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NONINVERTIBILITY PRESERVERS ON BANACH ALGEBRAS

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Abstract. It is proved that a linear surjection $\Phi\colon \mathcal{A}\to\mathcal{B}$, which preserves noninvertibility between semisimple, unital, complex Banach algebras, is automatically injective.

Keywords: linear preserver, noninvertible element, semisimple Banach algebra, socle

MSC 2000: 46H05, 46H10, 47B48

In a recent result, Brešar, Fošner, and Šemrl [3] extended Sourour's result [4] on the form of linear bijection, which preserve invertibility, from $\mathcal{B}(X)$ to arbitrary semisimple Banach algebras with 'large socle' (see also Aupetit and Mouton [2]). The present note was motivated by Sourour's question in [4]: Is a linear, unital surjection $\Phi\colon \mathcal{B}(X)\to\mathcal{B}(Y)$, which preserves invertibility, necessarily injective? We show below, with help of [3], that the answer is affirmative when 'invertibility' is replaced by 'noninvertibility'.

Before stating the result, we collect some terminology: If a is an element of a Banach algebra $\mathcal{A}$, we let $\operatorname{Sp}(a)$ be its *spectrum* and $\operatorname{soc}\mathcal{A}$ the *socle* of $\mathcal{A}$ (see [1]). Recall that an ideal I of $\mathcal{A}$ is called *essential* if it has a nonzero intersection with every nonzero ideal of $\mathcal{A}$; in semisimple Banach algebras this is equivalent to $a\cdot I = 0 \Rightarrow a = 0$ for each $a\in\mathcal{A}$. As an example, if $\mathcal{A}=\mathcal{B}(X)$ then $\operatorname{soc}\mathcal{A}$ equals the ideal of finite-rank operators, and *is essential.* Finally, a linear mapping Φ *preserves noninvertibility* (*in one direction*) if $\Phi(a)$ is not invertible whenever a is not invertible.

We will prove the following

Theorem 1. *Let $\Phi\colon \mathcal{A}\to\mathcal{B}$ be a linear surjection that preserves noninvertibility between semisimple, unital, complex Banach algebras $\mathcal{A}$ and $\mathcal{B}$ (in one direction*

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only). Then it is bounded and bijective. Moreover, there exists an invertible $a \in \mathcal{A}$ such that $\Psi(x) := \Phi(ax)$ satisfies

$$\Psi(\Psi^{-1}(y^2) - \Psi^{-1}(y)^2) \cdot \operatorname{soc} \mathcal{B} = 0 \quad \forall\, y \in \mathcal{B}. \tag{1}$$

In particular, if $\operatorname{soc} \mathcal{B}$ is an essential ideal of $\mathcal{B}$ then Ψ^{-1}, hence also Ψ, is a Jordan isomorphism. In this case, $\Phi(x) = \Psi(a^{-1}x) = b \cdot \Psi'(x)$, where $\Psi, \Psi' \colon \mathcal{A} \to \mathcal{B}$ are Jordan isomorphisms, and $b \in \mathcal{B}$ is invertible.

Proof. We claim that Φ is injective:

Indeed, suppose $\Phi(n) = 0$ for some nonzero $n \in \mathcal{A}$. As $\mathcal{A}$ is semisimple, we can then find some $c \in \mathcal{A}$ such that cn is not a quasinilpotent. By surjectivity, $\Phi(a) = \mathbf{1}$ for some a, which is necessarily invertible. Now, as $\operatorname{Sp}[((1-\xi)c + \xi a^{-1})n] \neq \{0\}$ at $\xi = 0$, the Scarcity Lemma (see [1, Theorem 3.4.25, and Corollary 3.4.18]) on the analytic multifunction $\xi \mapsto \operatorname{Sp}[((1-\xi)c + \xi a^{-1})n]$ implies that it differs from $\{0\}$ for any ξ off some subset $\Omega \subset \mathbb{C}$ with zero capacity. Observe that such Ω can contain no interval [1, Corollary A.1.27], and that a belongs to the open set of invertibles, while the mapping $x \mapsto x^{-1}$ is continuous at $x = a^{-1}$. Consequently, we may replace, if necessary, c by some $(1-\xi)c + \xi a^{-1}$ to ensure that, in addition to $\operatorname{Sp}(cn) \neq \{0\}$, the element c is invertible, and that the line-interval $[c^{-1}, a]$ contains solely invertible elements.

Let $b := c^{-1} - a$, and let $\mathcal{D} := \{\mu \in \mathbb{C};\ (a + \mu b) \text{ is invertible}\}$ be an open subset, which contains $[0,1]$. If $\mu \in \mathcal{D}$ is sufficiently small, the right-hand side of

$$\Phi(a + \mu b + \lambda n) = \Phi(a) + \mu\Phi(b) + 0 = \mathbf{1} + \mu\Phi(b),$$

is invertible for any λ. However,

$$a + \mu b + \lambda n = (a + \mu b) \cdot (\mathbf{1} + \lambda(a + \mu b)^{-1} n) \tag{2}$$

and the analytic function $\mu \mapsto (a + \mu b)^{-1} n$ has at least one nonzero spectral point at $\mu := 1$. By the Scarcity Lemma we may find arbitrarily small $\mu \in [0,1]$, such that $(a + \mu b)^{-1} n$ is not a quasinilpotent. Consequently, for any of these small μ, the right-hand side of (2) is noninvertible for at least some λ, contradicting the fact that it is mapped into *invertible* $\Phi(a + \mu b + \lambda n) = \mathbf{1} + \mu\Phi(b)$. Thus, Φ is injective, hence also bijective.

Since a is invertible, the mapping $\Psi(x) := \Phi(ax)$ is also bijective. Its inverse is unital and preserves invertibility between semisimple Banach algebras. Obviously then, $\operatorname{Sp}(\Psi^{-1}(y)) \subseteq \operatorname{Sp}(y)$, so Ψ^{-1} is bounded by [1, Theorem 5.5.2]. The same holds for $\Phi \colon x \mapsto \Psi(a^{-1}x)$ by the Open Mapping Theorem. Eq. (1) now follows from [3, Main Theorem], which proves the first part.

Finally, if $\operatorname{soc}\mathcal{B}$ is essential then, plainly, Ψ^{-1} and Ψ are Jordan. By [4, Proposition 1.3] such Ψ preserves invertibility in both directions. Hence, $b := \Phi(\mathbf{1}) = \Psi(a^{-1})$ is invertible, and the mapping $\Psi'(x) := b^{-1}\Phi(x)$ is a unital bijection, whose inverse preserves invertibility. As before we derive (1) for Ψ' in place of Ψ, and then conclude that Ψ' is Jordan. $\square$

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