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AN UPPER BOUND FOR DOMINATION NUMBER OF 5-REGULAR GRAPHS

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Abstract. Let $G=(V,E)$ be a simple graph. A subset $S\subseteq V$ is a dominating set of G, if for any vertex $u\in V-S$, there exists a vertex $v\in S$ such that $uv\in E$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. In this paper we will prove that if G is a 5-regular graph, then $\gamma(G)\leqslant \frac{5}{14}n$.

Keywords: domination number, 5-regular graph, upper bounds

MSC 2000: 05C69

1. Introduction

Let $G=(V,E)$ be a simple graph and v be a vertex in V. The open neighborhood of v, denoted by $N(v)$, is the set of vertices adjacent to v, i.e., $N(v)=\{u\in V\colon\ uv\in E\}$. Let $S\subseteq V$, $G[S]$ denotes the subgraph of G induced by S. For any two disjoint vertex subsets $V_1,V_2\subseteq V$, $E[V_1,V_2]$ denotes the set of edges between V_1 and V_2. $\delta(G)$ denotes the minimum degree of the vertices of G. A subset $S\subseteq V$ is a dominating set of G, if for any vertex $u\in V-S$, there exists a vertex $v\in S$ such that $uv\in E$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of cardinality $\gamma(G)$ is called a γ-set of G.

Theorem 1 ([1], [2]). *For any graph G,*

$$\gamma(G)\leqslant n\Big[1-\delta(G)\Big(\frac{1}{\delta(G)+1}\Big)^{1+1/\delta(G)}\Big].$$

When $\delta(G)$ is small, the best upper bounds on $\gamma(G)$ have been obtained.

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Theorem 2 ([6]). *If G is a graph with $\delta(G) \geqslant 1$, then $\gamma(G) \leqslant n/2$.*

Theorem 3 ([5]). *Let G be a connected graph of order $n \geqslant \delta$. If $\delta(G) \geqslant 2$ and $G \notin \mathcal{A}$, then $\gamma(G) \leqslant \frac{2}{5}n$.*

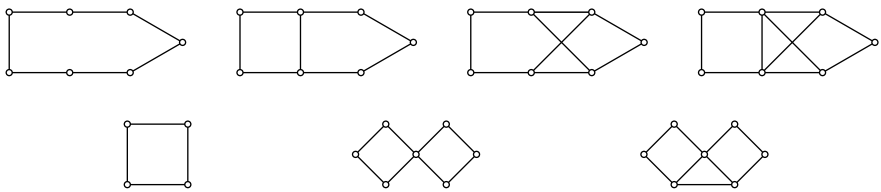

Figure 1. Graphs in family $\mathcal{A}$.

Theorem 4 ([7]). *If G is a graph with $\delta(G) \geqslant 3$, then $\gamma(G) \leqslant \frac{3}{8}n$.*

According to the above conclusions, Haynes et al. posed the following conjecture.

Conjecture 1 ([3]). *For any graph G with $\delta(G) \geqslant k$ $(k \geqslant 4)$, $\gamma(G) \leqslant kn/(3k-1)$.*

By Theorem 1, Conjecture 1 is true for $k \geqslant 7$. So Conjecture 1 is open only for the graphs G with minimum degree $\delta(G) \in \{4, 5, 6\}$. In [4], Liu and Sun proved that Conjecture 1 is true for 4-regular graphs. In this paper we will prove that Conjecture 1 is true for 5-regular graphs.

2. Main results

Let G be a simple graph, and let S be a γ-set of G. For a vertex $u \in V - S$, if $|N(u) \cap S| = k$, then u is a k-neighbor of S. Define $N_k(S) = \{u \in V - S\colon u \text{ is a } k\text{-neighbor of } S\}$. If $u \in N_1(S)$ and v is the only vertex in $N(u) \cap S$, then u is a private neighbor of v (with respect to S). For any vertex $v \in S$, denote $N_k(v) = N(v) \cap N_k(S)$. For $\{v_0, v_1\} \subseteq S$, denote $N_k(v_0, v_1) = N_k(v_0) \cup N_k(v_1)$. Let J_0 be the set of vertices in S with no private neighbors, let J_1 be the set of vertices in S with one private neighbor and let J_2 be the set of vertices in S with at least two private neighbors. Thus J_0, J_1 and J_2 is a partition of S. For $v \in J_1$, let $P(v)$ denote the only private neighbor of the vertex v. Let $i(S)$ denote the number of isolates in $G[S]$.

Theorem 5. *If G is a 5-regular graph with n vertices, then $\gamma(G) \leqslant \frac{5}{14}n$.*

Proof. Without loss of generality, amongst all γ-sets of G, let S be chosen so that

(1) $i(S)$ is maximized.
(2) Subject to (1), $|N_1(S)|$ is minimized.
(3) Subject to (2), $|N_2(S)|$ is minimized.

Before proceeding further, we prove the following claims.

Claim 1. *Each vertex $v \in J_0 \cup J_1$ is an isolate in $G[S]$.*

Proof. If $v \in J_0$, then by the definition of J_0, v is an isolate in $G[S]$. Suppose that there is a vertex $v \in J_1$ such that v is adjacent to a vertex of S. Let $S' = (S - \{v\}) \cup \{P(v)\}$. Then S' is a γ-set of G such that $i(S') > i(S)$. This contradicts our choice of S. $\square$

Claim 2. *For any vertex $u \in V - S$, if $v_1, v_2 \in N(u) \cap J_0$ $(v_1 \neq v_2)$, then $|N_2(v_1) \cap N_2(v_2)| \geqslant 2$.*

Proof. If $|N_2(v_1) \cap N_2(v_2)| = 0$, then $S' = (S - \{v_1, v_2\}) \cup \{u\}$ is a dominating set of G such that $|S'| < |S|$, a contradiction. If $|N_2(v_1) \cap N_2(v_2)| = 1$, let $S' = (S - \{v_1, v_2\}) \cup (N_2(v_1) \cap N_2(v_2))$. Then S' is a dominating set of G such that $|S'| < |S|$, a contradiction. Thus we have $|N_2(v_1) \cap N_2(v_2)| \geqslant 2$. $\square$

Claim 3. *For any vertex $v \in J_1$, if $N_2(v) = \emptyset$ and $N_4(v) \cup N_5(v) \neq \emptyset$, then $N(P(v)) \cap (N_3(S) \cup N_4(S)) \neq \emptyset$.*

Proof. First we prove that $N(P(v)) \cap N_1(S) = \emptyset$. Suppose, to the contrary, that $N_2(v) = \emptyset$ but $|N(P(v)) \cap N_1(S)| \geqslant 1$. Let $S' = (S - \{v\}) \cup \{P(v)\}$. Then S' is a γ-set of G such that $i(S') = i(S)$ and $|N_1(S')| < |N_1(S)|$, a contradiction.

Now we prove that $|N(P(v)) \cap N_2(S)| \leqslant 3$. Suppose, to the contrary, that $|N(P(v)) \cap N_2(S)| = 4$. Let $S' = (S - \{v\}) \cup \{P(v)\}$. Then S' is a γ-set of G and $i(S') = i(S)$. Since $N_2(v) = \emptyset$, $|N_1(S')| = |N_1(S)|$. Since $N_4(v) \cup N_5(v) \neq \emptyset$, $|N_2(S')| < |N_2(S)|$, also a contradiction.

So, $|N(P(v)) \cap (N_1(S) \cup N_2(S))| \leqslant 3$. Then $N(P(v)) \cap (N_3(S) \cup N_4(S)) \neq \emptyset$. $\square$

Claim 4. *Assume $v_0 \in J_0$, $u_1 \in N(v_0)$, $v_1 \in N(u_1) \cap J_1$, $N_2(v_1) = \emptyset$ and $N_4(v_1) \cup N_5(v_1) \neq \emptyset$. If for any $v \in J_0$ and $v \neq v_0$, $N(v_0) \cap N(v) = \emptyset$, then there exists a vertex $w \in N(P(v_1)) \cap (N_3(S) \cup N_4(S))$ such that $N(w) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$.*

Proof. Since $N_2(v_1) = \emptyset$ and $N_4(v_1) \cup N_5(v_1) \neq \emptyset$, by Claim 3, $N(P(v_1)) \cap (N_3(S) \cup N_4(S)) \neq \emptyset$. Assume $w \in N(P(v_1)) \cap (N_3(S) \cup N_4(S))$. Then $N(w) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$. Suppose that $N(w) \cap (J_0 - \{v_0\}) \neq \emptyset$. Without loss of generality, assume $b \in N(w) \cap (J_0 - \{v_0\})$. We claim that $N(v_0) \cap N(b) \neq \emptyset$. Suppose, to the contrary, that $N(v_0) \cap N(b) = \emptyset$. Let $S' = (S - \{b, v_0, v_1\}) \cup \{u_1, w\}$. Then S' is a dominating set of G such that $|S'| < |S|$, a contradiction. Then $N(v_0) \cap N(b) \neq \emptyset$, which is a contradiction with the assumption. $\square$

In order to prove the theorem, we only need to prove that $n - |S| \geqslant \frac{9}{5}|S|$. To prove that $n - |S| \geqslant \frac{9}{5}|S|$, we are going to work out a function $g\colon E[V-S, S] \to [0,1]$ such that

(a) $n - |S| \geqslant \sum\limits_{e \in E[V-S,S]} g(e)$ and

(b) for each $v \in S$, $\gamma(v) = \sum\limits_{u \in N(v)-S} g(uv) \geqslant \frac{9}{5}$.

If such a function g exists, then $n - |S| \geqslant \sum\limits_{e \in E[V-S,S]} g(e) = \sum\limits_{v \in S} \Big(\sum\limits_{u \in N(v)-S} g(uv) \Big) = \sum\limits_{v \in S} \gamma(v) \geqslant \frac{9}{5}|S|$. Therefore, $\gamma(G) = |S| \leqslant \frac{5}{14}n$.

First we define two auxiliary functions.

For any edge $e = uv \in E[V-S, S]$, let

$$f(uv) = \begin{cases} 1, & u \in N_1(v), \\ \frac{1}{5}, & \text{otherwise.} \end{cases}$$

For any vertex $u \in V - S$, let

$$\varphi(u) = \begin{cases} \frac{3}{5}, & u \in N_2(v), \\ \frac{2}{5}, & u \in N_3(v), \\ \frac{1}{5}, & u \in N_4(v), \\ 0, & u \in N_1(v) \cup N_5(v). \end{cases}$$

It is easy to verify that

$$\sum_{e \in E[V-S,S]} f(e) + \sum_{u \in V-S} \varphi(u) = n - |S|.$$

Now we are going to construct the function g. In each step, we will guarantee that $\sum\limits_{e \in E[V-S,S]} g(e) \leqslant \sum\limits_{e \in E[V-S,S]} f(e) + \sum\limits_{u \in V-S} \varphi(u) = n - |S|$ and for any vertex $v_0 \in S$, $\gamma(v_0) \geqslant \frac{9}{5}$.

If $v_0 \in J_2$, let

$$g(uv_0) = \begin{cases} f(uv_0), & u \in N_1(v_0), \\ 0, & \text{otherwise.} \end{cases}$$

Since v_0 has at least two private neighbors and $f(uv_0) = 1$ for $u \in N_1(v_0)$, $\gamma(v_0) = \sum\limits_{u \in N(v_0)-S} g(uv_0) \geqslant 2 > \frac{9}{5}$.

If $v_0 \in J_1$, by Claim 1, v_0 is an isolate in $G[S]$. For $u \in N(v_0)$, let $g(uv_0) = f(uv_0)$. Then

$$f(uv_0) = \begin{cases} 1, & u \in N_1(v_0), \\ \frac{1}{5}, & \text{otherwise.} \end{cases}$$

Hence we have $\gamma(v_0) = g(P(v_0)v_0) + \sum\limits_{u \in N(v_0)-\{P(v_0)\}} g(uv_0) = 1 + 4 \times \frac{1}{5} = \frac{9}{5}$.

In the following we assume $v_0 \in J_0$. By Claim 1, v_0 is an isolate in $G[S]$. First we prove the following claim.

Claim 5. *Assume $v_0 \in J_0$, $u_1 \in N(v_0)$, $v_1 \in N(u_1) \cap J_1$ and $|N(u_1) \cap J_0| = t$ ($t \in \{1, 2\}$). If $u_2 \in N_2(v_1)$, then the edge u_1v_0 can gain at least $\frac{1}{10t}$ from u_2 without obstructing the other vertices v of S such that $\gamma(v) \geqslant \frac{9}{5}$. If there exists a vertex $w \in N(P(v_1)) \cap (N_3(S) \cup N_4(S))$ such that $N(w) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$, then the edge u_1v_0 can gain at least $\frac{1}{20t}$ from w without obstructing other vertices v of S such that $\gamma(v) \geqslant \frac{9}{5}$.*

P r o o f. First assume $u_2 \in N_2(v_1)$. Then $\varphi(u_2) = \frac{3}{5}$. Let $N(v_1) - \{u_1, u_2, P(v_1)\} = \{u_3, u_4\}$. Since $\gamma(v_1) = g(P(v_1)v_1) + \sum\limits_{i=1}^{4} g(u_iv_1) = 1 + 4 \times \frac{1}{5} = \frac{9}{5}$, the vertex u_2 has no contribution to $\gamma(v_1)$. First we divide equally the amount $\varphi(u_2)$ between the two edges joining u_2 to S. Thus u_2v_1 can gain $\frac{1}{2}\varphi(u_2)$ from u_2. Then we divide equally $\frac{1}{2}\varphi(u_2)$ obtained by u_2v_1 among the edges u_1v_1, u_3v_1 and u_4v_1. Thus u_1v_1 can gain $\frac{1}{6}\varphi(u_2)$. Finally we divide equally $\frac{1}{6}\varphi(u_2)$ obtained by u_1v_1 among the edges joining u_1 to J_0. Therefore the edge u_1v_0 can gain $\frac{1}{6t}\varphi(u_2) = \frac{1}{10t}$ from u_2.

Now assume $w \in N(P(v_1)) \cap (N_3(S) \cup N_4(S))$ such that $N(w) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$. Thus

$$\varphi(w) = \begin{cases} \frac{2}{5}, & \text{if } w \in N_3(S), \\ \frac{1}{5}, & \text{if } w \in N_4(S). \end{cases}$$

Let $N(v_1) - \{u_1, P(v_1)\} = \{u_2, u_3, u_4\}$. Let $b \in N(w) \cap (S - \{v_0\})$, since $N(w) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$, $\gamma(b) = \sum\limits_{u \in N(b)-S} g(ub) \geqslant \frac{9}{5}$. Thus the vertex w has no contribution to $\gamma(b)$. If $w \in N_3(S)$, first we divide equally $\varphi(w)$ between the two edges joining w to $V - S$. Thus the edge $P(v_1)w$ can gain $\frac{1}{2}\varphi(w)$ from w. Then we divide equally $\frac{1}{2}\varphi(w)$ obtained by the edge $P(v_1)w$ among the edges u_1v_1, u_2v_1,

u_3v_1 and u_4v_1. Thus the edge u_1v_1 can gain $\frac{1}{8}\varphi(w)$. Finally we divide equally $\frac{1}{8}\varphi(w)$ obtained by the edge u_1v_1 among the edges joining u_1 to J_0. Thus the edge u_1v_0 can gain $\frac{1}{8t}\varphi(w) = \frac{1}{20t}$ from w. If $w \in N_4(S)$, first we divide equally $\varphi(w)$ among the edges u_1v_1, u_2v_1, u_3v_1 and u_4v_1. Thus the edge u_1v_1 can gain $\frac{1}{4}\varphi(w)$ from w. Then we divide equally $\frac{1}{4}\varphi(w)$ obtained by u_1v_1 among the edges joining u_1 to J_0. So the edge u_1v_0 can also gain $\frac{1}{4t}\varphi(w) = \frac{1}{20t}$ from w. □

For $uv_0 \in E[V-S,S]$, if the edge uv_0 can gain the amount α without obstructing the other vertices v of S such that $\gamma(v) \geqslant \frac{9}{5}$, we say that v_0 can gain the amount α.

Let $N(v_0) = \{u_1, u_2, u_3, u_4, u_5\}$.

Case 1. $\left|\bigcup\limits_{i=1}^{5} N(u_i) \cap J_0\right| \geqslant 3$.

Assume there are three different vertices $v_0, v_1, v_2 \in \bigcup\limits_{i=1}^{5} N(u_i) \cap J_0$. By Claim 2, $N(v_0) \cap N(v_i) \neq \emptyset$, $(i = 1, 2)$. By Claim 2, $|N_2(v_0) \cap N_2(v_i)| \geqslant 2$ $(i = 1, 2)$. Thus $|N_2(v_0)| \geqslant 4$, $|N_2(v_1)| \geqslant 2$ and $|N_2(v_2)| \geqslant 2$. Since for $u \in N_2(S)$, $\varphi(u) = \frac{3}{5}$. Let $\alpha = \sum\limits_{u\in N(v_0)} (f(uv_0) + \frac{1}{2}\varphi(u)) + \sum\limits_{u\in N(v_1)} (f(uv_1) + \frac{1}{2}\varphi(u)) + \sum\limits_{u\in N(v_2)} (f(uv_2) + \frac{1}{2}\varphi(u))$. Then $\alpha \geqslant 15\times\frac{1}{5}+8\times\frac{1}{2}\times\frac{3}{5} = \frac{27}{5} = 3\times\frac{9}{5}$. So for $i \in \{0, 1, 2\}$, we can define $g(uv_i)$ such that $\sum\limits_{u\in N(v_i)} g(uv_i) = \frac{1}{3}\alpha$. Then $\gamma(v_0) = \gamma(v_1) = \gamma(v_2) = \sum\limits_{u\in N(v_i)} g(uv_i) = \frac{1}{3}\alpha \geqslant \frac{9}{5}$.

Case 2. $\left|\bigcup\limits_{i=1}^{5} N(u_i) \cap J_0\right| = 2$.

Assume $\bigcup\limits_{i=1}^{5} N(u_i) \cap J_0 = \{v_0, v_1\}$. By Claim 2, $|N_2(v_0) \cap N_2(v_1)| \geqslant 2$. Without loss of generality, we assume $u_1, u_2 \in N_2(v_0) \cap N_2(v_1)$. Then $\varphi(u_1) = \varphi(u_2) = \frac{3}{5}$. Let $N(v_1) - \{u_1, u_2\} = \{u_6, u_7, u_8\}$.

Case 2.1 $|(N(v_0) \cup N(v_1)) \cap N_3(S)| \geqslant 1$.

With loss of generality, assume $u_3 \in N_3(S)$. Then $\varphi(u_3) = \frac{2}{5}$. Let

$$g(u_iv_0) = \begin{cases} f(u_iv_0) + \frac{1}{2}\varphi(u_i), & i = 1, 2, 3, \\ f(u_iv_0), & i = 4, 5, \end{cases}$$

and

$$g(u_iv_1) = \begin{cases} f(u_iv_1) + \frac{1}{2}\varphi(u_i), & i = 1, 2, \\ f(u_iv_1) + \frac{1}{2}\varphi(u_3), & i = 6, \\ f(u_iv_1), & i = 7, 8. \end{cases}$$

Then $\gamma(v_0) = \gamma(v_1) = \sum\limits_{i=1}^{5} g(u_iv_0) = 5 \times \frac{1}{5} + 2 \times \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{9}{5}$.

Case 2.2. $|(N(v_0) \cup N(v_1)) \cap N_3(S)| = 0$.

Case 2.2.1. $|(N_2(v_0, v_1) \cup N_4(v_0, v_1)) - \{u_1, u_2\}| \geqslant 2$.

Without loss of generality, assume $u_3, u_4 \in N_2(S) \cup N_4(S)$. For $i \in \{3, 4\}$,

$$\varphi(u_i) = \begin{cases} \frac{3}{5}, & \text{if } u_i \in N_2(S), \\ \frac{1}{5}, & \text{if } u_i \in N_4(S). \end{cases}$$

If $u_i \in N_2(S)$, then the vertices v_0 and v_1 can gain $\frac{1}{2}\varphi(u_i) = \frac{3}{10}$ from u_i. If $u_i \in N_4(S)$, then v_0 and v_1 can gain $\varphi(u_i) = \frac{1}{5}$ from u_i. Thus v_0 and v_1 can gain at least $2 \times \frac{1}{5}$ from u_3 and u_4. So we let

$$g(u_iv_0) = \begin{cases} f(u_iv_0) + \frac{1}{2}\varphi(u_i), & i = 1, 2, \\ f(u_iv_0) + \frac{1}{5}, & i = 3, \\ f(u_iv_0), & i = 4, 5, \end{cases}$$

and

$$g(u_iv_1) = \begin{cases} f(u_iv_1) + \frac{1}{2}\varphi(u_i), & i = 1, 2, \\ f(u_iv_1) + \frac{1}{5}, & i = 6, \\ f(u_iv_1), & i = 7, 8. \end{cases}$$

Then $\gamma(v_0) = \gamma(v_1) = \sum_{i=1}^{5} g(u_iv_0) = 5 \times \frac{1}{5} + 2 \times \frac{1}{2} \times \frac{3}{5} + \frac{1}{5} = \frac{9}{5}$.

Case 2.2.2. $|(N_2(v_0, v_1) \cup N_4(v_0, v_1)) - \{u_1, u_2\}| \leqslant 1$.

Assume $u_4, u_5 \in N_5(S)$. Then $u_3 \in N_2(S) \cup N_4(S) \cup N_5(S)$.

Firstly, we look at u_4, we will prove that v_0 and v_1 can gain at least $\frac{1}{10}$. Denote $N(u_4) = \{v_0, v_1, v_2, v_3, v_4\}$. Since $\bigcup_{i=1}^{8} N(u_i) \cap J_0 = \{v_0, v_1\}$, $\{v_2, v_3, v_4\} \subseteq J_1 \cup J_2$. If $\{v_2, v_3, v_4\} \cap J_2 \neq \emptyset$, without loss of generality, assume $v_2 \in J_2$ and u', u'' are two private neighbors of v_2. Then $f(u_4v_2) = \frac{1}{5}$. Since $\gamma(v_2) \geqslant g(u'v_2) + g(u''v_2) = 2$, the edge u_4v_2 has no contribution to $\gamma(v_2)$. Thus v_0 and v_1 can gain $\frac{1}{5}$ from u_4v_2.

If $\{v_2, v_3, v_4\} \subseteq J_1$, then we consider the following two subcases.

Case 2.2.2.1. $|E[\{v_2, v_3, v_4\}, N_2(S)]| \geqslant 1$.

Without loss of generality, assume $v_2u' \in E[\{v_2\}, N_2(S)]$. By Claim 5, v_0 and v_1 can gain $\frac{1}{10}$ from u'.

Case 2.2.2.2. $|E[\{v_2, v_3, v_4\}, N_2(S)]| = 0$.

For $i \in \{2, 3, 4\}$, since $N_5(v_i) \neq \emptyset$, by Claim 3, $N(P(v_i)) \cap (N_3(S) \cup N_4(S)) \neq \emptyset$. Let $w_i \in N(P(v_i)) \cap (N_3(S) \cup N_4(S))$. By the definition of $\varphi(w_i)$,

$$\varphi(w_i) = \begin{cases} \frac{2}{5}, & \text{if } w_i \in N_3(S), \\ \frac{1}{5}, & \text{if } w_i \in N_4(S). \end{cases}$$

If $\bigcup_{i=2}^{4} N(w_i) \cap S \subseteq J_1 \cup J_2$, then, by Claim 5, v_0 and v_1 can gain $3 \times \frac{1}{20}$ from w_2, w_3 and w_4.

If $\bigcup_{i=2}^{4} N(w_i) \cap J_0 \neq \emptyset$, without loss of generality, assume $b \in N(w_2) \cap J_0$. If $b \neq v_0$, we claim that $N(v_0) \cap N(b) \neq \emptyset$. Suppose, to the contrary, that $N(v_0) \cap N(b) = \emptyset$. Let $S' = (S - \{b, v_0, v_2\}) \cup \{u_4, w_2\}$. Then S' is a dominating set of G such that $|S'| < |S|$, a contradiction. Thus $b = v_1$. Since $|N_4(v_0, v_1)| \leqslant 1$, we have $|N_4(v_0, v_1)| = 1$. Without loss of generality, we can assume $w_2 = u_3$. If $b = v_0$, then $w_2 = u_3$ also. Since $|N_4(v_0, v_1)| = 1$, $(N(w_3) \cup N(w_4)) \cap S \subseteq J_1 \cup J_2$. By Claim 5, v_0 and v_1 can gain $2 \times \frac{1}{20} = \frac{1}{10}$ from w_3 and w_4.

Therefore, we have proved that if $|N_4(v_0, v_1)| = 1$, v_0 and v_1 can gain at least $\frac{1}{10}$ from w_2, w_3, w_4, and if $|N_4(v_0, v_1)| = 0$, v_0 and v_1 can gain at least $\frac{3}{20}$ from w_2, w_3 and w_4.

Secondly, we look at u_5. Similar to u_4, we can prove that if $|N_4(v_0, v_1)| = 1$, v_0 and v_1 can gain at least $\frac{1}{10}$ and if $|N_4(v_0, v_1)| = 0$, v_0 and v_1 can gain at least $\frac{3}{20}$.

Finally, we look at u_3. If $u_3 \in N_2(S)$, then $\varphi(u_3) = \frac{3}{5}$. We divide equally $\varphi(u_3)$ between the two edges of $E[V - S, S]$ incident with u_3. Thus v_0 and v_1 can gain at least $\frac{1}{2}\varphi(u_3) = \frac{3}{10}$ from u_3. If $u_3 \in N_4(S)$, then $\varphi(u_3) = \frac{1}{5}$. Thus v_0 and v_1 can gain $\varphi(u_3) = \frac{1}{5}$ from u_3. Therefore, v_0 and v_1 can gain at least $\varphi(u_3) = \frac{1}{5}$ from u_3. If $u_3 \in N_5(S)$, then $|N_5(v_0)| = 3$. Similar to u_4 and u_5, v_0 and v_1 can gain $\frac{3}{20}$.

Now we give a brief summary. If $|N_2(v_0, v_1) \cup N_4(v_0, v_1)| = 1$, v_0 and v_1 can gain at least $\frac{1}{5} + 2 \times \frac{1}{10} = \frac{2}{5}$. If $|N_2(v_0, v_1) \cup N_4(v_0, v_1)| = 0$, v_0 and v_1 can gain at least $3 \times \frac{3}{20} = \frac{9}{20} > \frac{2}{5}$. Therefore, v_0 and v_1 can gain at least $\frac{2}{5}$. So let

$$g(u_i v_0) = \begin{cases} f(u_i v_0) + \frac{1}{2}\varphi(u_i), & i = 1, 2, \\ f(u_i v_0) + \frac{1}{5}, & i = 3, \\ f(u_i v_0), & i = 4, 5, \end{cases}$$

and

$$g(u_i v_1) = \begin{cases} f(u_i v_1) + \frac{1}{2}\varphi(u_i), & i = 1, 2, \\ f(u_i v_1) + \frac{1}{5}, & i = 6, \\ f(u_i v_1), & i = 7, 8. \end{cases}$$

Then $\gamma(v_0) = \gamma(v_1) = \sum_{i=1}^{5} g(u_i v_0) = 5 \times \frac{1}{5} + 2 \times \frac{1}{2} \times \frac{3}{5} + \frac{1}{5} = \frac{9}{5}$.

Case 3. $\left| \bigcup_{i=1}^{5} N(u_i) \cap J_0 \right| = 1$.

In this case, we have $\bigcup_{i=1}^{5} N(u_i) \cap J_0 = \{v_0\}$. Thus $\bigcup_{i=1}^{5} N(u_i) - \{v_0\} \subseteq J_1 \cup J_2$.

Case 3.1. $|N_2(v_0)| \geqslant 3$.

Without loss of generality, assume $u_1, u_2, u_3 \in N_2(v_0)$. Then $\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = \frac{3}{5}$. For each u_i ($i \in \{1,2,3\}$), we divide equally $\varphi(u_i)$ between the two edges of $E[V-S,S]$ incident with u_i. Then the edge u_iv_0 gains $\frac{1}{2}\varphi(u_i) = \frac{3}{10}$ from u_i. So let

$$g(u_iv_0) = \begin{cases} f(u_iv_0) + \frac{1}{2}\varphi(u_i), & i = 1,2,3, \\ f(uv_0), & i = 4,5. \end{cases}$$

Then $\gamma(v_0) = \sum\limits_{i=1}^{5} g(u_iv_0) = 5 \times \frac{1}{5} + 3 \times \frac{1}{2} \times \frac{3}{5} = \frac{19}{10} > \frac{9}{5}$.

Case 3.2. $|N_2(v_0)| = 2$.

Assume $N_2(v_0) = \{u_1, u_2\}$. Then $\varphi(u_1) = \varphi(u_2) = \frac{3}{5}$.

Case 3.2.1. $|N_3(v_0) \cup N_4(v_0)| \geqslant 1$.

Assume $u_3 \in N_3(v_0) \cup N_4(v_0)$. Then

$$\varphi(u_3) = \begin{cases} \frac{2}{5}, & \text{if } u_3 \in N_3(S), \\ \frac{1}{5}, & \text{if } u_3 \in N_4(S). \end{cases}$$

Thus the edge u_3v_0 can gain at least $\frac{1}{5}$ from u_3. So let

$$g(u_iv_0) = \begin{cases} f(u_iv_0) + \frac{1}{2}\varphi(u_i), & i = 1,2, \\ f(u_iv_0) + \frac{1}{5}, & i = 3, \\ f(u_iv_0), & i = 4,5. \end{cases}$$

Then $\gamma(v_0) = \sum\limits_{i=1}^{5} g(u_iv_0) = 5 \times \frac{1}{5} + 2 \times \frac{1}{2} \times \frac{3}{5} + \frac{1}{5} = \frac{9}{5}$.

Case 3.2.2. $|N_5(v_0)| = 3$.

Then $u_3 \in N_5(v_0)$. Denote $N(u_3) = \{v_0, v_1, v_2, v_3, v_4\}$. Then for $i \in \{1,2,3,4\}$, $v_i \in J_1 \cup J_2$. If $v_i \in J_2$, then the edge u_iv_0 can gain $f(u_3v_0) = \frac{1}{5}$ from u_3v_i. Otherwise, $v_i \in J_1$. If $N_2(v_i) \neq \emptyset$, assume $u' \in N_2(v_i)$. By Claim 5, the edge u_3v_0 can gain $\frac{1}{10}$ from u'. If $N_2(v_3) = \emptyset$, by Claim 3, there exists a vertex $w_i \in N_3(S) \cup N_4(S)$ such that $N(w_i) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$. By Claim 5, the edge u_3v_0 can gain $\frac{1}{20}$ from w_i. Hence, the edge u_3v_0 can gain at least $4 \times \frac{1}{20} = \frac{1}{5}$ altogether. So let

$$g(u_iv_0) = \begin{cases} f(u_iv_0) + \frac{1}{2}\varphi(u_i), & i = 1,2, \\ f(u_iv_0) + \frac{1}{5}, & i = 3, \\ f(u_iv_0), & i = 4,5. \end{cases}$$

Then $\gamma(v_0) = \sum\limits_{i=1}^{5} g(u_iv_0) = 5 \times \frac{1}{5} + 2 \times \frac{1}{2} \times \frac{3}{5} + \frac{1}{5} = \frac{9}{5}$.

Case 3.3. $|N_2(v_0)| \leqslant 1$.

Case 3.3.1. $|N_3(v_0)| \geqslant 2$.

Assume $u_1, u_2 \in N_3(v_0)$. Then $\varphi(u_1) = \varphi(u_2) = \frac{2}{5}$. Let

$$g(u_i v_0) = \begin{cases} f(u_i v_0) + \varphi(u_i), & i = 1, 2, \\ f(u_i v_0), & i = 3, 4, 5. \end{cases}$$

Then $\gamma(v_0) = \sum\limits_{i=1}^{5} g(u_i v_0) = 5 \times \frac{1}{5} + 2 \times \frac{2}{5} = \frac{9}{5}$.

Case 3.3.2. $|N_3(v_0)| = 1$.

Case 3.3.2.1. $|N_2(v_0) \cup N_4(v_0)| \geqslant 2$.

Since

$$\varphi(u) = \begin{cases} \frac{3}{5}, & u \in N_2(v_0), \\ \frac{2}{5}, & u \in N_3(v_0), \\ \frac{1}{5}, & u \in N_4(v_0), \end{cases}$$

for $i \in \{1, 2, 3, 4, 5\}$, let

$$g(u_i v_0) = \begin{cases} f(u_i v_0) + \frac{1}{2}\varphi(u_i), & \text{if } u_i \in N_2(v_0), \\ f(u_i v_0) + \varphi(u_i), & \text{if } u_i \in N_3(v_0) \cup N_4(v_0), \\ f(u_i v_0), & \text{if } u_i \in N_2(v_0). \end{cases}$$

Then $\gamma(v_0) = \sum\limits_{i=1}^{5} g(u_i v_0) \geqslant 5 \times \frac{1}{5} + \frac{2}{5} + 2 \times \frac{1}{5} = \frac{9}{5}$.

Case 3.3.2.2. $|N_2(v_0) \cup N_4(v_0)| \leqslant 1$.

Assume $N_3(v_0) = \{u_2\}$ and $u_3, u_4, u_5 \in N_5(v_0)$. Then $\varphi(u_2) = \frac{2}{5}$ and $u_1 \in N_2(S) \cup N_4(S) \cup N_5(S)$. Denote $N(u_2) \cap S = \{v_0, v_{21}, v_{22}\}$ and $N(u_k) \cap S = \{v_0, v_{k1}, v_{k2}, v_{k3}, v_{k4}\}$ $(k \in \{3, 4, 5\})$. Then $\bigcup\limits_{k=2}^{5} N(u_k) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$.

For any $v_{ij} \in \bigcup\limits_{k=2}^{5} N(u_k) \cap (S - \{v_0\})$ $(i \in \{2, 3, 4, 5\})$, if $v_{ij} \in J_2$, then the edge $u_i v_0$ can gain $f(u_i v_{ij}) = \frac{1}{5}$ from $u_i v_{ij}$. Otherwise, $v_{ij} \in J_1$. If $N_2(v_{ij}) \neq \emptyset$, by Claim 5, the edge $u_i v_0$ can gain $\frac{1}{10}$. If $N_2(v_{ij}) = \emptyset$, by Claim 4, there exists a vertex $w_{ij} \in N_3(S) \cup N_4(S)$ such that $N(w_{ij}) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$. By Claim 5, the edge $u_i v_0$ can gain $\frac{1}{20}$ from w_{ij}. Since $\left|\bigcup\limits_{k=2}^{5} N(u_k) \cap (S - \{v_0\})\right| = 14$, the edges $u_2 v_0$, $u_3 v_0$, $u_4 v_0$ and $u_5 v_0$ can gain at least $14 \times \frac{1}{20}$ altogether.

Next we look at u_1. If $u_1 \in N_2(S)$, then $\varphi(u_1) = \frac{3}{5}$. We divide equally $\varphi(u_1)$ between the two edges of $E[V - S, S]$ incident with u_1. Thus the edge $u_1 v_0$ can gain $\frac{1}{2}\varphi(u_1) = \frac{3}{10}$ from u_1. If $u_1 \in N_4(S)$, then $\varphi(u_1) = \frac{1}{5}$. Denote $N(u_1) \cap S = \{v_0, v_{11}, v_{12}, v_{13}\}$. Then $\{v_{11}, v_{12}, v_{13}\} \subseteq J_1 \cup J_2$. Similarly to u_3v_0, u_4v_0 and u_5v_0, the edge u_1v_0 can gain at least $3 \times \frac{1}{20}$. If $u_1 \in N_5(S)$, denote $N(u_1) \cap S = \{v_0, v_{11}, v_{12}, v_{13}, v_{14}\}$. Then $\{v_{11}, v_{12}, v_{13}, v_{14}\} \subseteq J_1 \cup J_2$. Similarly to u_3v_0, u_4v_0 and u_5v_0, the edge u_1v_0 can gain at least $4 \times \frac{1}{20}$.

Hence, for $u_1 \in N_2(S) \cup N_4(S) \cup N_5(S)$, the edge u_1v_0 can gain at least $\frac{3}{20}$. Let

$$g(u_iv_0) = \begin{cases} f(u_iv_0) + \frac{3}{20}, & i = 1, \\ f(u_iv_0) + 2 \times \frac{1}{20}, & i = 2, \\ f(u_iv_0) + 4 \times \frac{1}{20}, & i = 3, 4, 5. \end{cases}$$

Then $\gamma(v_0) = \sum_{i=1}^{5} g(u_iv_0) = 5 \times \frac{1}{5} + 17 \times \frac{1}{20} > \frac{9}{5}$.

Case 3.3.3. $|N_3(v_0)| = 0$.

Case 3.3.3.1. $|N_2(v_0) \cup N_4(v_0)| \geqslant 4$.

For $i \in \{1, 2, 3, 4, 5\}$, let

$$g(u_iv_0) = \begin{cases} f(u_iv_0) + \frac{1}{2}\varphi(u_i), & \text{if } u_i \in N_2(v_0), \\ f(u_iv_0) + \varphi(u_i), & \text{if } u_i \in N_3(v_0) \cup N_4(v_0), \\ f(u_iv_0), & \text{if } u_i \in N_5(v_0). \end{cases}$$

Then $\gamma(v_0) = \sum_{i=1}^{5} g(u_iv_0) = 5 \times \frac{1}{5} + 4 \times \frac{1}{5} = \frac{9}{5}$.

Case 3.3.3.2. $|N_2(v_0)| = 1$ and $|N_4(v_0)| \leqslant 2$.

Let $N_2(v_0) = \{u_1\}$ and let $N_4(v_0) \cup N_5(v_0) = \{u_2, u_3, u_4, u_5\}$. Then $\varphi(u_1) = \frac{3}{5}$. Let $|N_4(v_0)| = t$. Then $|N_5(v_0)| = 4 - t$ and $t \in \{0, 1, 2\}$. Thus $\bigcup_{k=2}^{5} N(u_k) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$ and $\left|\bigcup_{k=2}^{5} N(u_k) \cap (S - \{v_0\})\right| = 3t + 4(4 - t) = 16 - t \geqslant 14$. For any $v_{ij} \in \bigcup_{k=2}^{5} N(u_k) \cap (S - \{v_0\})$ $(i \in \{2, 3, 4, 5\})$, if $v_{ij} \in J_2$, then the edge u_iv_0 can gain $f(u_iv_{ij}) = \frac{1}{5}$ from u_iv_{ij}. Otherwise, $v_{ij} \in J_1$. If $N_2(v_{ij}) \neq \emptyset$, by Claim 5, the edge u_iv_0 can gain $\frac{1}{10}$. If $N_2(v_{ij}) = \emptyset$, by Claim 4, there exists a vertex $w_{ij} \in N_3(S) \cup N_4(S)$ such that $N(w_{ij}) \cap (S - \{v_0\}) \subseteq J_1 \cup J_2$. By Claim 5, the edge u_3v_0 can gain $\frac{1}{20}$ from w_{ij}. Since there are $16 - t$ vertices in $\bigcup_{k=2}^{5} N(u_k) \cap (S - \{v_0\})$, the edges u_2v_0, u_3v_0, u_4v_0 and u_5v_0 can gain at least $(16-t)\times\frac{1}{20}$ altogether. Let

$$g(u_iv_0)=\begin{cases} f(u_iv_0)+\frac{1}{2}\varphi(u_i), & i=1,\\ f(u_iv_0)+\frac{1}{4}(16-t)\times\frac{1}{20}, & i=2,3,4,5.\end{cases}$$

Then $\gamma(v_0)=\sum\limits_{i=1}^{5} g(u_iv_0)=5\times\frac{1}{5}+\frac{1}{2}\times\frac{3}{5}+(16-t)\times\frac{1}{20}>\frac{9}{5}$.

Case 3.3.3.3. $|N_2(v_0)|=0$ and $|N_4(v_0)|\leqslant 3$.

Let $|N_4(v_0)|=t$. Then $|N_5(v_0)|=5-t$ and $t\in\{0,1,2,3\}$. Thus $\bigcup\limits_{k=1}^{5} N(u_k)\cap(S-\{v_0\})\subseteq J_1\cup J_2$ and $\left|\bigcup\limits_{k=1}^{5} N(u_k)\cap(S-\{v_0\})\right|=3t+4(5-t)=20-t\geqslant 17$. For any $v_{ij}\in\bigcup\limits_{k=1}^{5} N(u_k)\cap(S-\{v_0\})$ $(i\in\{1,2,3,4,5\})$, if $v_{ij}\in J_2$, then the edge u_iv_0 can gain $f(u_iv_{ij})=\frac{1}{5}$ from u_iv_{ij}. Otherwise, $v_{ij}\in J_1$. If $N_2(v_{ij})\neq\emptyset$, by Claim 5, the edge u_iv_0 can gain $\frac{1}{10}$. If $N_2(v_{ij})=\emptyset$, by Claim 4, there exists a vertex $w_{ij}\in N_3(S)\cup N_4(S)$ such that $N(w_{ij})\cap(S-\{v_0\})\subseteq J_1\cup J_2$. By Claim 5, the edge u_iv_0 can gain $\frac{1}{20}$ from w_{ij}. Since there are $20-t$ vertices in $\bigcup\limits_{k=1}^{5} N(u_k)\cap(S-\{v_0\})$, the edges u_1v_0, u_2v_0, u_3v_0, u_4v_0 and u_5v_0 can gain at least $(20-t)\times\frac{1}{20}$ altogether. For $i\in\{1,2,3,4,5\}$, let $g(u_iv_0)=f(u_iv_0)+\frac{1}{5}(20-t)\times\frac{1}{20}$. Then $\gamma(v_0)=\sum\limits_{i=1}^{5} g(u_iv_0)=5\times\frac{1}{5}+(20-t)\times\frac{1}{20}>\frac{9}{5}$.

We have finished the definition of the function g, which satisfies conditions (a) and (b). Therefore the proof of the theorem is completed. $\square$

By a similar method, we can prove that Conjecture 1 is true for 6-regular graphs [8]. Therefore Conjecture 1 is true for all k-regular graphs, where $k\geqslant 3$.

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