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CLOSED IDEALS IN TOPOLOGICAL ALGEBRAS: A CHARACTERIZATION OF THE TOPOLOGICAL Φ-ALGEBRA $C_k(X)$

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Abstract. Let A be a uniformly closed and locally m-convex Φ-algebra. We obtain internal conditions on A stated in terms of its closed ideals for A to be isomorphic and homeomorphic to $C_k(X)$, the Φ-algebra of all the real continuous functions on a normal topological space X endowed with the compact convergence topology.

Keywords: locally m-convex algebra, Φ-algebra, compact convergence topology

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0. Introduction

For X a Hausdorff completely regular space, $C(X)$ denotes the set of all real-valued continuous functions on X. We consider on $C(X)$ the usual pointwise operations and order. The problem of characterizing the space $C(X)$ depends on which structures of this space we are interested in. For instance, Gelfand [4] characterizes $C(X)$ as a Banach algebra for X a compact space; Henriksen-Johnson [8] as a Φ-algebra for X a Lindelöf space (an improvement of this characterization was given by Plank [14]); and Anderson [1] as an l-group and l-ring for the general case (see the comments made by Hager [6] and Henriksen [7]).

In this paper we view $C(X)$ as both a Φ-algebra (with the point-wise order relationship) and a topological algebra (with the topology of compact convergence). We posed ourselves the following problem: Given a topological Φ-algebra A, when is A isomorphic and homeomorphic to $C(X)$ for some topological space X? We have already obtained some partial answers to this question. Namely, in [15] we solve it for X a hemicompact k-space, and in [11] for X a realcompact k_r-space. Apart from these studies, the only other different algebraic-topological characterization of $C(X)$

that we know of is that of Gelfand. One finds that the algebras that these studies work with are complete and are endowed with the "order topology". In this article, we will characterize the topological Φ-algebras that are isomorphic and homeomorphic to some $C(X)$ for X normal. It is clear that such an algebra might not be complete ($C(X)$ is complete if and only if X is a k_r-space [20]), and that its topology is not necessarily the order topology (the topology of $C(X)$ coincides with its order topology if and only if X is realcompact [3]).

The article is organized into four sections. The first sets out the terminology and basic notions on topological algebras that we will need. In particular, given a topological algebra A, here one defines its topological spectrum X_A and considers the spectral representation $A \to C(X_A)$. Our aim is to find internal conditions on A that allow us to identify A with $C(X_A)$, i.e., we look for conditions on A such that the morphism $A \to C(X_A)$ is injective, continuous, etc. These will be stated in terms that involve the closed sets of X_A. The less the explanation of a condition on A refers to X_A or to the morphism $A \to C(X_A)$, the more internal it is. With this in mind, in Section 2 we analyse the relationship between the closed sets of X_A and the closed ideals of A. Thus, we will be able to express properties of the topological spectrum of A or of its spectral representation in terms of the closed ideals of A. For instance, we express in this way the condition that A separates any two disjoint closed subsets of X_A, or the continuity of the morphism $A \to C(X_A)$.

In the first two sections, we only consider the topological algebra structure, without concerning ourselves about a possible order structure. Section 3 gives the definition of a uniformly closed Φ-algebra and sets out its properties that will be used later. In Section 4 we prove our main result (Theorem 4.3), which characterizes $C(X)$, for X a normal space, as a uniformly closed and locally m-convex Φ-algebra. All the hypotheses of this theorem are expressed in terms of closed ideals, and an essential part of its proof is that the spectral representation of a locally m-convex algebra is an injective morphism of lattices when this algebra is also a uniformly closed Φ-algebra.

1. Locally m-convex algebras

1.1. We will take as known the basic notions of topological vector space theory: convex, balanced, absolutely convex, absorbent, bounded, seminorm, weak topology, and so forth (see [18] for example). We never require a locally convex space to be Hausdorff, and will only consider vector spaces over the field $\mathbb{R}$ of the real numbers.

In the following, every ring will be assumed to be commutative and to possess an identity, and every morphism of rings will carry the identity into the identity. We shall denote by $\mathbb{R}$-*algebra* (henceforth simply *algebra*) every ring A endowed with a morphism of rings $\mathbb{R} \to A$ (the *structural morphism* of the algebra) which must be injective and allow $\mathbb{R}$ to be identified with a subring of A; in particular 1 will denote indistinctly the identity of $\mathbb{R}$ and the identity of A. Given algebras A and B, a map $A \to B$ is a *morphism of algebras* if it is a morphism of rings that leaves $\mathbb{R}$ invariant.

A *topological algebra* is an algebra A endowed with a (not necessarily Hausdorff) topology for which A is a topological vector space, the product of A is continuous, and the map $a \to a^{-1}$ (defined over the invertible elements) is continuous. An important class of topological algebras are the *locally m-convex algebras*, i.e., those in which there exists a basis of neighbourhoods of 0 formed by absolutely m-convex sets (a subset U of an algebra is *m-convex* if it is convex and $UU \subseteq U$). Every locally m-convex algebra is a locally convex space. If q is a seminorm on an algebra A and $U = \{a \in A\colon\ q(a) \leqslant 1\}$ is its closed unit ball, it is easy to see that $UU \subseteq U$ if and only if q is an m-seminorm (q is an *m-seminorm* if $q(ab) \leqslant q(a)q(b)$ for all $a, b \in A$); thus a topological algebra is a locally m-convex algebra when its topology may be defined by a family of m-seminorms.

1.2. Let A be a topological algebra. We shall call the set of morphisms of algebras of A in $\mathbb{R}$ that are continuous the *topological spectrum* of A, and shall denote it by $\operatorname{Spec}_t A$. Each element $a \in A$ defines on $\operatorname{Spec}_t A$ the function $a\colon \operatorname{Spec}_t A \to \mathbb{R}$, $x \mapsto a(x) := x(a)$. The initial topology that these functions define on $\operatorname{Spec}_t A$ is known as the *Gelfand topology*. Except when otherwise specified, we shall assume that $\operatorname{Spec}_t A$ is endowed with this topology. Thus, it is clear that $\operatorname{Spec}_t A$ is a completely regular Hausdorff topological space (it may be that $\operatorname{Spec}_t A = \emptyset$).

Let us assume that $\operatorname{Spec}_t A \neq \emptyset$ and let $C(\operatorname{Spec}_t A)$ be the algebra of all the real-valued continuous functions on $\operatorname{Spec}_t A$. There is a natural morphism of algebras $T\colon\ A \to C(\operatorname{Spec}_t A)$, known as the *spectral representation* of A. A is said to be *semisimple* when its spectral representation is injective.

A maximal ideal M of A is *real* if the residue class field A/M is $\mathbb{R}$. If $x\colon\ A \to \mathbb{R}$ is a morphism of algebras, then its kernel $\operatorname{Ker} x$ is a real maximal ideal of A, and x is continuous if and only if $\operatorname{Ker} x$ is closed. Hence there is a one-to-one correspondence between the points of $\operatorname{Spec}_t A$ and the set of all the closed real maximal ideals of A. The *radical* of A, denoted by $\operatorname{rad} A$, is defined as the intersection of all its closed real maximal ideals. Clearly the kernel of the spectral representation is $\operatorname{rad} A$, so that A is semisimple if and only if $\operatorname{rad} A = 0$.

1.3. Let us now consider the topological algebra that interests us most. Let X be a Hausdorff topological space. For each compact subset K of X, we have the m-seminorm q_K on $C(X)$ defined by the equality $q_K(f) = \max\{|f(x)|\colon\ x \in K\}$ ($f \in C(X)$). The topology that the family $\{q_K\colon\ K \text{ compact subset of } X\}$ defines in $C(X)$ is known as the *topology of uniform convergence on compact sets* (in brief, *compact convergence topology*). We shall denote this topology by τ_k, and the locally m-convex algebra $(C(X), \tau_k)$ we shall denote by $C_k(X)$.

1.4. An algebra A is said to be *strictly real* if $1 + a^2$ is invertible for all $a \in A$. The usual algebras of functions (continuous, continuous and bounded, differentiable) are strictly real. The following lemma is proved in [13] for complex algebras; a proof of the real case may be found in [17, Chapter II, Example 1.6 and Theorem 3.10].

Lemma 1.5. *Let A be a locally m-convex, Hausdorff and strictly real algebra. Every non-dense ideal of A is contained in some closed real maximal ideal. Hence, every closed maximal ideal of A is real.*

2. Closed ideals in a topological algebra

Throughout this paper, X will be a completely regular and Hausdorff topological space.

2.1. Let A be a topological algebra. For every subset S of A we have the closed set $(S)_0 := \{x \in \operatorname{Spec}_t A \colon a(x) = 0 \text{ for every } a \in S\}$ of $\operatorname{Spec}_t A$, and for every subset Y of $\operatorname{Spec}_t A$ we have in A the closed ideal $I_Y := \{a \in A \colon a(Y) = 0\}$. Writing $\mathscr{I} = \{\text{closed ideals of } A\}$ and $\mathscr{C} = \{\text{closed sets of } \operatorname{Spec}_t A\}$, then, given $I \in \mathscr{I}$ and $F \in \mathscr{C}$, we shall say that $(I)_0$ is the *zero set* of the ideal I and that I_F is the *associated ideal* corresponding to the closed set F. We have the maps

$$\begin{array}{ll} \mathscr{I} \xrightarrow{h} \mathscr{C}, & \mathscr{C} \xrightarrow{k} \mathscr{I}, \\ I \longmapsto (I)_0, & F \longmapsto I_F, \end{array}$$

where $(A)_0 = \emptyset$ and $I_\emptyset = A$. It is easy to see that the following are satisfied:

(i) $I \subseteq I_{(I)_0}$ and $F \subseteq (I_F)_0$ for any $I \in \mathscr{I}$ and $F \in \mathscr{C}$;

(ii) if $J_1, J_2 \in \mathscr{I}$ such that $J_1 \subseteq J_2$, then $(J_2)_0 \subseteq (J_1)_0$; consequently, it follows from (i) that $I_F = I_{(I_F)_0}$ for every closed set F of $\operatorname{Spec}_t A$;

(iii) if $F_1, F_2 \in \mathscr{C}$ such that $F_1 \subseteq F_2$, then $I_{F_2} \subseteq I_{F_1}$; consequently, it follows from (i) that $(J)_0 = (I_{(J)_0})_0$ for every closed ideal J of A.

2.2. Let A be an algebra and $\operatorname{Spec}_m A = \{\text{maximal ideals of } A\}$ be the *maximal spectrum* of A. If for every ideal I of A we write $[I]_0 := \{\text{maximal ideals of } A \text{ that contain } I\}$, then the sets of the family $\{[I]_0 \colon I \text{ ideal of } A\}$ are the closed sets of a topology on $\operatorname{Spec}_m A$, known as the *Zariski topology*. Under this topology $\operatorname{Spec}_m A$

is a compact topological space (not necessarily Hausdorff). One basis of closed sets for this topology is the collection $\{[a]_0\colon a \in A\}$, where $[a]_0 := [(a)]_0$ and (a) is the ideal of A generated by a.

If A is also a topological algebra, then on $\operatorname{Spec}_t A$ we can also consider the Zariski topology (that induced by the topology of $\operatorname{Spec}_m A$). It is clear that if I is an ideal of A then $(I)_0 = [I]_0 \cap \operatorname{Spec}_t A$; furthermore, if $\bar{I}$ is the closure of I in A, then $\bar{I}$ is a closed ideal of A such that $(I)_0 = (\bar{I})_0$. In sum, the closed sets of $\operatorname{Spec}_t A$ for its Zariski topology are the zero sets of the closed ideals of A, and a basis of closed sets for the said topology is the collection $\{(a)_0\colon a \in A\}$.

Definition 2.3. We shall say that a topological algebra A is *regular* if its elements separate points and closed sets of $\operatorname{Spec}_t A$ in the following sense: if $x \in \operatorname{Spec}_t A$ and F is a non-empty closed set of $\operatorname{Spec}_t A$ such that $x \notin F$, then there exists $a \in A$ satisfying $a(F) = 0$ and $a(x) = 1$.

It follows from the definition that A is regular if and only if $\{(a)_0\colon a \in A\}$ is a basis of closed sets in $\operatorname{Spec}_t A$; i.e., A regular is equivalent to the coincidence of the Gelfand and the Zariski topologies in $\operatorname{Spec}_t A$.

Lemma 2.4. *A topological algebra A is regular if and only if it satisfies one of the following three equivalent statements:*

(i) *k is injective;*

(ii) *h is epijective;*

(iii) *hk is the identity map of $\mathscr{C}$.*

Proof. On the one hand, the condition (iii) means that A separates points of closed sets of $\operatorname{Spec}_t A$: given a closed set F of $\operatorname{Spec}_t A$, as $F \subseteq (I_F)_0$ always holds, one will have $F = (I_F)_0$ if and only if for each $x \in \operatorname{Spec}_t A$, $x \notin F$, there exists $a \in I_F$ such that $a(x) = 1$. On the other, it is clear that condition (ii) means that the Gelfand and the Zariski topologies coincide in $\operatorname{Spec}_t A$. Therefore, (ii) and (iii) are equivalent.

(i) $\Rightarrow$ (iii) Given $F \in \mathscr{C}$ one has $I_F = I_{(I_F)_0}$, so that if k is injective then it must be that $F = (I_F)_0$. Therefore hk is the identity.

(iii) $\Rightarrow$ (i) This is immediate. □

Definition 2.5. We shall say that a topological algebra A has the property I_1 if every closed ideal of A is an intersection of closed real maximal ideals.

Note 2.6. Let A be a topological algebra. If A is semisimple (which implies, by definition, that $\operatorname{Spec}_t A \neq \emptyset$), then its topology is Hausdorff. If A is Hausdorff and has the property I_1, then the ideal 0 must be an intersection of closed real maximal ideals. Therefore $\operatorname{Spec}_t A \neq \emptyset$ and A is semisimple.

Lemma 2.7. *A topological algebra A has the property* I_1 *if and only if it satisfies one of the following three equivalent statements:*

(i) *k is epijective;*

(ii) *h is injective;*

(iii) *kh is the identity map of $\mathscr{I}$.*

Proof. Clearly (i) is just the definition that A should have the property I_1.

(i) $\Rightarrow$ (iii) Given $J \in \mathscr{I}$, by hypothesis, there exists $C \in \mathscr{C}$ such that $J = I_C$, so that $I_{(J)_0} = I_{(I_C)_0} = I_C = J$. Therefore kh is the identity.

(iii) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (i) Given $J \in \mathscr{I}$, one has $(J)_0 = (I_{(J)_0})_0$, so that if h is injective it must be that $J = I_{(J)_0}$, i.e., J is an intersection of closed real maximal ideals. □

Corollary 2.8. *A topological algebra is regular and has the property* I_1, *if and only if the maps h and k establish a bijection between the set of closed ideals of A and the set of closed subsets of* $\operatorname{Spec}_t A$.

Example 2.9. Each $x \in X$ defines the continuous morphism of algebras $\delta_x\colon C_k(X) \to \mathbb{R}$, $\delta_x(f) := f(x)$, and so we have the natural map $i\colon X \to \operatorname{Spec}_t C_k(X)$, $i(x) := \delta_x$. On the one hand, $i\colon X \to i(X)$ is a homeomorphism because X is completely regular. On the other, if for each closed set C in X we denote $I_C = \{f \in C(X)\colon f(C) = 0\}$, then the closed ideals in $C_k(X)$ are in one-to-one correspondence (via $C \mapsto I_C$) with the closed subsets of X (see [12]), and consequently the closed maximal ideals in $C_k(X)$ are in one-to-one correspondence with the points of X. Therefore, $X = \operatorname{Spec}_t C_k(X)$, the spectral representation of $C_k(X)$ is an algebraic isomorphism (in particular $C_k(X)$ is semisimple), and $C_k(X)$ is regular and has the property I_1.

We are interested in the characterization of $C_k(X)$ with X normal, so that we shall now show how one can express that X is normal in terms of the closed ideals of $C_k(X)$.

Definitions 2.10. We shall say that a topological algebra A has the property I_3 if there do not exist two closed ideals in A whose sum is dense and proper. We shall say that A is *normal* if its elements separate disjoint closed sets of $\operatorname{Spec}_t A$ in the following sense: if F, G are disjoint non-empty closed sets of $\operatorname{Spec}_t A$, then there exists $a \in A$ such that $a(F) = 0$ and $a(G) = 1$.

Clearly, if A is normal then it is regular. According to Urysohn's Lemma, X is normal if and only if $C_k(X)$ is normal.

Lemma 2.11. *$C_k(X)$ is normal if and only if it has the property* I_3.

P r o o f. Assume that $C_k(X)$ is normal. Let I, J be closed ideals of $C_k(X)$ such that $I+J$ is dense. Let us show that then $I+J=C_k(X)$. We have that $I+J$ is not contained in any closed real maximal ideal, i.e., $(I)_0\cap(J)_0=(I+J)_0=\emptyset$. If F, G are closed sets of X such that $I=I_F$ and $J=I_G$, then $F\cap G=(I)_0\cap(J)_0=\emptyset$ and there therefore exists $f\in C(X)$ such that $f(F)=0$ and $f(G)=1$; if one defines $g=1-f$, then $f\in I$, $g\in J$ and $f+g=1$, so that $I+J=C_k(X)$.

Let us now assume that $C_k(X)$ has the property I_3 and let F, G be disjoint closed sets of X. One has $(I_F+I_G)_0=(I_F)_0\cap(I_G)_0=F\cap G=\emptyset$, i.e., I_F+I_G is not contained in any closed real maximal ideal. Since $C_k(X)$ has the property I_1, every non-dense ideal of $C_k(X)$ is contained in some closed real maximal ideal, so that it follows that I_F+I_G is dense; therefore $I_F+I_G=C_k(X)$. If $f\in I_F$, $g\in I_G$ such that $f+g=1$, then $f(F)=0$ and $f(G)=1$. □

For a topological algebra A, the above equivalence is, in general, not true. The reason for presenting the proof of the above lemma in detail is to give meaning to the following definition, and to make the proof of the subsequent lemma immediate.

Definition 2.12. We shall say that a topological algebra A has the property I_2 if each non-dense ideal is contained in some closed real maximal ideal.

It is obvious that if A has the property I_1 then A has the property I_2, and that if A has the property I_2 then every closed maximal ideal of A is real. According to Lemma 1.5, every Hausdorff locally m-convex and strictly real algebra (and in particular $C_k(X)$) has the property I_2.

Lemma 2.13. *Let A be a topological algebra.*

(i) *If A is normal and has the property* I_1, *then A has the property* I_3.

(ii) *If A is regular and has the properties* I_2 *and* I_3, *then A is normal.*

Consequently, if A is regular and has the property I_1, *then A is normal if and only if it has the property* I_3.

2.14. The question then arises as to how, given a topological algebra A, one can express in terms of its ideals the following property (which, when $A=C_k(X)$, is well known): An element $a\neq 0$ of A is invertible if and only if $a(x)\neq 0$ for all $x\in\operatorname{Spec}_t A$. Since the statement "$a(x)\neq 0$ for all $x\in\operatorname{Spec}_t A$" means "$a$ belongs to no closed real maximal ideal of A", the proof of the following lemma is immediate (cf. [13, Chapter I]):

Lemma 2.15. *If A is a topological algebra with the property* I_2, *then the following are equivalent:*

(i) *an element $a \neq 0$ of A is invertible if and only if $a(x) \neq 0$ for all $x \in \operatorname{Spec}_t A$;*

(ii) *there exist no principal ideals in A that are proper and dense.*

2.16. Let us now consider the spectral representation $A \to C(\operatorname{Spec}_t A)$ of a topological algebra A, and investigate when it is continuous by assuming that $C(\operatorname{Spec}_t A)$ is endowed with the compact convergence topology. For each compact subset K of $\operatorname{Spec}_t A$ consider the set $S_K = \{a \in A \colon 0 \notin a(K)\}$. It is easy to see that the functions of $C(\operatorname{Spec}_t A)$ that do not vanish at any point of a given compact set form an open set of $C_k(\operatorname{Spec}_t A)$; therefore, if the spectral representation of A is continuous, then S_k is an open set of A for every compact subset K of $\operatorname{Spec}_t A$. One has that:

Proposition 2.17. *For a topological algebra A whose topology is locally convex, the following are equivalent:*

(i) *the spectral representation of A is continuous;*

(ii) *S_K is open for every compact subset K of* $\operatorname{Spec}_t A$.

P r o o f. According to the argument of 2.16, one only has to prove that (ii) $\Rightarrow$ (i). Hence let us assume that (ii) is satisfied. Given a compact subset K of $\operatorname{Spec}_t A$ and given $\varepsilon > 0$, we have to prove that $U = \{a \in A \colon |a(x)| < \varepsilon \text{ for all } x \in K\}$ is a neighbourhood of 0. Since S_K is an open neighbourhood of $\varepsilon \in A$, we have that $V = S_K - \varepsilon$ is a neighbourhood of 0; then $W = V \cap (-V)$ is another neighbourhood of 0 such that $\pm\varepsilon \notin a(K)$ for all $a \in W$. If W' is a convex neighbourhood of 0 such that $W' \subseteq W$, then $W' \subseteq U$, since every $x \in K$ is a continuous map of A in $\mathbb{R}$ that maps the connected set W' in an interval of $\mathbb{R}$ that contains 0 and does not contain $\pm\varepsilon$. $\square$

As our intention is to state an analogous result to 2.17 without making any reference to the compact subsets of the topological spectrum, we need to characterize the said compact subsets in some way.

2.18. We have to make some remarks with respect to the quotient of a topological algebra by an ideal. If A is a topological algebra and I is an ideal of A, we shall endow the quotient A/I with the finest topology for which A/I is a topological algebra and the quotient morphism $\pi \colon A \to A/I$ is continuous. If A is a locally m-convex algebra, this topology coincides with the quotient topology (see [17, I.2.5]). In that case, from the known correspondence between the ideals of A that contain I and the ideals of A/I, and from the properties of the quotient topology of A/I, it follows that there exists a bijection between the closed ideals of A that contain I and the closed ideals of A/I. If A is not locally m-convex this bijection may not exist. Nonetheless, it is easy to see that, in any case, there exists a bijection between the closed real maximal ideals of A that contain I and the closed real maximal ideals of A/I.

The properties set out in the following lemma follow straightforwardly from the definitions.

Lemma 2.19. *Let I be an ideal of a topological algebra A.*

(i) $\operatorname{Spec}_t(A/I) = (I)_0$ *(topological equality).*

(ii) *If A is regular then A/I is regular. Consequently, if A is regular and F is a closed set of $\operatorname{Spec}_t A$, then $\operatorname{Spec}_t(A/I_F) = F$ and I_F is the greatest of the ideals I of A such that $(I)_0 = F$, i.e., A/I_F is semisimple.*

In the following result one characterizes the compact subsets of X in terms of their associated closed ideals of $C_k(X)$.

Lemma 2.20. *For each closed set F of X, the following are equivalent:*

(i) *F is compact;*

(ii) *every maximal ideal that contains I_F is closed (and therefore real).*

Proof. (i) $\Rightarrow$ (ii) Let F be a compact subset in X. A known generalization of Tietze's Extension Theorem states that $C(X)/I_F = C(F)$, the quotient morphism $C(X) \to C(X)/I_F$ being just the restriction morphism $C(X) \to C(F)$. Moreover, it is easy to see that the said equality is also topological, i.e., $C_k(X)/I_F = C_k(F)$. The implication then follows since in $C_k(F)$ every maximal ideal is closed (see [5, 40.4]).

(ii) $\Rightarrow$ (i) Since $C_k(X)$ is regular we have that $C_k(X)/I_F$ is regular and therefore the topology of $F = \operatorname{Spec}_t(C_k(X)/I_F)$ coincides with the Zariski topology induced by $\operatorname{Spec}_m(C_k(X)/I_F)$. Since in $C_k(X)$ the concepts "closed maximal ideal" and "closed real maximal ideal" are equivalent, one deduces from the hypothesis that $\operatorname{Spec}_t(C_k(X)/I_F) = \operatorname{Spec}_m(C_k(X)/I_F)$. Therefore F is compact (see 2.2). □

2.21. The question that now arises is whether the above equivalence is true for arbitrary topological algebras, i.e., whether for a closed set F of the topological spectrum of a topological algebra A the conditions "F is compact" and "every maximal ideal of A that contains I_F is real and closed" are equivalent. It is easy to see by analysing the proof of the last lemma that if A is regular then (ii) $\Rightarrow$ (i) is true. Nevertheless, we will need to add some other hypothesis if we want (i) $\Rightarrow$ (ii).

Definition 2.22. An algebra A is said to be a *Gelfand algebra* if every prime ideal of A is contained in a unique maximal ideal.

It is known that $C(X)$ is a Gelfand algebra (see [5, Theorem 2.11]). It is also immediate to check that every quotient of a Gelfand algebra is a Gelfand algebra, and that if A is a Gelfand algebra then $\operatorname{Spec}_m A$, endowed with its Zariski topology, is Hausdorff. Furthermore, if A is a reduced algebra (i.e., if the intersection of all maximal ideals of A is null), then A is a Gelfand algebra if and only if $\operatorname{Spec}_m A$ is Hausdorff (see [2, § 6.1]). The intersection of all maximal ideals of an algebra A is known as the *Jacobson radical* of A; we shall denote it by $\operatorname{rad}_J A$.

Theorem 2.23. *Let A be a regular topological and Gelfand algebra. For each closed set F of $\operatorname{Spec}_t A$, the following are equivalent:*

(i) *F is compact;*

(ii) *every maximal ideal that contains I_F is real and closed.*

Proof. To prove this theorem, we shall use the following easily checked fact: if B is an algebra and Z is a non-empty subset of $\operatorname{Spec}_m B$, then Z is dense in $\operatorname{Spec}_m B$ if and only if $\operatorname{rad}_J B = \bigcap_{x \in Z} x$.

(i) $\Rightarrow$ (ii) Let us first see that $\operatorname{Spec}_t(A/I_F)$ is a dense subset of $\operatorname{Spec}_m(A/I_F)$. By the remark at the beginning of the proof, the above statement will be true if it is proved that $\operatorname{rad}_J(A/I_F) = \operatorname{rad}(A/I_F)$. It is obvious that $\operatorname{rad}_J(A/I_F) \subseteq \operatorname{rad}(A/I_F)$, and the equality is the case because A/I_F is semisimple and therefore $\operatorname{rad}(A/I_F) = 0$. Now, on the one hand, $\operatorname{Spec}_m(A/I_F)$ is Hausdorff because A/I_F is a Gelfand algebra, and on the other, the topology of $F = \operatorname{Spec}_t(A/I_F)$ coincides with that induced by $\operatorname{Spec}_m(A/I_F)$ because A/I_F is regular. Hence it follows that if F is compact, then $\operatorname{Spec}_t(A/I_F) = \operatorname{Spec}_m(A/I_F)$.

(ii) $\Rightarrow$ (i) As in Lemma 2.20. □

Definition 2.24. We shall say that an ideal I of a topological algebra A is a *C-ideal*, if I is closed and every maximal ideal of A that contains I is real and closed. Every closed real maximal ideal of A is trivially a C-ideal.

2.25. Let A be a regular topological and Gelfand algebra.

If every closed ideal of A is the intersection of closed real maximal ideals (property I_1), then there is a one-to-one correspondence between the closed ideals of A and the closed sets of $\operatorname{Spec}_t A$. As a consequence, Theorem 2.23 establishes a bijection between the C-ideals of A and the compact subsets of $\operatorname{Spec}_t A$.

In general, there will be more closed ideals than those of the form I_F with F a closed set of $\operatorname{Spec}_t A$, and there may therefore exist C-ideals that are not in the family $\{I_F\colon F \text{ compact subset of } \operatorname{Spec}_t A\}$. Whatever the case, if I is a C-ideal and $F = (I)_0$, then one has the inclusion $I \subseteq I_F$ and hence I_F is also a C-ideal. Hence, if I is a C-ideal, then $(I)_0$ is compact.

We are now ready to give a version of 2.17 in terms of closed ideals. For each ideal I of A, $\pi_I\colon A \to A/I$ will be the quotient morphism and $(A/I)^{-1}$ will denote the set of the invertible elements of A/I.

Theorem 2.26. *Let A be a topological regular and Gelfand algebra, whose topology is locally convex. The spectral representation of A is continuous if and only if $\pi_I^{-1}((A/I)^{-1})$ is an open set of A for every C-ideal I of A.*

Proof. If I is a C-ideal of A and $K = (I)_0$, then one has that $\pi_I^{-1}((A/I)^{-1}) = \{a \in A\colon 0 \notin a(K)\} = S_K$. Indeed, given $a \in A$, $\pi_I(a)$ is invertible in A/I if and only if $\pi_I(a)$ is not in any maximal ideal of A/I, if and only if a is not in any maximal ideal of A that contains I, if and only if a is not in any closed real maximal ideal of A that contains I, if and only if $a \in S_K$.

After what was seen in the previous paragraph, the theorem results by applying 2.17 with no more than taking into account that, if I is a C-ideal of A, then $(I)_0$ is compact (see 2.25), and that if F is a compact subset of $\operatorname{Spec}_t A$ then I_F is a C-ideal (see 2.23). □

Definition 2.27. A topological algebra A in which the set A^{-1} of its invertible elements is open is called a *Q-algebra*.

Let A be a locally m-convex algebra and I an ideal of A. In this case the topology of A/I is the quotient topology, and therefore $\pi_I((A/I)^{-1})$ is an open set of A if and only if A/I is a Q-algebra. As a consequence we have the following particular case of the above theorem:

Theorem 2.28. *Let A be a locally m-convex, regular and Gelfand algebra. The spectral representation of A is continuous if and only if A/I is a Q-algebra for every C-ideal I of A.*

3. Uniformly closed Φ-algebras

3.1. A *vector lattice* is a real vector space E endowed with an order relationship "$\leqslant$" with which it is a lattice (every non-empty finite subset has a supremum and an infimum) and is compatible with the vector structure (if $a, b \in E$ such that $a \leqslant b$, then $a + c \leqslant b + c$ for every $c \in E$, and $\lambda a \leqslant \lambda b$ for every $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$). For $C(X)$ we shall always consider its usual order with which it is a vector lattice: this is the point-wise defined natural order.

Let E be a vector lattice. The set $E_+ = \{a \in E\colon a \geqslant 0\}$ is called the *positive cone* of E. As is usual, the supremum and infimum of a finite subset $\{a_1, \dots, a_n\}$

of E will be denoted by $a_1 \vee \ldots \vee a_n$ and $a_1 \wedge \ldots \wedge a_n$, respectively. Given an element $a \in E$, its *positive part*, its *negative part*, and its *absolute value* are elements of E which are denoted by a^+, a^- and $|a|$, respectively, and are defined by the equalities $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, $|a| = a^+ \vee a^-$. A subset C of E is said to be *solid* if $a \in C$ implies $\{b \in E\colon\ |b| \leqslant |a|\} \subseteq C$. A map $T\colon\ E \to F$, where E and F are vector lattices, is a *morphism of vector lattices* if it is linear and is a morphism of lattices, i.e., if it is a linear map such that $T(a \vee b) = T(a) \vee T(b)$ and $T(a \wedge b) = T(a) \wedge T(b)$ for all $a, b \in E$. If $T\colon\ E \to F$ is a linear map, then it is clear that T is a morphism of vector lattices if and only if $T(|a|) = |T(a)|$ for all $a \in E$.

3.2. An *l-algebra* is an algebra A endowed with an order relationship "$\leqslant$" with which it is a lattice and is compatible with the algebraic structure (if $a, b \in A$ such that $a \leqslant b$, then $a + c \leqslant b + c$ for all $c \in A$, $\lambda a \leqslant \lambda b$ for all $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$, and $ac \leqslant bc$ for all $c \in A_+$). If A is an l-algebra, in particular it is a vector lattice, so that the notions given in 3.1 for vector lattices are valid in A. Let A and B be l-algebras. A map $A \to B$ is said to be a *morphism of l-algebras* if it is a morphism of algebras and a morphism of lattices. An ideal I of A is said to be an *l-ideal* if I is a solid set. A *maximal l-ideal* is a proper l-ideal that is not contained strictly in another proper l-ideal. With its usual order, $C(X)$ is an l-algebra, and each closed ideal of $C_k(X)$ is an l-ideal, since for each closed set F of X the ideal I_F is solid.

3.3. An l-algebra A is called *Archimedean* if, for $a, b \in A$, $na \leqslant b$ for all $n \in \mathbb{N}$ implies $a \leqslant 0$. An l-algebra A is called an *f-algebra* if, for $a, b, c \in A$, $a \wedge b = 0$ and $c \geqslant 0$ imply $ca \wedge b = 0$. A *Φ-algebra* is an Archimedean f-algebra. It is clear that $C(X)$ is a Φ-algebra.

3.4. Let A be an f-algebra. It is known that, for any $a \in A$, one has $a^2 = |a|^2 \geqslant 0$; in particular $1 \geqslant 0$ (in A), and therefore the order that A induces in $\mathbb{R}$ is the usual order of $\mathbb{R}$. According to the above, given $\alpha, \beta \in \mathbb{R}$, one will have $\alpha \leqslant \beta$ in $\mathbb{R}$ if and only if $\alpha \leqslant \beta$ in A, so that we will make no distinction. A sequence $(a_n)_n$ in A is said to be *Cauchy uniform* if for every real $\varepsilon > 0$ there exists a positive integer ν such that $|a_n - a_m| \leqslant \varepsilon$ for $n, m \geqslant \nu$. A sequence $(a_n)_n$ in A is said to be *uniformly convergent* to $a \in A$ if for each real $\varepsilon > 0$ there exists a positive integer ν such that $|a_n - a| \leqslant \varepsilon$ for $n \geqslant \nu$. It is easy to see that if $(a_n)_n$ is uniformly convergent to both a and b in A and A is Archimedean, then $a = b$. A subset S in A is said to be *uniformly closed* if each Cauchy uniform sequence in S is uniformly convergent in S. A subset S in A is said to be *uniformly dense* if for each element $a \in A$ there is a sequence in S that converges uniformly to a. It is easy to see that $C(X)$ is uniformly closed.

We shall say that an element $a \in A$ is *bounded*, if there exists a non-negative integer n such that $|a| < n$. We shall denote the set of all the bounded elements of A by A^*. It is clear that A^*, with the order induced by the order of A, is an f-algebra, and that the inclusion $A^* \to A$ is a morphism of l-algebras. Furthermore, it is easy to check that if A is a uniformly closed Φ-algebra, then A^* is also a uniformly closed Φ-algebra. $C(X)^*$ are the functions of $C(X)$ that are bounded in the usual sense. The l-algebra $C(X)^*$ is denoted by $C^*(X)$.

Lemma 3.5. *Let A be a uniformly closed Φ-algebra.*

(i) *If $a \in A$, $a \geqslant 1$, then a is an invertible element. As a consequence, A is a strictly real algebra.*
(ii) *A has square roots: given $a \in A_+$ there exists a unique $b \in A_+$ such that $b^2 = a$.*
(iii) *Every maximal ideal of A is an l-ideal.*
(iv) *The Jacobson radical of A is null,* $\operatorname{rad}_J A = 0$.
(v) *A is a Gelfand algebra.*

Proof. See [8] or [9] for (i) and (ii), and [14, Theorem 3.7] for (iii). According to (iii) the maximal l-ideals of A are just the maximal ideals of A, and in [10, Chapter II, Theorem 2.11] it is proved that the intersection of all maximal l-ideals of A is zero, and hence (iv) holds. Again, from (iii) one derives that $\operatorname{Spec}_m A$ is just the set of all the maximal l-ideals of A, and that it is a Hausdorff space (see [8, p. 79]). It follows that A satisfies (v) because $\operatorname{rad}_J A = 0$ (see 2.22). □

Lemma 3.6. *Let A and B be uniformly closed Φ-algebras. Every morphism of algebras $T\colon A \to B$ is a morphism of l-algebras.*

Proof. This follows from 3.5 (ii). Let $a \in A$. On the one hand, there exists $b \in A$ such that $b^2 = |a|$ and therefore $T(|a|) = T(b^2) = T(b)^2 \geqslant 0$; on the other hand we have $|T(a)|^2 = T(a)^2 = T(a^2) = T(|a|^2) = T(|a|)^2$. It then follows that $T(|a|) = |T(a)|$ and the proof is complete. □

Lemma 3.7. *Let A be a topological algebra. If A is also a uniformly closed Φ-algebra, then A is regular.*

Proof. If for each $a \in A$ we denote by $\operatorname{coz}(a)$ the complement of $(a)_0$ in $\operatorname{Spec}_t A$, we have to prove that a basis of open sets in $\operatorname{Spec}_t A$ is the collection $\{\operatorname{coz}(a)\colon a \in A\}$. By definition of the Gelfand topology, a basis of open sets in $\operatorname{Spec}_t A$ is formed by the finite intersections of sets of the form $\{x \in \operatorname{Spec}_t A\colon a(x) \in (\alpha, \beta)\}$ with $a \in A$ and $\alpha, \beta \in \mathbb{R}$. Given $x \in \operatorname{Spec}_t A$ we have

$$\begin{aligned} \alpha < a(x) < \beta &\iff (a-\alpha)(x) > 0 \quad \text{and} \quad (\beta - a)(x) > 0 \\ &\iff (a-\alpha)^+(x) \neq 0 \quad \text{and} \quad (\beta - a)^+(x) \neq 0, \end{aligned}$$

where $(a-\alpha)^+$ and $(\beta-a)^+$ are taken in $C(\operatorname{Spec}_t A)$. Since, according to 3.6, the spectral representation $A \to C(\operatorname{Spec}_t A)$ is a morphism of l-algebras, we obtain

$$\{x \in \operatorname{Spec}_t A\colon\ a(x) \in (\alpha,\beta)\} = \operatorname{coz}((a-\alpha)^+) \cap \operatorname{coz}((\beta-a)^+),$$

where now $(a-\alpha)^+$ and $(\beta-a)^+$ are taken in A. To conclude the proof, it is enough to take into account that, given $a_1,\dots,a_n \in A$, one has $\operatorname{coz}(a_1)\cap\ldots\cap\operatorname{coz}(a_n) = \operatorname{coz}(a_1\cdot\ldots\cdot a_n)$. □

4. Main result

Our main result in the present work is a characterization of $C_k(X)$ as a locally m-convex Φ-algebra for X normal. We shall need the following two lemmas:

Lemma 4.1 (Tietze [19]). *Let E be a vector subspace of $C^*(X)$ that contains the constant functions. If E S^1-separates disjoint closed sets of X (i.e., for each pair of non-empty disjoint closed sets F and G of X, there exists $h \in E$ such that $0 \leqslant h \leqslant 1$, $h(F)=0$ and $h(G)=1$), then E is uniformly dense in $C^*(X)$.*

Lemma 4.2 (Requejo [16]). *Let τ be a locally m-convex Hausdorff topology on $C(X)$. If X is normal and for each τ-closed ideal I of $C(X)$ there exists a closed subset F in X such that $I = I_F$, then τ is less fine than the topology of $C_k(X)$.*

Theorem 4.3. *Let A be a uniformly closed Φ-algebra endowed with a locally m-convex Hausdorff topology. A is l-isomorphic (isomorphic as an l-algebra) and homeomorphic with $C_k(X)$ for some normal topological space X, if and only if:*

(i) *each closed ideal of A is an intersection of closed maximal ideals;*
(ii) *in A there exist no principal ideals that are proper and dense;*
(iii) *in A there exist no two closed ideals whose sum is proper and dense;*
(iv) *for each C-ideal I of A, A/I is a Q-algebra.*

Proof. Throughout the article, it has been seen that, when X is normal, $C_k(X)$ is a uniformly closed Φ-algebra endowed with a locally m-convex Hausdorff topology, and which satisfies the conditions (i), (ii), (iii) and (iv).

Conversely, let A be a uniformly closed Φ-algebra endowed with a locally m-convex Hausdorff topology that satisfies the conditions (i), (ii), (iii) and (iv). Let us first note that each closed maximal ideal of A is real because A is strictly real (see 3.5 and 1.5); therefore, condition (i) states precisely that A has the property I_1, and as a consequence we have that $\operatorname{Spec}_t A \neq \emptyset$ and that A is semisimple (see 2.6).

Furthermore, as A is regular, condition (iii) implies that A is normal (see 3.7 and 2.13).

Since, according to 3.6, the spectral representation $A \to C(\operatorname{Spec}_t A)$ is a morphism of l-algebras, identifying A with its image we have that A is a uniformly closed l-subalgebra of $C(\operatorname{Spec}_t A)$ that separates disjoint closed sets of $\operatorname{Spec}_t A$. Then A^* S^1-separates them, since, if for $a \in A$ one has $a(F) = 0$ and $a(G) = 1$ (F and G closed sets of $\operatorname{Spec}_t A$), the same is the case for $|a| \wedge 1 \in A^*$. From 4.1 it follows that A^* is uniformly dense in $C^*(\operatorname{Spec}_t A)$, and as A^* is uniformly closed (since A is) we conclude that $A^* = C^*(\operatorname{Spec}_t A)$. Now, if $f \in C(\operatorname{Spec}_t A)$ then $f_1 = 1/(f^+ + 1)$ and $f_2 = 1/(f^- + 1)$ are functions of A^* that do not vanish at any point of $\operatorname{Spec}_t A$ and such that $f = 1/f_1 - 1/f_2$. But, by the hypothesis (ii), $1/f_1, 1/f_2 \in A$ (see 2.15), so that $f \in A$ and we conclude that $A = C(\operatorname{Spec}_t A)$, i.e., the spectral representation of A is an isomorphism of l-algebras.

Finally, let us show that the spectral representation of A is a homeomorphism. On the one hand, it is clear that $\operatorname{Spec}_t A$ is normal, so that from property (i) and 4.2 it follows that the topology of A is less fine than that of $C_k(\operatorname{Spec}_t A)$. On the other, as A is a Gelfand algebra according to 3.5, from Theorem 2.28 it follows that the property (iv) is equivalent to its spectral representation being continuous, i.e., the topology of A is finer than that of $C_k(\operatorname{Spec}_t A)$. □

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