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MAHLER MEASURES IN A CUBIC FIELD

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Abstract. We prove that every cyclic cubic extension E of the field of rational numbers contains algebraic numbers which are Mahler measures but not the Mahler measures of algebraic numbers lying in E. This extends the result of Schinzel who proved the same statement for every real quadratic field E. A corresponding conjecture is made for an arbitrary non-totally complex field E and some numerical examples are given. We also show that every natural power of a Mahler measure is a Mahler measure.

Keywords: Mahler measure, Pisot numbers, cubic extension

MSC 2000: 11R06, 11R09, 11R16.

1. Introduction

Let α be an algebraic number of degree d over the field of rational numbers $\mathbb{Q}$ with minimal polynomial

$$a_d X^d + \ldots + a_1 X + a_0 = a_d (X - \alpha_1) \ldots (X - \alpha_d) \in \mathbb{Z}[X].$$

Its *Mahler measure* is defined by $M(\alpha) = a_d \prod_{j=1}^{d} \max\{1, |\alpha_j|\}$. The set of all Mahler measures $\mathscr{M} = \{M(\alpha) \colon \alpha \in \overline{\mathbb{Q}}\}$ was studied on many occasions. (See, for instance, [1]–[8], [13].) Usually, the motivation for its study are the unsolved Lehmer's problem [10], its relations with ergodic theory and different applications of the lower bounds for measures to other problems of algebraic number theory. Throughout, we say that β is a *Mahler measure* if $\beta \in \mathscr{M}$.

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In [5] we showed that for any given β one can determine whether it is the Mahler measure of an integer polynomial or not. More precisely, we showed that it is sufficient to consider those polynomials which have all their roots in the normal closure of $\mathbb{Q}(\beta)$ over $\mathbb{Q}$ and whose degrees are bounded in a certain way. However, $\beta \in \mathscr{M}$ means that β is the Mahler measure of an integer irreducible polynomial. The word 'irreducible' makes a big difference. Nevertheless, it is shown in [5] that if a unit β belongs to $\mathscr{M}$ then $\beta = M(\alpha)$ with certain α in the normal closure of $\mathbb{Q}(\beta)$ over $\mathbb{Q}$. As above, this gives an effective procedure for determining whether a unit belongs to $\mathscr{M}$ or not. The same is true if $N(\beta)$ is not divisible by an sth power of an integer greater than 1, where $s = \deg \beta$ and where $N(\beta)$ is the norm of β. We asked therefore whether this is true without any condition on β.

Recently, Schinzel [13] considered some special quadratic numbers β and proved that some condition on β for the statement as above is necessary. In particular, he constructed certain quadratic numbers β which belong to $\mathscr{M}$ but which are not expressible as $\beta = M(\alpha)$ with $\alpha \in \mathbb{Q}(\beta)$. So the overall situation with $\mathscr{M}$ is much more difficult. It is not known, for instance, whether $1 + \sqrt{17}$ belongs to $\mathscr{M}$ or not.

Summarizing, we see that despite considerable progress towards determining the structure of $\mathscr{M}$ the question remains open even for the set $\mathscr{M} \cap \mathbb{Q}(\sqrt{D})$, where $D \geqslant 2$ is a square-free integer. Inspired by Schinzel's results, we begin with the following conjecture claiming that in any field the set of Mahler measures is quite complicated.

Conjecture. *Let E be an arbitrary non-totally complex extension of $\mathbb{Q}$ of degree $s \geqslant 2$. Then E contains infinitely many Mahler measures β of degree s that are not Mahler measures of any $\alpha \in F$, where F is the normal closure of E over $\mathbb{Q}$.*

Probably, the conjecture holds with F replaced by an arbitrary finite extension K of $\mathbb{Q}$. However, this stronger version seems to be very difficult already for $s = 2$.

As in [8], we say that $\alpha > 1$ is a *generalized Pisot number* if it is an algebraic number whose other conjugates (if any) all lie strictly inside the unit circle. Such numbers are a useful tool in the study of $\mathscr{M}$ (see [5], [7], [8]).

Theorem 1. *Let E be an arbitrary non-totally complex extension of $\mathbb{Q}$ of degree $s \geqslant 2$. Then E contains infinitely many Mahler measures β of degree s such that β is not the Mahler measure of any generalized Pisot number α.*

Note that if $E/\mathbb{Q}$ is normal then $E = F$. In particular, this is the case when E is a real quadratic or a cyclic cubic extension of $\mathbb{Q}$. Also, if β is a Mahler measure in a real quadratic field E and $\beta = M(\alpha)$ with $\alpha \in E$, where α is greater than or equal to its other conjugate α' in absolute value, then α or $-\alpha$ must be a generalized Pisot number. Similarly, if β is a Mahler measure in a cubic extension E and $\beta = M(\alpha)$

with $\alpha \in E$, where α is the unique conjugate lying either outside or inside the unit circle, then $\pm\alpha$ or $\pm\alpha^{-1}$ is a generalized Pisot number. Using $M(\alpha) = M(-\alpha)$, $M(\alpha) = M(\alpha^{-1})$ and applying the theorem we obtain the following corollary.

Corollary. *In every real quadratic and in every cubic extension E of $\mathbb{Q}$ there are infinitely many $\beta \in E \cap \mathscr{M}$ such that β is not the Mahler measure of any $\alpha \in E$.*

Note that the real quadratic case corresponds to Corollary 2 in Schinzel's paper [13]. The cubic case is new. It settles the conjecture for all cyclic cubic fields. Theorem 1 of [13] can also be generalized to a cubic field. Recall that an algebraic integer β is called *primitive* if β/k is not an algebraic integer for every rational integer $k \geqslant 2$.

Theorem 2. *A primitive real cubic integer β is in $\mathscr{M}$ if and only if there is a rational integer k such that $\beta > k > \max\{|\beta'|, |\beta''|\}$, $k \mid \beta\beta' + \beta\beta'' + \beta'\beta''$ and $k^2 \mid N(\beta)$, where β' and β'' are the conjugates of β.*

Since all definitive results for Mahler measures except for some special constructions are stated either for primitive integers or for numbers with some restrictions on their norm (see [5], [7], [13]), one may get an impression that there are no such results (without any restrictions) for $\mathscr{M}$. This is not the case. We conclude with a result of this kind. (Its partial case for units was obtained in [7].)

Theorem 3. *If $\beta \in \mathscr{M}$ then $\beta^m \in \mathscr{M}$ for every positive integer m.*

By Schinzel's results, there are primitive quadratic $\beta \in \mathscr{M}$ such that $p\beta \notin \mathscr{M}$ for certain primes p that do not split in $\mathbb{Q}(\beta)$ (See, for instance, Corollary 1 in [13].) Hence the additive version of Theorem 3 is false. It would be of interest to find out whether this is always the case.

Question. Is there a real β different from a positive integer such that $m\beta \in \mathscr{M}$ for every $m = 1, 2, 3, \ldots$?

Theorem 1 will be proved in Section 2. Afterwards we will prove Theorems 2 and 3. Some numerical examples can be found in Section 4.

2. Generalized Pisot numbers: the proof of Theorem 1

We will choose a real algebraic integer $\gamma \in E$ of degree s over $\mathbb{Q}$, an integer u and an infinite set of positive integers S with the following properties. Firstly,

$$1 + \max_{2 \leqslant j \leqslant s} |\gamma_j| < u < \gamma,$$

where $\gamma_1 = \gamma, \gamma_2, \dots, \gamma_s$ is the set of conjugates of γ over $\mathbb{Q}$. Secondly, $\gcd(u, N(\gamma)) = 1$, where $N(\gamma) = \gamma_1\gamma_2 \dots \gamma_s$. Finally, S will be the set of positive integers t such that $\sqrt{1 - 4\gamma u t^2}$ has degree $2s$ over $\mathbb{Q}$. We claim that the polynomial

$$P(X) = \prod_{j=1}^{s} (utX^2 + X + \gamma_j t)$$

is irreducible in $\mathbb{Z}[X]$, and its zero $\xi = \bigl(-1 + \sqrt{1 - 4\gamma u t^2}\bigr)/2ut$ lying in a quadratic extension of E has the Mahler measure equal to $t^s u^{s-1}\gamma$. Then we will show that $t^s u^{s-1}\gamma$ is not the Mahler measure of a generalized Pisot number for every t sufficiently large. (The theorem then follows by the infinity of the set S.)

We begin by choosing any Pisot number $\gamma \in E$ of degree s. It is well known that this is possible, because E is non-totally complex. (See, for instance, Salem's book [11] or the recent paper [9] for a stronger statement in a real field.) Then we multiply γ by a large positive integer k. Clearly, the gap between $1 + k \max_{2 \leqslant j \leqslant s} |\gamma_j| < 1 + k$ and $k\gamma$ will be of order ck, where c is an absolute positive constant, whereas $N(k\gamma) = k^s N(\gamma)$ is of order $c' k^s$. The product of all primes separating $1 + k$ and $k\gamma$ is therefore of order $\exp\{ck\}$, so at least one such prime u does not divide $k^s N(\gamma)$. On replacing γ by $k\gamma$ (without changing notation), we see that $1 + \max_{2 \leqslant j \leqslant s} |\gamma_j| < u < \gamma$ and $\gcd(u, N(\gamma)) = 1$.

By Hilbert's irreducibility theorem and Capelli's lemma (see, e.g., p. 298 and p. 92 in [12]), the set S of positive integers t for which $\sqrt{1 - 4\gamma u t^2}$ has 'generic' degree $2s$ over $\mathbb{Q}$ is infinite. Let t be one of these. Then $\xi = \bigl(-1 + \sqrt{1 - 4\gamma u t^2}\bigr)/2ut$ is of degree $2s$ and $P(\xi) = 0$. The polynomial P is reducible in $\mathbb{Z}[X]$ only if it has a constant factor, say $v > 1$. Since the coefficients of $P(X)$ are all divisible by t except for the coefficient at X^s which is equal to 1 modulo t, we have that $\gcd(v, t) = 1$. Two extreme coefficients of P are $u^s t^s$ and $N(\gamma) t^s$, hence $v \mid u^s$ and $v \mid N(\gamma)$. This is a contradiction with the fact that $\gcd(u, N(\gamma)) = 1$. Therefore $P(X)$ is irreducible in $\mathbb{Z}[X]$. Hence, by the multiplicative property of Mahler's measure,

$$M(\xi) = M(P) = \prod_{j=1}^{s} M(utX^2 + X + \gamma_j t).$$

From the inequality $u < \gamma$, we find that ξ is complex, so $M(utX^2+X+\gamma t) = ut\xi\overline{\xi} = t\gamma$. For each $j \geqslant 2$, we will show that $M(utX^2 + X + \gamma_j t) = ut$. Indeed $1 + |\gamma_j| < u$ implies that

$$|1 - 4\gamma_j ut^2| < 1 + 4(u-1)ut^2 \leqslant (2ut-1)^2,$$

hence the roots of $utX^2+X+\gamma_j t$, namely, $\left(-1 \pm \sqrt{1-4\gamma_j ut^2}\right)/2ut$ are both smaller than 1 in absolute value. It follows that $M(utX^2 + X + \gamma_j t) = ut$. Consequently, $M(\xi) = M(P) = t\gamma(ut)^{s-1} = t^s u^{s-1}\gamma$, as claimed.

It remains to show that $t^s u^{s-1}\gamma$ is not the Mahler measure of a generalized Pisot number α. Assume it is. Then $q\alpha = t^s u^{s-1}\gamma$, where q is the leading coefficient of the minimal polynomial for α. Since γ is an algebraic integer, we can write its minimal polynomial in the form $Q(X) = X^s + b_{s-1}X^{s-1} + \ldots + b_0$, where $b_0, \ldots, b_{s-1} \in \mathbb{Z}$. But $Q(q\alpha/t^s u^{s-1}) = 0$, so the minimal polynomial of α in $\mathbb{Z}[X]$ has the form

$$R(X) = qX^s + b_{s-1}t^s u^{s-1}X^{s-1} + b_{s-2}t^{2s}u^{2(s-1)}q^{-1}X^{s-2} + \ldots + b_0 t^{s^2}u^{s(s-1)}q^{-s+1}.$$

Now, since $\alpha = t^s u^{s-1}\gamma/q$ is a generalized Pisot number, we obtain the inequalities $t^s u^{s-1} \max\limits_{2\leqslant j\leqslant s} |\gamma_j| < q < t^s u^{s-1}\gamma$, giving $t^{s-1} < q < t^{s+1}$ for t sufficiently large. The upper bound $q < t^{s+1}$ implies that $t^{ks-1} \geqslant q^{k-1}$ for $1 \leqslant k \leqslant s$. It follows that all coefficients of R except for q are divisible by t, so $\gcd(q,t) = 1$, since $R(X) \in \mathbb{Z}[X]$ is irreducible. But then, by considering the constant coefficient of R, we see that q^{s-1} divides $b_0 u^{s(s-1)}$ which is impossible, because $q^{s-1} > t^{(s-1)^2} > b_0 u^{s(s-1)}$ This completes the proof of Theorem 1. $\square$

3. Primitive cubic integers and integer powers of a measure

P r o o f of Theorem 2. Assume that $\beta = M(\alpha)$. As in Lemma 2 of [7] and in Theorem 1 of [13] we may assume that α has $m = d/3$ conjugates $\alpha_1, \ldots, \alpha_m$ lying strictly outside the unit circle and $2d/3$ conjugates α_j satisfying $|\alpha_j| \leqslant 1$, where d is the degree of α over $\mathbb{Q}$. (Indeed, Lemma 2 of [7] implies that $m = d/3$ or $m = 2d/3$. However, in the second case, $m = 2d/3$, using the equality $M(\alpha) = M(\alpha^{-1})$ we can replace α by its reciprocal α^{-1} which will have $m = d/3$ conjugates lying strictly outside the unit circle.) If k is the leading coefficient of the minimal polynomial of α, then there is $\eta \in \{-1, 1\}$ such that $\beta = \eta k\alpha_1 \ldots \alpha_m$, $\beta' = \eta k\alpha_{m+1} \ldots \alpha_{2m}$, $\beta'' = \eta k\alpha_{2m+1} \ldots \alpha_{3m}$. Consequently, $\beta > k$, $|\beta'| < k$ and $|\beta''| < k$. Indeed, the equality $|\beta'| = k$ holds only if β' is complex. But then $\beta'' = \overline{\beta'}$ giving $\beta = N(\beta)/\beta'\beta'' = N(\beta)/k^2 \in \mathbb{Q}$, a contradiction. Also,

$$\beta\beta' + \beta\beta'' + \beta'\beta'' = k^2(\alpha_1 \ldots \alpha_{2m} + \alpha_1 \ldots \alpha_m\alpha_{2m+1} \ldots \alpha_{3m} + \alpha_{m+1} \ldots \alpha_{3m})$$

is a rational integer divisible by k, because $k\alpha_1\ldots\alpha_{2m}$, $k\alpha_1\ldots\alpha_m\alpha_{2m+1}\ldots\alpha_{3m}$ and $k\alpha_{m+1}\ldots\alpha_{3m}$ are algebraic integers, so their sum is a rational integer (because it is an algebraic integer and a rational number at the same time). Similarly, $N(\beta) = \beta\beta'\beta'' = \eta^3 k^3 N(\alpha) = \pm k^3 N(\alpha)$. But $kN(\alpha) \in \mathbb{Z}$, hence $k^2 \mid N(\beta)$ and the proof of necessity is completed.

For the proof of sufficiency, assume that there is an integer k, as claimed. Consider the polynomial

$$G(X) = kX^3 - (\beta+\beta'+\beta'')X^2 + (\beta\beta'+\beta\beta''+\beta'\beta'')X/k - N(\beta)/k^2 \in \mathbb{Z}[X].$$

Since $G(\beta/k) = 0$, we have that $M(\beta/k) = M(G) = \beta$. It remains to show that $G(X)$ is irreducible. The degree of β/k over $\mathbb{Q}$ is equal to 3, so the polynomial G can be reducible only if there is a prime number p dividing its coefficients. In particular, $p \mid k$. Hence $p \mid \beta+\beta'+\beta''$, $p^2 \mid \beta\beta'+\beta\beta''+\beta'\beta''$ and $p^3 \mid N(\beta)$. It follows that β/p is the root of the polynomial

$$k^2p^{-3}G(pX/k) = X^3 - (\beta+\beta'+\beta'')X^2/p + (\beta\beta'+\beta\beta''+\beta'\beta'')X/p^2 - N(\beta)/p^3$$

with integer coefficients, contrary to the primitivity of β. □

P r o o f of Theorem 3. Write $\beta = M(\alpha)$ and fix $m \geqslant 2$. Suppose that the degree of α over $\mathbb{Q}$ is d, and let d_n denote the degree of α^n over $\mathbb{Q}$, so that $d_1 = d$. Recall that the quantity $h(\gamma) = \log M(\gamma)/\deg\gamma$ is called the *Weil height* of $\gamma \in \overline{\mathbb{Q}}$. We will apply the formula $h(\gamma^n) = nh(\gamma)$ (see, e.g., Property 3.3 in [14]) to the powers of α. If $d_m = d$, then $h(\alpha^m) = mh(\alpha)$ implies immediately that $\beta^m = M(\alpha)^m = M(\alpha^m)$, so $\beta^m \in \mathscr{M}$.

We now turn to the case $d_m < d$. Set $t_1 = d/d_m$, $t_2 = d_m/d_{mt_1}$, $t_3 = d_{mt_1}/d_{mt_1t_2}$, etc. Since $t_1t_2\ldots t_k = d/d_{mt_1\ldots t_{k-1}} \leqslant d$, sooner or later in the sequence of positive integers $t_1, t_2, t_3, \ldots$ we will get an element equal to 1. Let $k \geqslant 2$ be the smallest positive integer for which $t_k = 1$. Using $h(\alpha^{mt_1\ldots t_{k-1}}) = mt_1\ldots t_{k-1}h(\alpha)$ and $d/d_{mt_1\ldots t_{k-1}} = t_1\ldots t_k = t_1\ldots t_{k-1}$, we obtain that

$$\beta^m = M(\alpha)^m = M(\alpha^{mt_1\ldots t_{k-1}}) \in \mathscr{M}.$$

□

4. Explicit examples

Some explicit quadratic examples were given already in [13]. For instance, $21 + 14\sqrt{2}$ is the Mahler measure of the quartic irreducible polynomial $7X^4 + 2X^3 + 41X^2 + 22X + 7$, but $21 + 14\sqrt{2} \neq M(\alpha)$ for any $\alpha \in \mathbb{Q}(\sqrt{2})$.

Similarly, taking $u = 4$, $t = 1$, $\gamma = 3 + \sqrt{7}$ in the definition of $P(X)$ in Section 2, we get

$$M(16X^4 + 8X^3 + 25X^2 + 6X + 2) = 4(3 + \sqrt{7}).$$

Here, $16X^4+8X^3+25X^2+6X+2$ is irreducible, although $\gcd(u, N(\gamma)) = \gcd(4, 2) = 2 \neq 1$. Suppose that $4(3 + \sqrt{7})$ (which is not of the form $p\beta$ with β primitive and p prime as in [13]) is the Mahler measure of a number $\alpha \in \mathbb{Q}(\sqrt{7})$. Write $\alpha = s+t\sqrt{7}$ with non-zero rational numbers s, t. Its conjugate $\alpha' = s - t\sqrt{7}$ must be smaller than 1 in absolute value. Suppose that q is the leading coefficient of the minimal polynomial of α. Write this polynomial as $G(X) = qX^2 - 2sqX + q(s^2 - 7t^2)$. The equality $4(3 + \sqrt{7}) = q(s + t\sqrt{7})$ implies that $s = 12/q$ and $t = 4/q$. Also, from $12-4\sqrt{7} < q < 12+4\sqrt{7}$ we obtain that $2 \leqslant q \leqslant 22$. But $G(X) = qX^2-24X+32/q$ is irreducible in $\mathbb{Z}[X]$ only if $q = 1$ or $q = 32$, a contradiction.

In order to get an example in a cyclic cubic field generated by θ, where $\theta^3-3\theta-1 = 0$, we consider $\gamma = 2+\theta$. Then γ is a root of $X^3 - 6X^2 + 9X - 3 = 0$. Setting $u = 3$ and $t = 1$ we get

$$P(X) = 27X^6 + 27X^5 + 63X^4 + 37X^3 + 33X^2 + 9X + 3$$

which is irreducible in $\mathbb{Z}[X]$ and has Mahler's measure 9γ. A potential $\alpha = 9\gamma/q$ satisfying $M(\alpha) = 9\gamma$ has minimal polynomial $qX^3 - 54X^2 + 3^6X/q - 3^7/q^2 \in \mathbb{Z}[X]$. This polynomial is irreducible only if $q = 1$. However, on the other hand, q must belong to the interval $9\max\{|\gamma'|, |\gamma''|\} < q < 9\gamma$ giving the bounds $15 \leqslant q \leqslant 34$, a contradiction. Hence, there is no $\alpha \in \mathbb{Q}(\theta) = \mathbb{Q}(\gamma)$ for which $9\gamma = M(\alpha)$, although $9\gamma \in \mathscr{M}$.

Finally, in connection with our question from Section 1, we remark that for each positive integer m the number $\frac{1}{2}m^2(1 + \sqrt{5})$ belongs to $\mathscr{M}$. Indeed, setting in our construction $u = 1$ and $t = m$, we see that $M(mX^2 + X + m\beta) = m\beta$ and $M(mX^2 + X + m\beta') = m$, where $\beta = \frac{1}{2}(1 + \sqrt{5})$ and $\beta' = \frac{1}{2}(1 - \sqrt{5})$. Hence,

$$M(m^2X^4 + 2mX^3 + (m^2 + 1)X^2 + mX - m^2) = m^2\beta.$$

It is easy to see that the polynomial $m^2X^4 + 2mX^3 + (m^2 + 1)X^2 + mX - m^2$ is irreducible for every positive integer m since it has coprime coefficients and the degree of $\sqrt{1 - 4m^2\beta}$ over $\mathbb{Q}$ is equal to 4. (If it were equal to 2, then its conjugate would have be $\eta\sqrt{1 - 4m^2\beta'}$ with $\eta = 1$ or $\eta = -1$. But their product $\eta\sqrt{1 - 4m^2 - 16m^4}$ is irrational for every $m \geqslant 1$, which is impossible.)

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