpha \times T_\alpha}$ and $(k,c) \in S$, by Corollary 5.5 and (C.7) (i), we have

$$(k,c)(i,a) = (i(c,k)^{\#}, ca) \ \varrho \ (j(c,k)^{\#}, cb) = (k,c)(j,b).$$

Thus $\varrho \in \mathcal{LC}(S)$. By using (C.6) (ii), one can easily show that

$$(\forall\,(i,a) \in T_\alpha \times I_\alpha) \quad (i, 1_{T_\alpha}) = (i,a)^{+}_{\varrho}.$$

Therefore, by Theorem 4.6, S is left C-ϱ-rpp.

Conversely, if S is a left C-ϱ-rpp semigroup, then, by Theorem 5.3, it is a left C-liberal semigroup. There is no harm if we denote $S = I \times_{Y,\eta} T$, where $T = [Y; T_\alpha]$ is a C-full Ehresmann semigroup and $I = [Y; I_\alpha]$ is a left regular band. For any $\alpha \in Y$, let

$$\delta_\alpha = \varrho|_{I_\alpha \times T_\alpha} \mathcal{P}_I.$$

For any $(i, 1_{T_\alpha}) \in E(I_\alpha \times T_\alpha)$, it follows by Lemma 5.4 that the relation

$$\varrho_\alpha = \varrho|_{S_{(1_{T_\alpha})}} \mathcal{P}_{T_\alpha}$$

on T_α is independent of i. It is easy to check that $\delta = \bigcup_{\alpha \in Y} \delta_\alpha$ and ϱ_α $(\alpha \in Y)$ satisfy condition (C.7). $\square$

Lemma 5.7 ([3]). *Let S be a semigroup. The following statements are equivalent:*

(1) *S is a C-rpp semigroup;*
(2) *S is a strongly C-rpp semigroup;*
(3) *S is a semilattice of left cancellative monoids;*
(4) *S is a strong semilattice of left cancellative monoids.*

Corollary 5.8 ([23]). *Let S be a semigroup. The following statements are equivalent:*

(1) *S is a C-wrpp semigroup;*
(2) *S is a strongly C-wrpp semigroup;*
(3) *S is a semilattice of $\mathcal{R}$-left cancellative monoids;*
(4) *S is a strong semilattice of $\mathcal{R}$-left cancellative monoids.*

By using Theorem 5.6, Lemma 5.7 and Lemma 5.8, we can easily establish the following two results:

Corollary 5.9 ([14]). *Let S be a semigroup. The following statements are equivalent:*

(1) *S is a left C-rpp semigroup;*
(2) *S is the semilattice Y of the direct products $I_\alpha \times T_\alpha$ $(\alpha \in Y)$ of left zero bands I_α and left cancellative momoids T_α such that, for any $(i, a) \in I_\alpha \times T_\alpha$, $j \in I_\beta$ and $k \in I_\gamma$,*

$$((i,a)(j,1_{T_\beta}))_I = ((i,a)(k,1_{T_\gamma}))_I \Longrightarrow ((i,1_{T_\alpha})(j,1_{T_\beta}))_I = ((i,1_{T_\alpha})(k,1_{T_\gamma}))_I;$$

(3) *S is a left semi-spined product $= I \times_{Y,\eta} T$ of a left regular band $I = [Y; I_\alpha]$ and a C-rpp semigroup $T = [Y; T_\alpha]$ in which the structural homomorphism η satisfies the condition*

$$(\forall\, (i,a) \in I_\alpha \times T_\alpha)(\forall\, j,k \in I) \quad j(i,a)^{\#} = k(i,a)^{\#} \Longrightarrow j(i,1_{T_\alpha})^{\#} = k(i,1_{T_\alpha})^{\#}.$$

Corollary 5.10. *Let S be a semigroup. The following statements are equivalent:*

(1) *S is a left C-wrpp semigroup;*

(2) *S is the semilattice Y of the direct products $I_\alpha \times T_\alpha$ ($\alpha \in Y$) of left zero bands I_α and $\mathcal{R}$-left cancellative monoids T_α such that, for any $(i,a) \in I_\alpha \times T_\alpha$, $j \in I_\beta$ and $k \in I_\gamma$,*

$$((i,a)(j,1_{T_\beta}))_I = ((i,a)(k,1_{T_\gamma}))_I \Longrightarrow ((i,1_{T_\alpha})(j,1_{T_\beta}))_I = ((i,1_{T_\alpha})(k,1_{T_\gamma}))_I;$$

(3) *S is a left semi-spined product $= I \times_{Y,\eta} T$ of a left regular band $I = [Y; I_\alpha]$ and a C-wrpp semigroup $T = [Y; T_\alpha]$ in which the structural homomorphism η satisfies the condition*

$$(\forall\, (i,a) \in I_\alpha \times T_\alpha)(\forall\, j,k \in I) \quad j(i,a)^{\#} = k(i,a)^{\#} \Longrightarrow j(i,1_{T_\alpha})^{\#} = k(i,1_{T_\alpha})^{\#}.$$

By using Theorem 3.6, we can also obtain the band-formal constructions of left C-liberal semigroups, left C-rpp semigroups and left C-wrpp semigroups. Furthermore, by using the characterizations for left C-liberal semigroups, all results on left C-semigroups given by Zhu, Guo and Shum [27] and Guo, Ren and Shum [8] can be obtained as well.

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References

[1] *L. Du, Y. He*: On the **-Green's relations of semigroups and C-broad semigroups. J. Northwest Univ. (Natural Sci. Edt.) *29* (1999), 9–12.

[2] *A. El-Qallali*: Structure theory for abundant and related semigroups. PhD. Thesis. York, 1980.

[3] *J. B. Fountain*: Rpp monoids with central idempotents. Semigroup Forum *13* (1977), 229–237. Zbl 0353.20051

[4] *J. B. Fountain*: Adequate semigroups. Proc. Edinburgh Math. Soc. *44* (1979), 113–125. Zbl 0414.20048

[5] *J. B. Fountain*: Abundant semigroups. Proc. London Math. Soc. *44* (1982), 103–129. Zbl 0481.20036

[6] *J. B. Fountain, G. M. S. Gomes, and V. Gould*: A Munn type representation for a class of E-semiadequate semigroups. J. Algebra *218* (1999), 693–714. Zbl 0940.20064

[7] *G. M. S. Gomes, V. Gould*: Fundamental Ehresmann semigroups. Semigroup Forum *63* (2001), 11–33. Zbl 0998.20051

[8] *Y. Q. Guo, X. M. Ren, and K. P. Shum*: Another structure of left C-semigroups. Adv. Math. *24* (1995), 39–43.

[9] *Y. Q. Guo, K. P. Shum, P. Y. Zhu*: The structure of left C-rpp semigroups. Semigroup Forum *50* (1995), 9–23. Zbl 0821.20046

[10] *Y. He*: A construction for $\mathcal{P}$-regular semigroups (announcement). Adv. Math. *29* (2000), 566–568.

[11] *Y. He*: Partial kernel normal systems in regular semigroups. Semigroup Forum *64* (2002), 325–328. Zbl 1002.20037

[12] *Y. He*: Some studies on regular and generalized regular semigroups. PhD. Thesis. Zhongshan Univ., Guangzhou, 2002.

[13] *Y. He*: A construction for $\mathcal{P}$-regular semigroups. Commun. Algebra *31* (2003), 1–27. Zbl 1026.20053

[14] *Y. He, Y. Q. Guo, and K. P. Shum*: The construction of orthodox supper rpp semigroups. Sci. China *47* (2004), 552–565. Zbl 1083.20051

[15] *J. M. Howie*: Fundamentals of Semigroup Theory. Oxford University Press Inc., New York, 1995. Zbl 0835.20077

[16] *M. Kil'p*: On monoids over which all strongly flat cyclic right acts are projective. Semigroup Forum *52* (1996), 241–245. Zbl 0844.20051

[17] *M. V. Lawson*: Rees matrix semigroups. Proc. Edinburgh Math. Soc. *33* (1990), 23–37. Zbl 0668.20049

[18] *M. V. Lawson*: Semigroups and ordered categories. I. The reduced case. J. Algebra *141* (1991), 422–462. Zbl 0747.18007

[19] *D. B. McAlister*: One-to one partial right translations of a right cancellative semigroup. J. Algebra *43* (1976), 231–251. Zbl 0349.20025

[20] *M. Petrich*: Inverse Semigroups. John Wiley & Sons, New York, 1984. Zbl 0546.20053

[21] *M. Petrich, N. Reilly*: Completely Regular Semigroups. John Wiley & Sons, New York, 1999. Zbl 0967.20034

[22] *F. Shao, Y. He*: Partial kernel normal systems for eventually regular semigroups. Semigroup Forum *71* (2005), 401–410.

[23] *X. D. Tang*: On a theorem of C-wrpp semigroups. Comm. Algebra *25* (1997), 1499–1504. Zbl 0879.20030

[24] *M. Yamada*: P-systems in regular semigroups. Semigroup Forum *24* (1982), 173–187. Zbl 0479.20030

[25] *M. Yamada, M. K. Sen*: $\mathcal{P}$-regular semigroups. Semigroup Forum *39* (1989), 157–178. Zbl 0664.20039

[26] *M. C. Zhang, Y. He*: The structure of $\mathcal{P}$-regular semigroups. Semigroup Forum *54* (1997), 278–291. Zbl 0887.20028

[27] *P. Y. Zhu, Y. Q. Guo, K. P. Shum*: Structure and characterizations of left Clifford semigroups. Science in China, Series A *35* (1992), 791–805. Zbl 0781.20035

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