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SIMULTANEOUS APPROXIMATION FOR SZÁSZ-MIRAKIAN QUASI-INTERPOLANTS

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Abstract. We obtain simultaneous approximation equivalence theorem for Szász-Mirakian quasi-interpolants.

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MSC 2000: 41A25, 41A36

1. Introduction

The Szász-Mirakian operator is defined by

$$S_n(f,x)=\sum_{k=0}^{\infty}s_{n,k}(x)f\Big(\frac{k}{n}\Big),\quad s_{n,k}(x)=\mathrm{e}^{-nx}\frac{(nx)^k}{k!}. \tag{1.1}$$

It is known that for $f\in C_B[0,\infty)$ (the set of continuous and bounded functions), $\varphi(x)=\sqrt{x}$ and $0<\alpha<1$ (cf. [4])

$$\|S_nf-f\|=O(n^{-\alpha})\Leftrightarrow\omega_\varphi^2(f,t)=O(t^{2\alpha}), \tag{1.2}$$

where $\omega_\varphi^2(f,t)$ is Ditzian-Totik modulus. But this result can not include the case of the classical modulus $\omega^2(f,t)$. In [3] Ditzian used the unified modulus $\omega^2_{\varphi^\lambda}(f,t)=\sup\limits_{0<h\leqslant t}\|\Delta^2_{h\varphi^\lambda}f\|$ $(0\leqslant\lambda\leqslant1)$ to bridge the gap between the classical moduli $(\lambda=0)$

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and the Ditzian-Totik moduli ($\lambda = 1$). With $\omega^2_{\varphi^\lambda}(f,t)$ we have (cf. [5])

$$(1.3)\quad |S_n(f,x) - f(x)| = O\Big(\Big(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big)^{-\alpha}\Big) \Leftrightarrow \omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha) \quad (0 < \alpha < 2).$$

In order to obtain faster convergence, quasi-interpolants $S_n^{\langle r\rangle}$ of S_n in the sense of Sablonnière are considered (cf. [1], [2], [7]). We recall their construction. Π_n denotes the space of algebraic polynomials of degree at most n. On Π_n the Szász-Mirakian operator S_n and its inverse S_n^{-1} can be expressed as linear differential operators with polynomials coefficients in the form $S_n = \sum\limits_{j=0}^{n} \beta_j^n D^j$ and $S_n^{-1} = \sum\limits_{j=0}^{n} \alpha_j^n D^j$ with $D = \mathrm{d}/\mathrm{d}x$ and $D^0 = \mathrm{id}$. The left Szász-Mirakian quasi-interpolants of r degree are defined by

$$(1.4)\qquad S_n^{\langle r\rangle}(f,x) = \sum_{j=0}^{r} \alpha_j^n D^j S_n(f,x).$$

Some basic properties can be found in [1], [2]:

(1) $S_n^{\langle 0\rangle} = S_n$, $S_n^{\langle n\rangle} = \mathrm{id}$.

(2) For $0 \leqslant r \leqslant n$, $p \in \Pi_r$, one has

$$(1.5)\qquad S_n^{\langle r\rangle} p = p.$$

(3) $\alpha_0^n(x) = 1$, $\alpha_1^n(x) = 0$,

$$(1.6)\qquad \alpha_j^n(x) = C_{j-1}^n \frac{x}{n^{j-1}} + C_{j-2}^n \frac{x^2}{n^{j-2}} + \ldots + C_{j'}^n \frac{x^{j-j'}}{n^{j'}},$$

where $j' = \left[\frac{1}{2}(j+1)\right]$ and C_j^n are constants independent of n.

(4) For $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x}$, $n \geqslant 2r-1$, $r \in \mathbb{N}$, we have

$$(1.7)\qquad \|S_n^{\langle 2r-1\rangle} f - f\|_\infty \leqslant C\omega_\varphi^{2r}\Big(f, \frac{1}{\sqrt{n}}\Big)_\infty.$$

We note that there are no inverse and equivalence results in [2]. In this paper we will consider the simultaneous approximation for $S_n^{\langle 2r-1\rangle}(f)$ and give an equivalent result with the unified modulus $\omega^{2r}_{\varphi^\lambda}(f,t)$ ($0 \leqslant \lambda \leqslant 1$).

Theorem 1.1. *Let $f^{(s)} \in C_B[0,\infty)$, $s \in \mathbb{N} \cup \{0\}$, $0 \leqslant s \leqslant 2r-1$, $\varphi(x) = \sqrt{x}$, $\delta_n(x) = \max\{\varphi(x), 1/\sqrt{n}\}$, $n \geqslant 4r$, $r \in \mathbb{N}$, $0 \leqslant \lambda \leqslant 1$. Then for $0 < \alpha < 2r-s$ the following two statements are equivalent*

$$(1.8) \qquad (1)\quad |D^s S_n^{\langle 2r-1\rangle}(f,x) - f^{(s)}(x)| = O\Big(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\alpha}\Big),$$

$$(2)\quad \omega_{\varphi^\lambda}^{2r-s}(f^{(s)},t) = O(t^\alpha).$$

Now we give the definitions of the unified modulus and K-functional:

$$(1.9) \qquad \omega_{\varphi^\lambda}^r(f,t) = \sup_{0<h\leqslant t}\ \sup_{x-\frac{r}{2}h\varphi^\lambda \in [0,\infty)} |\Delta_{h\varphi^\lambda}^r f(x)|,$$

$$(1.10) \qquad K_{\varphi^\lambda}^r(f,t^r) = \inf_{g \in w^r(\varphi,[0,\infty))} \{\|f-g\|_\infty + t^r\|\varphi^{r\lambda} g^{(r)}\|_\infty\},$$

$$(1.11) \quad \overline{K}_{\varphi^\lambda}^r(f,t^r) = \inf_{g \in w^r(\varphi,[0,\infty))} \{\|f-g\|_\infty + t^r\|\varphi^{r\lambda} g^{(r)}\|_\infty + t^{r/(1-\lambda/2)}\|g^{(r)}\|_\infty\}$$

where

$$w^r(\varphi,[0,\infty)) = \{g\colon\ g \in C[0,\infty),\ g^{(r-1)} \in \mathrm{A.C.}_{\mathrm{loc}}, \|\varphi^{r\lambda}g^{(r)}\|_\infty < \infty,\ \|g^{(r)}\|_\infty < \infty\}.$$

It was proved in [4] that

$$(1.12) \qquad \omega_{\varphi^\lambda}^r(f,t) \sim K_{\varphi^\lambda}^r(f,t^r) \sim \overline{K}_{\varphi^\lambda}^r(f,t^r).$$

Throughout this paper $\|\cdot\|$ denotes $\|\cdot\|_\infty$, and C denotes a positive constant, not necessarily the same at each occurrence.

2. Preliminaries

By [4, (9.4.3)], we have for $f^{(s)} \in C_B[0,\infty)$

$$(2.1) \quad D^s S_n(f,x) = \sum_{k=0}^{\infty} n^s s_{n,k}(x) \vec{\Delta}_{\frac{1}{n}}^s f\Big(\frac{k}{n}\Big)$$

$$= \sum_{k=0}^{\infty} n^s s_{n,k}(x) \int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} f^{(s)}\Big(\frac{k}{n} + u_1 + \ldots + u_s\Big)\, \mathrm{d}u_1 \ldots \mathrm{d}u_s.$$

Noting that $\alpha_j^n \in \Pi_j$, from (1.4) and (2.1) we have for $0 \leqslant s \leqslant 2r-1$ and $f^{(s)} \in C_B[0,\infty)$

$$\begin{aligned}(2.2)\qquad D^s S_n^{\langle 2r-1\rangle}(f,x) &= \sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\binom{s}{i} D^i\alpha_j^n(x) D^{j+s-i} S_n(f,x)\\ &= \sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\binom{s}{i} D^i\alpha_j^n(x) D^{j-i}\sum_{k=0}^{\infty} n^s s_{n,k}(x)\\ &\quad \int_0^{\frac{1}{n}}\dots\int_0^{\frac{1}{n}} f^{(s)}\Big(\frac{k}{n}+u_1+\ldots+u_s\Big)\,\mathrm{d}u_1\ldots\mathrm{d}u_s,\end{aligned}$$

where $j\wedge s = \min\{j,s\}$.

Observe that

$$\begin{aligned}(2.3)\qquad S_{n,s}^{\langle 2r-1\rangle}(g,x) &= \sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\binom{s}{i} D^i\alpha_j^n(x) D^{j-i}\sum_{k=0}^{\infty} n^s s_{n,k}(x)\\ &\quad \int_0^{\frac{1}{n}}\dots\int_0^{\frac{1}{n}} g\Big(\frac{k}{n}+u_1+\ldots+u_s\Big)\,\mathrm{d}u_1\ldots\mathrm{d}u_s\\ &= \sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\binom{s}{i} D^i\alpha_j^n(x) D^{j-i}\sum_{k=0}^{\infty} s_{n,k}(x)\bar{g}\Big(\frac{k}{n}\Big),\end{aligned}$$

where $\bar{g}\big(\frac{k}{n}\big) = n^s\int_0^{\frac{1}{n}}\dots\int_0^{\frac{1}{n}} g\big(\frac{k}{n}+u_1+\ldots+u_s\big)\,\mathrm{d}u_1\ldots\mathrm{d}u_s$. Thus we see that

$$(2.4)\qquad D^s S_n^{\langle 2r-1\rangle}(f,x) = S_{n,s}^{\langle 2r-1\rangle}(f^{(s)},x).$$

Since $\alpha_j^n(x) \in \Pi_{[\frac{j}{2}]}$ (see (1.6)), it is easy to see by (2.3) that

$$(2.5)\qquad S_{n,s}^{\langle 2r-1\rangle}(1,x) = 1.$$

For $x \in E_n = \big[\frac{1}{n},\infty\big)$ by [4, (9.4.9)], we can deduce that

$$(2.6)\qquad |D^m s_{n,k}(x)| \leqslant C\sum_{l=0}^{m}\Big(\frac{n}{x}\Big)^{\frac{m+l}{2}}\Big|\frac{k}{n}-x\Big|^l s_{n,k}(x).$$

Next we give some lemmas.

Lemma 2.1. *For $\alpha_j^n(x)$ and $r \leqslant j$, we have*
(1) $x \in E_n^c = \left[0, \frac{1}{n}\right)$,

$$|\alpha_j^n(x)| \leqslant Cn^{-j}, \tag{2.7}$$

$$|D^r \alpha_j^n(x)| \leqslant Cn^{-j+r}. \tag{2.8}$$

(2) $x \in E_n = \left[\frac{1}{n}, \infty\right)$,

$$|\alpha_j^n(x)| \leqslant Cn^{-j/2}\varphi^j(x), \tag{2.9}$$

$$|D^r \alpha_j^n(x)| \leqslant Cn^{-\frac{j}{2}+\frac{r}{2}}\varphi^{j-r}(x). \tag{2.10}$$

Proof. This follows easily from (1.6). □

Lemma 2.2. *The operator $S_{n,s}^{\langle 2r-1\rangle}(g,x)$ is bounded, that is,*

$$\|S_{n,s}^{\langle 2r-1\rangle}(g,x)\| \leqslant C\|g\|. \tag{2.11}$$

Proof. (1) For $x \in E_n^c = \left[0, \frac{1}{n}\right)$, from (2.3), (2.1) and (2.8), we have

$$|S_{n,s}^{\langle 2r-1\rangle}(g,x)| \leqslant C \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} n^{-j+i} n^{j-i} \sum_{k=0}^{\infty} \left|\vec{\Delta}_{\frac{1}{n}}^{j-i} \bar{g}\left(\frac{k}{n}\right)\right| s_{n,k}(x).$$

Noting that $\left|\vec{\Delta}_{\frac{1}{n}}^{j-i} \bar{g}\left(\frac{k}{n}\right)\right| \leqslant C\|\bar{g}\| \leqslant C\|g\|$, we have

$$|S_{n,s}^{\langle 2r-1\rangle}(g,x)| \leqslant C\|g\|.$$

(2) For $x \in E_n = \left[\frac{1}{n}, \infty\right)$, from (2.3), (2.6), (2.10) and [4, (9.4.14)], we have

$$\begin{aligned}
|S_{n,s}^{\langle 2r-1\rangle}(g,x)| &= \left|\sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} \binom{s}{i} D^i \alpha_j^n(x) D^{j-i} \sum_{k=0}^{\infty} s_{n,k}(x)\bar{g}\left(\frac{k}{n}\right)\right| \\
&\leqslant C \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} n^{-\frac{j}{2}+\frac{i}{2}} \varphi^{j-i}(x) \sum_{k=0}^{\infty} |D^{j-i} s_{n,k}(x)| \left|\bar{g}\left(\frac{k}{n}\right)\right| \\
&\leqslant C \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} n^{-\frac{j}{2}+\frac{i}{2}} \varphi^{j-i}(x) \sum_{k=0}^{\infty} \sum_{l=0}^{j-i} \left(\frac{n}{x}\right)^{\frac{j-i+l}{2}} \left|\frac{k}{n} - x\right|^l s_{n,k}(x) \left|\bar{g}\left(\frac{k}{n}\right)\right| \\
&\leqslant C\|g\| \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} n^{-\frac{j}{2}+\frac{i}{2}} \varphi^{j-i}(x) \sum_{l=0}^{j-i} \left(\frac{n}{x}\right)^{\frac{j-i+l}{2}} n^{-\frac{l}{2}} \varphi^l(x) \leqslant C\|g\|.
\end{aligned}$$

□

Now we give the estimate of the moments for Szász operators (cf. [4, p. 138 (9.5.10)]) which will be used later:

$$S_n((\cdot - x)^{2j}, x) \leqslant \begin{cases} Cn^{-2j}, & \text{for } x \in E_n^c = \left[0, \frac{1}{n}\right); \\ C\frac{\varphi^{2j}(x)}{n^j}, & \text{for } x \in E_n = \left[\frac{1}{n}, \infty\right). \end{cases} \tag{2.12}$$

3. Direct Theorem

In this section we give a direct approximation theorem.

Theorem 3.1. *If* $\varphi(x) = \sqrt{x}$, $\delta_n(x) = \max\{\varphi(x), \frac{1}{\sqrt{n}}\}$, $0 \leqslant \lambda \leqslant 1$, $n \geqslant 2r-1$, $0 \leqslant s \leqslant 2r-1$, *then for* $f^{(s)} \in C_B[0,\infty)$, *we have*

$$|D^s S_n^{\langle 2r-1\rangle}(f,x) - f^{(s)}(x)| \leqslant C\omega_{\varphi^\lambda}^{2r-s}\left(f^{(s)}, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{3.1}$$

P r o o f. By the definition of $\overline{K}_{\varphi^\lambda}^{2r-s}(f, t^{2r-s})$ for fixed n, x, λ, we can choose $g(t) = g_{\lambda,n,x}(t)$ such that

$$\begin{aligned} \|f^{(s)} - g\| &+ \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{x}}\right)^{2r-s} \|\varphi^{(2r-s)\lambda} g^{(2r-s)}\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{x}}\right)^{\frac{2r-s}{1-\lambda/2}} \|g^{(2r-s)}\| \\ &\leqslant 2\overline{K}_{\varphi^\lambda}^{2r-s}\left(f, \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r-s}\right). \end{aligned} \tag{3.2}$$

Using $f^{(s)} = f^{(s)} - g + g$, by (2.4) and (2.11), we have

$$\begin{aligned} |D^s S_n^{\langle 2r-1\rangle}(f,x) - f^{(s)}(x)| &= |S_{n,s}^{\langle 2r-1\rangle}(f^{(s)}, x) - f^{(s)}(x)| \\ &\leqslant |S_{n,s}^{\langle 2r-1\rangle}(f^{(s)} - g, x)| + |f^{(s)}(x) - g(x)| + |S_{n,s}^{\langle 2r-1\rangle}(g,x) - g(x)| \\ &\leqslant C\|f^{(s)} - g\| + |S_{n,s}^{\langle 2r-1\rangle}(g,x) - g(x)| \\ &\leqslant C\overline{K}_{\varphi^\lambda}^{2r-s}\left(f, \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r-s}\right) + |S_{n,s}^{\langle 2r-1\rangle}(g,x) - g(x)|. \end{aligned} \tag{3.3}$$

Therefore we only need to estimate $|S_{n,s}^{\langle 2r-1\rangle}(g,x) - g(x)|$.

Since $S_n^{\langle 2r-1\rangle}(f,x)$ is exact on Π_{2r-1}, we have for all $1 \leqslant j \leqslant 2r-1$

$$S_n^{\langle 2r-1\rangle}((t-x)^j, x) = 0. \tag{3.4}$$

Note that $D^s S_n^{\langle 2r-1\rangle}(f,x) = S_{n,s}^{\langle 2r-1\rangle}(f^{(s)}, x)$, so

$$D^s S_n^{\langle 2r-1\rangle}((t-x)^j, x) = j(j-1)\dots(j-s+1) S_{n,s}^{\langle 2r-1\rangle}((t-x)^{j-s}, x).$$

Therefore for $1 \leqslant j \leqslant 2r-1-s$, we have

$$S_{n,s}^{\langle 2r-1\rangle}((t-x)^j, x) = 0. \tag{3.5}$$

Now using Taylor formula, we write

$$g(t) = g(x) + (t-x)g'(x) + \ldots + \frac{(t-x)^{2r-1-s}}{(2r-1-s)!} g^{(2r-1-s)}(x) + R_{2r-s}(g,t,x)$$

where $R_{2r-s}(g,t,x) = \frac{1}{(2r-1-s)!}\int_x^t (t-u)^{2r-1-s} g^{(2r-s)}(u)\,\mathrm{d}u$.

By (2.5) and (3.5), we have

$$|S_{n,s}^{\langle 2r-1\rangle}(g,x) - g(x)| = |S_{n,s}^{\langle 2r-1\rangle}(R_{2r-s}(g,\cdot,x),x)| =: I.$$

We will estimate I.

For $x \in E_n^c = \left[0, \frac{1}{n}\right)$, by (2.3), (2.1) and (2.8), we have

$$\begin{aligned} &|S_{n,s}^{\langle 2r-1\rangle}(R_{2r-s}(g,\cdot,x),x)| \\ &\quad = \Big|\sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\binom{s}{i} D^i\alpha_j^n(x) D^{j-i}\sum_{k=0}^{\infty} s_{n,k}(x)\overline{R}_{2r-s}\Big(g,\frac{k}{n},x\Big)\Big| \\ &\quad \leqslant C\sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s} n^{-j+i} n^{j-i}\sum_{k=0}^{\infty}\Big|\vec{\Delta}_{\frac{1}{n}}^{j-i}\overline{R}_{2r-s}\Big(g,\frac{k}{n},x\Big)\Big| s_{n,k}(x), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &\overline{R}_{2r-s}\Big(g,\frac{k}{n},x\Big) \\ &\quad = n^s\int_0^{\frac{1}{n}}\cdots\int_0^{\frac{1}{n}}\frac{1}{(2r-1-s)!} \\ &\quad\times\int_x^{\frac{k}{n}+u_1+\ldots+u_s}\Big(\frac{k}{n}+u_1+\ldots+u_s-u\Big)^{2r-1-s} g^{(2r-s)}(u)\,\mathrm{d}u\,\mathrm{d}u_1\ldots\mathrm{d}u_s. \end{aligned}$$

As

$$\begin{aligned} &\Big|\vec{\Delta}_{\frac{1}{n}}^{j-i}\overline{R}_{2r-s}\Big(g,\frac{k}{n},x\Big)\Big| \\ &\quad = \Big|\sum_{m=0}^{j-i}(-1)^{j-i-m}\binom{j-i}{m} n^s\int_0^{\frac{1}{n}}\cdots\int_0^{\frac{1}{n}}\frac{1}{(2r-1-s)!} \\ &\qquad \int_x^{\frac{k+m}{n}+u_1+\ldots+u_s}\Big(\frac{k+m}{n}+u_1+\ldots+u_s-u\Big)^{2r-1-s} g^{(2r-s)}(u)\,\mathrm{d}u\,\mathrm{d}u_1\ldots\mathrm{d}u_s\Big| \end{aligned}$$

and (cf. [4, (9.6.1)])

$$
\begin{aligned}
|R_{2r-s}(g,t,x)| &= \left|\frac{1}{(2r-1-s)!}\int_x^t (t-u)^{2r-1-s} g^{(2r-s)}(u)\,\mathrm{d}u\right| \\
&\leqslant \frac{|t-x|^{2r-s-1}}{\delta_n^{(2r-s)\lambda}(x)}\left|\int_x^t \delta_n^{(2r-s)\lambda}(u)|g^{(2r-s)}(u)|\,\mathrm{d}u\right| \leqslant \|\delta_n^{(2r-s)\lambda} g^{(2r-s)}\|\cdot\frac{|t-x|^{2r-s}}{\delta_n^{(2r-s)\lambda}(x)},
\end{aligned}
$$

we have by [4, (1.1.3)]

$$
\begin{aligned}
(3.7)\quad \Big|\vec{\Delta}_{\frac{1}{n}}^{j-i}\overline{R}_{2r-s}\Big(g,\frac{k}{n},x\Big)\Big| &= \Big|\sum_{l=0}^{j-i}(-1)^l\binom{j-i}{l}\overline{R}_{2r-s}\Big(g,\frac{k}{n}+\frac{j-i-l}{n},x\Big)\Big| \\
&= \Big|\sum_{l=0}^{j-i}(-1)^l\binom{j-i}{l}\frac{n^s}{(2r-1-s)!}\int_0^{\frac{1}{n}}\dots\int_0^{\frac{1}{n}}\int_x^{\frac{k+j-i-l}{n}+u_1+\ldots+u_s} \\
&\qquad \Big(\frac{k+j-i-l}{n}+u_1+\ldots+u_s-u\Big)^{2r-1-s} g^{(2r-s)}(u)\,\mathrm{d}u\,\mathrm{d}u_1\ldots\mathrm{d}u_s\Big| \\
&\leqslant C\Big|\sum_{l=0}^{j-i} n^s\int_0^{\frac{1}{n}}\dots\int_0^{\frac{1}{n}}\|\delta_n^{(2r-s)\lambda}g^{(2r-s)}\| \\
&\qquad\qquad \frac{\left|\frac{k+j-i-l}{n}+u_1+\ldots+u_s-x\right|^{2r-s}}{\delta_n^{(2r-s)\lambda}(x)}\,\mathrm{d}u_1\ldots\mathrm{d}u_s\Big| \\
&\leqslant C\|\delta_n^{(2r-s)\lambda}g^{(2r-s)}\|\frac{1}{\delta_n^{(2r-s)\lambda}(x)} \\
&\qquad \times\sum_{l=0}^{j-i}\max\Big\{\Big|\frac{k+j-i-l+s}{n}-x\Big|^{2r-s},\Big|\frac{k+j-i-l}{n}-x\Big|^{2r-s}\Big\} \\
&\leqslant C\|\delta_n^{(2r-s)\lambda}g^{(2r-s)}\|\frac{1}{\delta_n^{(2r-s)\lambda}(x)}\Big(\Big|\frac{k}{n}-x\Big|^{2r-s}+\Big(\frac{s+j-i}{n}\Big)^{2r-s}\Big).
\end{aligned}
$$

Hence we get with $\delta_n(x)\sim\frac{1}{\sqrt{n}}$ for $x\in E_n^c$ by (3.6), (3.7), (3.2) and (2.12)

$$
\begin{aligned}
(3.8)\quad I &\leqslant C\frac{1}{\delta_n^{(2r-s)\lambda}(x)}\|\delta_n^{(2r-s)\lambda}g^{(2r-s)}\|\sum_{k=0}^{\infty}s_{n,k}(x)\Big[\Big|\frac{k}{n}-x\Big|^{2r-s}+\Big(\frac{1}{n}\Big)^{2r-s}\Big] \\
&\leqslant C\frac{1}{\delta_n^{(2r-s)\lambda}(x)}\Big\|\Big(\varphi+\frac{1}{\sqrt{n}}\Big)^{(2r-s)\lambda}g^{(2r-s)}\Big\|\,n^{-(2r-s)} \\
&\leqslant C\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{2r-s}\big(\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|+n^{-\frac{(2r-s)\lambda}{2}}\|g^{(2r-s)}\|\big) \\
&\leqslant C\Big[\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{2r-s}\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|+\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\frac{2r-s}{1-\lambda/2}}\|g^{(2r-s)}\|\Big] \\
&\leqslant C\overline{K}_{\varphi^\lambda}^{2r-s}\Big(f,\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{2r-s}\Big).
\end{aligned}
$$

For $x \in E_n = \left[\frac{1}{n}, \infty\right)$, from (2.3), (2.6) and (2.10), we have

$$I \leqslant C \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} \frac{\varphi^{j-i}(x)}{n^{(j-i)/2}} \sum_{k=0}^{\infty} n^s \sum_{l=0}^{j-i} \Big(\frac{n}{x}\Big)^{\frac{j-i+1}{2}} s_{n,k}(x) \Big|\frac{k}{n} - x\Big|^l \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}}$$
$$\left|\int_x^{\frac{k}{n}+u_1+\ldots+u_s} \Big(\frac{k}{n} + u_1 + \ldots + u_s - u\Big)^{2r-1-s} g^{(2r-s)}(u)\,\mathrm{d}u\right| \mathrm{d}u_1 \ldots \mathrm{d}u_s.$$

Using [4, (9.6.1)]

$$\left|\int_x^t (t-u)^{m-1} g^{(m)}(u)\,\mathrm{d}u\right| \leqslant \frac{|t-x|^m}{\varphi^{m\lambda}(x)} \|\varphi^{m\lambda} g^{(m)}\|,$$

we get (cf. [4, (9.4.14)])

$$\begin{aligned}
(3.9)\quad I &\leqslant C \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} \sum_{k=0}^{\infty} \sum_{l=0}^{j-i} n^s \Big(\frac{n}{x}\Big)^{\frac{l}{2}} s_{n,k}(x) \Big|\frac{k}{n} - x\Big|^l \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} \\
&\qquad \times \big\|\varphi^{(2r-s)\lambda} g^{(2r-s)}\big\| \frac{\big|\frac{k}{n} + u_1 + \ldots + u_s - x\big|^{2r-s}}{\varphi^{(2r-s)\lambda}(x)} \mathrm{d}u_1 \ldots \mathrm{d}u_s \\
&\leqslant C \big\|\varphi^{(2r-s)\lambda} g^{(2r-s)}\big\| \varphi^{-(2r-s)\lambda}(x) \\
&\quad \times \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} \sum_{l=0}^{j-i} \Big(\frac{n}{x}\Big)^{\frac{l}{2}} \sum_{k=0}^{\infty} s_{n,k}(x) \Big|\frac{k}{n} - x\Big|^l \left(\Big|\frac{k}{n} - x\Big|^{2r-s} + \Big(\frac{s}{n}\Big)^{2r-s}\right) \\
&\leqslant C \big\|\varphi^{(2r-s)\lambda} g^{(2r-s)}\big\| \varphi^{-(2r-s)\lambda}(x) \\
&\quad \times \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} \sum_{l=0}^{j-i} \Big(\frac{n}{x}\Big)^{\frac{l}{2}} \left[\Big(\frac{\varphi(x)}{\sqrt{n}}\Big)^{l+2r-s} + \Big(\frac{1}{n}\Big)^{2r-s} \Big(\frac{\varphi(x)}{\sqrt{n}}\Big)^l\right] \\
&\leqslant C \varphi^{(2r-s)(1-\lambda)}(x)/(\sqrt{n})^{2r-s} \big\|\varphi^{(2r-s)\lambda} g^{(2r-s)}\big\|.
\end{aligned}$$

From (3.8) and (3.9), we have

$$\begin{aligned}
(3.10)\qquad I &\leqslant C\left(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{2r-s} \|\varphi^{(2r-s)\lambda} g^{(2r-s)}\| + \Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\frac{2r-s}{1-\lambda/2}} \|g^{(2r-s)}\|\right) \\
&\leqslant C \overline{K}_{\varphi^\lambda}^{2r-s}\left(f, \Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{2r-s}\right).
\end{aligned}$$

Therefore from (3.8) and (3.10) we obtain

$$\begin{aligned}
|D^s S_n^{\langle 2r-1\rangle}(f,x) - f^{(s)}(x)| &= |S_{n,s}^{\langle 2r-1\rangle}(f^{(s)},x) - f^{(s)}(x)| \\
&\leqslant C\|f^{(s)} - g\| + |S_{n,s}^{\langle 2r-1\rangle}(g,x) - g(x)| \\
&\leqslant C \overline{K}_{\varphi^\lambda}^{2r-s}\left(f^{(s)}, \Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{2r-s}\right) \leqslant C \omega_{\varphi^\lambda}^{2r-s}\Big(f^{(s)}, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big).
\end{aligned}$$

The proof of the theorem is complete. □

4. Inverse Theorem

In this section we will give an inverse result as follows.

Theorem 4.1. *Let* $f^{(s)} \in C_B[0,\infty)$, $0 \leqslant s \leqslant 2r-1$, $n \geqslant 4r$, $0 \leqslant \lambda \leqslant 1$, $0 < \alpha < 2r-s$, *then*

$$|D^s S_n^{\langle 2r-1\rangle}(f,x) - f^{(s)}(x)| = O\Big(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\alpha}\Big)$$

implies

$$\omega_{\varphi^\lambda}^{2r-s}(f^{(s)},t) = O(t^\alpha).$$

To prove Theorem 4.1, we need the following lemma.

Lemma 4.2. *For* $n \geqslant 4r$, *we have*

$$|\varphi^{(2r-s)\lambda}(x) D^{2r-s} S_{n,s}^{\langle 2r-1\rangle}(g,x)| \leqslant C n^{\frac{2r-s}{2}} \delta_n^{(2r-s)(\lambda-1)}(x)\|g\|, \quad (g \in C_B[0,\infty)), \tag{4.1}$$

$$|\varphi^{(2r-s)\lambda}(x) D^{2r-s} S_{n,s}^{\langle 2r-1\rangle}(g,x)| \leqslant C\|\varphi^{2r-s} g^{(2r-s)}\|, \quad (g \in w^{2r-s}(\varphi,[0,\infty))) \tag{4.2}$$

P r o o f. First let us prove (4.1). We consider two cases: $x \in E_n^c$ and $x \in E_n$. For $x \in E_n^c\big(\frac{1}{\sqrt{n}} \sim \delta_n(x)\big)$, from (2.1), (2.3) and (2.8), we have

$$\begin{aligned}
&|\varphi^{(2r-s)\lambda}(x) D^{2r-s} S_{n,s}^{\langle 2r-1\rangle}(g,x)| \\
&\quad = \Bigg|\varphi^{(2r-s)\lambda}(x) \sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\binom{s}{i}\sum_{l=0}^{j-i}\binom{2r-s}{l} \\
&\qquad \times D^{i+l}\alpha_j^n(x) D^{2r-s-l+j-i}\sum_{k=0}^{\infty} s_{n,k}(x)\bar{g}\Big(\frac{k}{n}\Big)\Bigg| \\
&\quad \leqslant C n^{-\frac{2r-s}{2}\lambda}\sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\sum_{l=0}^{j-i} n^{-j+l+i} \\
&\qquad \times \Bigg|\sum_{k=0}^{\infty} n^{2r-s-l+j-i} s_{n,k}(x)\vec{\Delta}_{\frac{1}{n}}^{2r-s-l+j-i}\bar{g}\Big(\frac{k}{n}\Big)\Bigg| \\
&\quad \leqslant C n^{-\frac{2r-s}{2}\lambda} n^{2r-s}\|\bar{g}\| \leqslant C n^{\frac{2r-s}{2}} \delta_n^{(2r-s)(\lambda-1)}(x)\|g\|.
\end{aligned} \tag{4.3}$$

For $x \in E_n$ ($\varphi(x) \sim \delta_n(x)$) and $\lambda = 1$, from (2.3), (2.6) and (2.10), we have

$$\begin{aligned}
(4.4)\qquad & |\varphi^{2r-s}(x) D^{2r-s} S_{n,s}^{\langle 2r-1\rangle}(g,x)| \\
& \leqslant C\varphi^{2r-s}(x) \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} \sum_{l=0}^{j-i} \left| D^{i+l} \alpha_j^n(x) D^{2r-s-l+j-i} \sum_{k=0}^{\infty} s_{n,k}(x) \bar{g}\Big(\frac{k}{n}\Big) \right| \\
& \leqslant C\|g\|\varphi^{2r-s}(x) \sum_{j=0}^{2r-1} \sum_{i=0}^{j\wedge s} \sum_{l=0}^{j-i} n^{-\frac{j}{2}+\frac{l+i}{2}} \varphi^{j-i-l}(x) \\
& \quad \times \sum_{k=0}^{\infty} \sum_{m=0}^{2r-s+j-i-l} \Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{2r-s+j-i-l+m} \left|\frac{k}{n} - x\right|^m s_{n,k}(x) \\
& \leqslant Cn^{\frac{2r-s}{2}} \|g\|.
\end{aligned}$$

Using (4.4) for $0 \leqslant \lambda < 1$, we have

$$\begin{aligned}
(4.5)\qquad & |\varphi^{(2r-s)\lambda}(x) D^{2r-s} S_{n,s}^{\langle 2r-1\rangle}(g,x)| \\
& \quad = \varphi^{(2r-s)(\lambda-1)}(x) |\varphi^{2r\lambda}(x) D^{2r-s} S_{n,s}^{\langle 2r-1\rangle}(g,x)| \\
& \quad \leqslant Cn^{\frac{2r-s}{2}} \delta_n^{(2r-s)(\lambda-1)}(x) \|g\|.
\end{aligned}$$

By (4.3), (4.4) and (4.5), we get (4.1).

Now we prove (4.2). First of all we have (cf. [4, p. 154]) for $k = 1, 2, \ldots$

$$\begin{aligned}
(4.6)\quad \left|\vec{\Delta}_{\frac{1}{n}}^{2r-s} \bar{g}\Big(\frac{k}{n}\Big)\right| &= n^s \left| \int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} \vec{\Delta}_{\frac{1}{n}}^{2r-s} g\Big(\frac{k}{n} + u_1 + \ldots + u_s\Big) \, \mathrm{d}u_1 \ldots \mathrm{d}u_s \right| \\
& \leqslant Cn^s \int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} \sup_{1\leqslant i\leqslant 2r-s} \int_0^{\frac{i}{n}} \left| y - \frac{i}{n} \right|^{2r-s-1} \\
& \qquad \times \left| g^{(2r-s)}\Big(\frac{k}{n} + y + u_1 + \ldots + u_s\Big)\right| \mathrm{d}y \, \mathrm{d}u_1 \ldots \mathrm{d}u_s \\
& \leqslant Cn^{-(2r-s)+1} \int_0^{\frac{2r}{n}} \left| g^{(2r-s)}\Big(\frac{k}{n} + u\Big)\right| \mathrm{d}u \\
& \leqslant Cn^{-(2r-s)} \Big(\frac{k}{n}\Big)^{-\frac{2r-s}{2}\lambda} \|\varphi^{(2r-s)\lambda} g^{(2r-s)}\|,
\end{aligned}$$

and for $k = 0$

$$\begin{aligned}
(4.7)\qquad \left|\vec{\Delta}_{\frac{1}{n}}^{2r-s} \bar{g}(0)\right| &\leqslant C \int_0^{\frac{2r}{n}} u^{2r-s-1} |g^{(2r-s)}(u)| \, \mathrm{d}u \\
& \leqslant Cn^{-(2r-s)+\frac{2r-s}{2}\lambda} \|\varphi^{(2r-s)\lambda} g^{(2r-s)}\|.
\end{aligned}$$

Similarly we have for $k \neq 0$, $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}(4.8)\qquad \left|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\bar{g}\Big(\frac{k+m}{n}\Big)\right| &\leqslant Cn^{-(2r-s)+1}\int_0^{\frac{2r+m}{n}}\left|g^{(2r-s)}\Big(\frac{k}{n}+u\Big)\right|\,\mathrm{d}u\\ &\leqslant Cn^{-(2r-s)}\Big(\frac{k}{n}\Big)^{-\frac{2r-s}{2}\lambda}\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|,\end{aligned}$$

and for $k = 0$, $m \neq 0$,

$$\begin{aligned}(4.9)\qquad \left|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\bar{g}\Big(\frac{m}{n}\Big)\right| &\leqslant Cn^{-(2r-s)+1}\int_0^{\frac{2r}{n}}\left|g^{(2r-s)}\Big(\frac{m}{n}+u\Big)\right|\,\mathrm{d}u\\ &\leqslant Cn^{-(2r-s)+\frac{2r-s}{2}\lambda}\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|.\end{aligned}$$

Noting that for $p \in \mathbb{N}$ and $p \geqslant \frac{2r-s}{2}\lambda$

$$\begin{aligned}(4.10)\qquad \sum_{k=1}^{\infty}s_{n,k}(x)\Big(\frac{k}{n}\Big)^{-\frac{2r-s}{2}\lambda} &\leqslant \Big(\sum_{k=1}^{\infty}s_{n,k}(x)\Big(\frac{n}{k}\Big)^{p}\Big)^{\frac{2r-s}{2p}\lambda}\\ &\leqslant C\Big(\frac{1}{x^p}\sum_{k=1}^{\infty}s_{n,k+p}(x)\Big)^{\frac{2r-s}{2p}\lambda} \leqslant C\varphi^{-(2r-s)\lambda}(x),\end{aligned}$$

we have by (4.6)–(4.10)

$$\begin{aligned}(4.11)\qquad &\left|D^{2r-s+j-i-l}\sum_{k=0}^{\infty}s_{n,k}(x)\bar{g}\Big(\frac{k}{n}\Big)\right|\\ &\leqslant Cn^{2r-s+j-i-l}\sum_{k=0}^{\infty}s_{n,k}(x)\sum_{m=0}^{j-i-l}\left|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\bar{g}\Big(\frac{k+m}{n}\Big)\right|\\ &= Cn^{2r-s+j-i-l}\Bigg(s_{n,0}(x)\left|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\bar{g}(0)\right| + s_{n,0}(x)\sum_{m=1}^{j-i-l}\left|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\bar{g}\Big(\frac{m}{n}\Big)\right|\\ &\qquad + \sum_{m=0}^{j-i-l}\sum_{k=1}^{\infty}s_{n,k}(x)\left|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\bar{g}\Big(\frac{k+m}{n}\Big)\right|\Bigg)\\ &\leqslant Cn^{2r-s+j-i-l}\Big(n^{-(2r-s)+\frac{2r-s}{2}\lambda} + n^{-(2r-s)}\varphi^{-(2r-s)\lambda}(x)\Big)\\ &\qquad \times \|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|.\end{aligned}$$

For $x \in E_n^c$, $\varphi(x) \leqslant \frac{1}{\sqrt{n}}$, from (2.3), (2.8) and (4.11) we have

$$\begin{aligned}(4.12)\qquad &\left|\varphi^{(2r-s)\lambda}(x)D^{2r-s}S_{n,s}^{\langle 2r-1\rangle}(f,x)\right|\\ &= \left|\varphi^{(2r-s)\lambda}(x)\sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\sum_{l=0}^{j-i}D^{i+l}\alpha_j^n(x)D^{2r-s-l+j-i}\sum_{k=0}^{\infty}s_{n,k}(x)\bar{g}\Big(\frac{k}{n}\Big)\right|\\ &\leqslant C\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|.\end{aligned}$$

For $x \in E_n$ by (2.6) one has

$$\text{(4.13)}\quad \Big|D^{2r-s+j-i-l}\sum_{k=0}^{\infty}s_{n,k}(x)\overline{g}\Big(\frac{k}{n}\Big)\Big| = \Big|\sum_{k=0}^{\infty}(D^{j-i-l}s_{n,k}(x))n^{2r-s}\vec{\Delta}_{\frac{1}{n}}^{2r-s}\overline{g}\Big(\frac{k}{n}\Big)\Big|$$
$$\leqslant C\sum_{m=0}^{j-i-l}\Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j-i-l+m}\sum_{k=0}^{\infty}s_{n,k}(x)\cdot\Big|\frac{k}{n}-x\Big|^m n^{2r-s}\Big|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\overline{g}\Big(\frac{k}{n}\Big)\Big|.$$

On the other hand,

$$\varphi^{2r}(x)s_{n,k}(x) \leqslant \begin{cases} \dfrac{r!}{n^r}s_{n,r}(x), & k=0; \\ C\Big(\dfrac{k}{n}\Big)^r s_{n,k+r}(x), & k \neq 0, \end{cases}$$

thus by (4.6) and (4.7), we have

(4.14)
$$\varphi^{2r}(x)\sum_{k=0}^{\infty}s_{n,k}(x)\Big|\frac{k}{n}-x\Big|^m\Big|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\overline{g}\Big(\frac{k}{n}\Big)\Big|$$
$$\leqslant C\bigg(\frac{1}{n^r}\mathrm{e}^{-nx}\frac{(nx)^r}{r!}x^m\Big|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\overline{g}(0)\Big| + \sum_{k=1}^{\infty}s_{n,k+r}(x)\Big(\frac{k}{n}\Big)^r\Big|\frac{k}{n}-x\Big|^m\Big|\vec{\Delta}_{\frac{1}{n}}^{2r-s}\overline{g}\Big(\frac{k}{n}\Big)\Big|\bigg)$$
$$\leqslant C\bigg(\frac{1}{n^{r+m}}n^{-(2r-s)+\frac{2r-s}{2}\lambda} + \sum_{k=1}^{\infty}s_{n,k+r}(x)\Big(\frac{k}{n}\Big)^r\Big|\frac{k}{n}-x\Big|^m n^{-(2r-s)}\Big(\frac{k}{n}\Big)^{-\frac{2r-s}{2}\lambda}\bigg)$$
$$\times\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|;$$

here we have used that $\max \mathrm{e}^{-nx}x^{r+m}$ is achieved at $x = \frac{r+m}{n}$.

It is easy to see that

(4.15)
$$\sum_{k=1}^{\infty}s_{n,k+r}(x)\Big(\frac{k}{n}\Big)^{r(1-\lambda)+\frac{s}{2}\lambda}\Big|\frac{k}{n}-x\Big|^m$$
$$\leqslant C\bigg(\sum_{k=1}^{\infty}s_{n,k+r}(x)\Big(\frac{k}{n}\Big)^{2(r(1-\lambda)+\frac{s}{2}\lambda)}\bigg)^{\frac{1}{2}}\bigg(\sum_{k=1}^{\infty}s_{n,k+r}(x)\Big[\Big(\frac{k+r}{n}-x\Big)^{2m}+\Big(\frac{r}{n}\Big)^{2m}\Big]\bigg)^{\frac{1}{2}}$$
$$\leqslant Cx^{r(1-\lambda)+\frac{s}{2}\lambda}\Big[\Big(\frac{\varphi(x)}{\sqrt{n}}\Big)^m+\Big(\frac{1}{n}\Big)^m\Big]$$
$$\leqslant C\varphi^{2r(1-\lambda)+s\lambda}(x)\Big(\frac{\varphi(x)}{\sqrt{n}}\Big)^m,$$

here we have used that $\sum\limits_{k=1}^{\infty}s_{n,k+r}(x)\big(\frac{k}{n}\big)^{2p} \leqslant Cx^{2p}$ $(p>0)$.

Combining (4.13)–(4.15) we get

$$(4.16)\quad \Big|\varphi^{2r}(x)D^{2r-s+j-i-l}\sum_{k=0}^{\infty}s_{n,k}(x)\bar{g}\Big(\frac{k}{n}\Big)\Big| \leqslant C\sum_{m=0}^{j-i-l}\Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j-i-l+m}\Big(n^{\frac{2r-s}{2}\lambda-r-m}+\varphi^{2r(1-\lambda)+s\lambda}\Big(\frac{\varphi(x)}{\sqrt{n}}\Big)^{m}\Big) \times\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|.$$

Therefore, noting that $n\varphi^2(x)\geqslant 1$ for $x\in E_n$, we obtain

$$(4.17)\quad |\varphi^{(2r-s)\lambda}(x)D^{2r-s}S_{n,s}^{\langle 2r-1\rangle}(f,x)| \leqslant C\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|\varphi^{(2r-s)\lambda-2r}(x)\sum_{j=0}^{2r-1}\sum_{i=0}^{j\wedge s}\sum_{l=0}^{j-i}n^{-\frac{j}{2}+\frac{i+j}{2}}\varphi^{j-i-l}(x) \times\sum_{m=0}^{j-i-l}\Big(\frac{\sqrt{n}}{\varphi(x)}\Big)^{j-i-l+m}\Big[n^{\frac{2r-s}{2}\lambda-r-m}+\varphi^{2r(1-\lambda)+s\lambda}(x)\Big(\frac{\varphi(x)}{\sqrt{n}}\Big)^{m}\Big] \leqslant C\|\varphi^{(2r-s)\lambda}g^{(2r-s)}\|.$$

With (4.12) and (4.17) we get (4.2). The proof of the lemma is completed. □

P r o o f of Theorem 4.1. Using Lemma 4.2 in a similar way as in [6, p. 145, "⇐"], we can prove Theorem 4.1. Here we omit the details. □

Remark 1. If $s=0$ we obtain for $0<\alpha<2r$

$$|S_n^{\langle 2r-1\rangle}(f,x)-f(x)|=O\bigg(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\alpha}\bigg)\Leftrightarrow\omega_{\varphi^\lambda}^{2r}(f,t)=O(t^\alpha).$$

This relation contains the result of [2].

Remark 2. If $s=0$, $r=1$ we get for $0<\alpha<2$

$$|S_n(f,x)-f(x)|=O\bigg(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\alpha}\bigg)\Leftrightarrow\omega_{\varphi^\lambda}^{2}(f,t)=O(t^\alpha).$$

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