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ESTIMATES OF GLOBAL DIMENSION

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Abstract. In this note we show that for a $*^n$-module, in particular, an almost n-tilting module, P over a ring R with $A = \operatorname{End}_R P$ such that P_A has finite flat dimension, the upper bound of the global dimension of A can be estimated by the global dimension of R and hence generalize the corresponding results in tilting theory and the ones in the theory of $*$-modules. As an application, we show that for a finitely generated projective module over a VN regular ring R, the global dimension of its endomorphism ring is not more than the global dimension of R.

Keywords: global dimension, $*$-module

MSC 2000: 16D90

Introduction

The theory of $*$-modules has been studied extensively (see for instance [8], [1], [5], [10] etc.). A $*$-module is a left R-module P with $A = \operatorname{End}_R P$ such that there is an equivalence

$$\operatorname{Hom}_R(P,-)\colon\ \mathcal{C} \rightleftharpoons \mathcal{D}\colon\ P_A \otimes -.$$

between full subcategories $\mathcal{C} \subseteq R$-Mod and $\mathcal{D} \subseteq A$-Mod with $\mathcal{C}$ closed under direct sums and epimorphic images, $\mathcal{D}$ closed under submodules and $A \in \mathcal{D}$. $*$-modules generalize both tilting modules of projective dimension $\leqslant 1$ and quasi-progenerators [1], [2]. In fact, Colpi [2] proved that tilting modules of projective dimension $\leqslant 1$ coincide with $*$-modules which generate all injectives, while quasi-progenerators are just the $*$-modules which generate all of their submodules [1]. Trlifaj [11] showed that $*$-modules are finitely generated. Trlifaj [10] also showed that for a $*$-module ${}_RP$ the

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upper bound of the global dimension of $A = \operatorname{End}_R P$ can be estimated by the global dimension of R. Fuller [5] studied $*$-modules over ring extensions.

Noting the fact that a tilting module of projective dimension $\leqslant n$ (in the sense of Miyashita [7], see Section 1 for details) is a $*$-module if and only if it is classical i.e. if and only if $n = 1$, we introduced $*^n$-modules as generalizations of both $*$-modules and tilting modules of projective dimension $\leqslant n$ [14]. A $*^n$-module is a left R-module P with $A = \operatorname{End}_R P$ such that the functor

$$\operatorname{Hom}_R(P, -)\colon\ \mathcal{C} \rightleftharpoons \mathcal{D}\colon\ P_A \otimes -.$$

define an equivalence between full subcategory $\mathcal{C} \subseteq R$-Mod and $\mathcal{D} \subseteq A$-Mod with $\mathcal{C}$ consisting of modules n-presented by P (see Section 1 for the definition), and $\mathcal{D}$ consisting of modules M such that $\operatorname{Tor}_i^A(P, M) = 0$ for all $i \geqslant 1$. Note that $*^1$-modules are just $*$-modules by [3]. It was also shown in [14] that tilting modules of projective dimension $\leqslant n$ are $*^n$-modules n-presenting all injective modules ([14], Theorem 3.8). Examples of $*^n$-modules contain also all finitely generated projective R-modules P with $A = \operatorname{End}_R P$ such that P_A has finite flat dimension [13]. Corresponding to the notion of quasi-progenerators, a special class of $*^n$-modules, i.e. 2-quasi-progenerators, was introduced in [13]. Examples of 2-quasi-progenerators contain all finitely generated projective modules over VN regular rings.

In this note, we study the global dimension estimates for $*^n$-modules following ideas of Trlifaj [10]. We extend both results about global dimension estimates of $*$-modules and those of the tilting theory to $*^n$-modules. So far, there are many unsolved questions in the theory of $*^n$-modules. For example, we even don't know whether or not $*^n$-modules are finitely generated. We don't know if the flat dimension of P_A is finite, where ${}_RP$ is a $*^n$-module and $A = \operatorname{End}_R P$. In contrast, we easily check that P_A has flat dimension $\leqslant 1$ if ${}_RP$ is a $*$-module (see also [10]). We hope that this short note would be helpful to the study of this theory. We now remark that the above-mentioned two questions were answered in 2005, see [12]. However, it rises another question: "are all $*^n$-modules countably generated?".

1. Preliminaries

Throughout this note, all rings have non zero identity and all modules are unitary. For every ring R, R-Mod (Mod-R) denotes the category of all left (right) R-modules. Modules will mean left modules without explicit reference. By a subcategory we mean a full subcategory closed under isomorphisms.

Let R be a ring, P will always mean an R-module with the endomorphism ring $A = \operatorname{End}_R P$. Hence P is an R-A-bimodule. We say a left R-module M is n-presented by P if there exists an exact sequence $P^{(X_{n-1})} \to P^{(X_{n-2})} \to \ldots \to P^{(X_1)} \to P^{(X_0)} \to M \to 0$ with X_i, $0 \leqslant i \leqslant n-1$, sets. Denote by $\operatorname{Pres}^n(P)$ the category of all modules n-presented by P. Note that there is a clear inclusion between categories: $\operatorname{Pres}^{n+1}(P) \subseteq \operatorname{Pres}^n(P)$. We denote $\operatorname{Pres}^2(P)$ by $\operatorname{Pres}(P)$ and $\operatorname{Pres}^1(P)$ by $\operatorname{Gen}(P)$ as usual.

An R-module P is said to be selfsmall if, for any set X, there is the canonical isomorphism $\operatorname{Hom}_R(P, P^{(X)}) \simeq \operatorname{Hom}_R(P,P)^{(X)}$. Clearly, every finitely generated module is selfsmall. But the converse is generally false, see [4]. An R-module P is said to be $(n,1)$-quasi-projective if for any exact sequence $0 \to L \to P^{(X)} \to N \to 0$ with X a set and $L \in \operatorname{Pres}^{n-1} P$, the induced sequence $0 \to \operatorname{Hom}_R(P,L) \to \operatorname{Hom}_R(P, P^{(X)} \to \operatorname{Hom}_R(P,N) \to 0$ is also exact. An equivalent definition of $*^n$-modules is the following. An R-module P is said to be a $*^n$-module if P is selfsmall, $(n+1,1)$-quasi-projective and $\operatorname{Pres}^n(P) = \operatorname{Pres}^{n+1}(P)$ [14]. An R-module P is said to be a 2-quasi-progenerator if P is a $*^2$-module and P is semi-Σ-quasi-projective (see [9] for the definition) and $\operatorname{Pres}^2(P) = \operatorname{Pres}^3(P)$.

Let P be a $*^n$-module. Then the functor $\operatorname{Hom}_R(P,-)$ preserves all short exact sequences in $\operatorname{Pres}^n(P)$ [14].

Let R be a ring, $P \in R$-Mod and $A = \operatorname{End}_R P$. We use the following notions.

$\operatorname{Ker} T_P^{i\geqslant 1} =: \{M\colon\ \operatorname{Tor}_i^A(P,M) = 0 \text{ for all } i \geqslant 1\}$.

$\operatorname{Ker} E_P^{i\geqslant 1} =: \{M\colon\ \operatorname{Ext}_R^i(P,M) = 0 \text{ for all } i \geqslant 1\}$.

Note that $H_P = \operatorname{Hom}_R(P,-)$ and $T_P = P \otimes_A -$. It is well known that (T_P, H_P) is a pair of adjoint functors and there are the following canonical homomorphisms:

$$\varrho_M\colon\ T_P H_P M \to M \quad \text{by } f \otimes p \to f(p);$$
$$\sigma_N\colon\ N \to H_P T_P N \quad \text{by } n \to [p \to n \otimes p].$$

Following Miyashita [7], we say an R-module P is n-tilting provided

(i) there is an exact sequence $0 \to F_n \to \ldots \to F_0 \to P \to 0$ with F_i's finitely generated projective,

(ii) $\operatorname{Ext}_R^i(T,T) = 0$ for all $i \geqslant 1$, and

(iii) there is an exact sequence $0 \to R \to P_0 \to \ldots \to P_n \to 0$ with P_i's direct summands of finite direct sums of copies of P.

2. Estimates of global dimension

Denote by $\operatorname{Add}_R T$ the full subcategory of direct summands of sums of copies of T. Clearly $\operatorname{Add}_R R$ is just the full subcategory of all projective R-modules.

Lemma 2.1. *Let P be a $*^n$-module. Then every $M \in \operatorname{Pres}^n(P)$ has a resolution $\ldots \to P_1 \to P_0 \to M \to 0$, where $P_i \in \operatorname{Add}_R P$.*

P r o o f. Since P is a $*^n$-module, $\operatorname{Pres}^n(P) = \operatorname{Pres}^{n+1}(P)$. Hence we have an exact sequence $0 \to M_1 \to P^{(X_0)} \to M \to 0$ with $M_1 \in \operatorname{Pres}^n(P)$. Applying the same arguments to M_1 we obtain the conclusion. □

Let P be a $*^n$-module. For any $M \in \operatorname{Pres}^n(P)$, put P-res.dim$(M) = m$ (called P-resolution dimension of M) where m is the smallest integer such that there is an exact sequence in R-Mod: $0 \to P_m \to \ldots \to P_0 \to M \to 0$ with $P_i \in \operatorname{Add}_R P$ for $0 \leqslant i \leqslant m$. If there is no such integer m, put P-res.dim$(M) = \infty$. By Lemma 2.1, the definition of P-resolution dimension is consistent.

For a ring R, we denote by $\operatorname{pd}_R T$ the projective dimension of the R-module T and by $\operatorname{gd} R$ the global dimension of R.

Lemma 2.2. *Let P be a $*^n$-module. Then P-res.dim$(M) = \operatorname{pd}_A(H_P M)$ for any $M \in \operatorname{Pres}^n(P)$.*

P r o o f. Since P is selfsmall, we see that there is an equivalence

$$H_P : \operatorname{Add}_R P \rightleftharpoons \operatorname{Add}_A A : T_P.$$

Hence $H_P N \in \operatorname{Add}_A A$ for any $N \in \operatorname{Add}_R P$.

Let now $M \in \operatorname{Pres}^n(P)$. Assume that P-res.dim$(M) < \infty$, then we have an exact sequence

$$0 \to P_m \to \ldots \to P_0 \to M \to 0 \tag{1}$$

with $P_i \in \operatorname{Add}_R P$. Since $M \in \operatorname{Pres}^n(P)$ and P is a $*^n$-module, the sequence (1) is exact under the functor H_P. Hence we have the following exact sequence

$$0 \to A_m \to \ldots \to A_0 \to H_P M \to 0 \tag{2}$$

$A_i \in \operatorname{Add}_A A$. Therefore $\operatorname{pd}_A(H_P M) \leqslant m$.

Conversely, consider an exact sequence of the form (2). Since P is a $*^n$-module and $M \in \operatorname{Pres}^n(P)$, we have $H_P M \in \operatorname{Ker} T_P^{i \geqslant 1}$. Hence by applying the functor T_P to the exact sequence (2) we have an exact sequence of the form (1). Thus $m = P$-res.dim$(M) \leqslant \operatorname{pd}_A(H_P M)$. Combining the arguments above we conclude that P-res.dim$(M) = \operatorname{pd}_A(H_P M)$ for any $M \in \operatorname{Pres}^n(P)$. □

Proposition 2.3. *Assume P is a $*^n$-module with $A = \operatorname{End}_R P$ such that P_A has flat dimension $\leqslant t$. Let $d = \operatorname{pd}_R P$, $A_P = \{M \in \operatorname{Pres}^n(P)\colon M \notin \operatorname{Add}_R P$ and $\operatorname{pd}_R M \leqslant d\}$ and $D = \max\Big\{\Big(\sup\limits_{M\in A_P} P\text{-}\operatorname{res.dim}(M) - d\Big), 0\Big\}$. Then $\operatorname{gd} A \leqslant \operatorname{gd} R + D + t$.*

P r o o f. We may assume $d < \infty$ and $D < \infty$. For any $0 \neq N \in A$-mod, consider the following exact sequence

$$\text{(T)} \qquad 0 \to X \to A_{t-1} \to A_{t-2} \to \ldots \to A_0 \to N \to$$

with $A_i \in \operatorname{Add}_A A$ for $0 \leqslant i \leqslant t-1$. Since P_A has flat dimension $\leqslant t$, we see that $X \in \operatorname{Ker} T_P^{i\geqslant 1}$. Moreover, since ${}_RP$ is a $*^n$-module we have $X = H_P M$ for some $M \in \operatorname{Pres}^n(P)$.

We claim that $\operatorname{pd}_A X \leqslant D + \max\{d, \operatorname{pd}_R M\}$. We use the induction on $\operatorname{pd}_R M$ to prove the assertion. It's easy to check that the assertion holds if $M \in \operatorname{Add}_R P$ (since in this case $P\text{-}\operatorname{res.dim}(M) = 0$ and hence $\operatorname{pd}_A X = 0$ by Lemma 2.2). If $\operatorname{pd}_R M \leqslant d$, we have $\operatorname{pd}_A X = P\text{-}\operatorname{res.dim}(M) \leqslant \max\Big\{\sup\limits_{M\in A_P} P\text{-}\operatorname{res.dim}(M), d\Big\} = \max\Big\{\sup\limits_{M\in A_P} P\text{-}\operatorname{res.dim}(M) - d, 0\Big\} + d = D + \max\{d, \operatorname{pd}_R M\}$. Now consider the case $\operatorname{pd}_R M > d$. Note that we have an exact sequence $0 \to M_1 \to P_1 \to M \to 0$ with $M_1 \in \operatorname{Pres}^n(P)$ and $P_1 \in \operatorname{Add}_R P$. It follows that the sequence $0 \to H_P M_1 \to A_1' \to X \to 0$ is exact where $A_1' = H_P P_1 \in \operatorname{Add}_A A$. Note that $\operatorname{pd}_R M = \operatorname{pd}_R M_1 + 1$ since $\operatorname{pd}_R P = d < \operatorname{pd}_R M$, so the induction assumptions work and we have $\operatorname{pd}_A X = \operatorname{pd}_A(H_P M_1) + 1 \leqslant D + \max\{d, \operatorname{pd}_R M_1\} + 1 = D + \max\{d, \operatorname{pd}_R M\}$ (note that $\operatorname{pd}_R M_1 \geqslant d$).

From the above arguments we see that $\operatorname{pd}_A X \leqslant \operatorname{gd} R + D$. Using the exact sequence (T) we obtain that $\operatorname{gd} A \leqslant \operatorname{gd} R + D + t$. □

Definition 2.4. An R-module P is said to be almost n-tilting if P is a $*^n$-module such that $\operatorname{Pres}^n(P) \subseteq \operatorname{Ker} E_P^{i\geqslant 1}$.

When $n = 1$, the above notions coincide with the notion of almost-tilting module defined in [10].

Obviously, every tilting module of projective dimension $\leqslant n$ is almost n-tilting. The converse doesn't hold in general. We also note that, if P is a $*^n$-module and P is projective, then it's almost n-tilting. Therefore, every selfsmall projective R-module (hence always countably generated by the structure theorem for projective modules due to Kaplansky [6] with $A = \operatorname{End}_R P$ such that P_A has finite flat dimension is almost n-tilting for some integer n by [13]. Note also that, for many rings, including semiperfect or VN regular ones, selfsmall projective modules are finitely generated.

It is still an open question whether almost n-tilting modules, or selfsmall projective modules whose flat dimension is finite over its endomorphism ring, are finitely generated.

The following gives an example of almost n-tilting modules which are neither projective nor tilting.

Example 2.5. Let R be the algebra defined by the quiver $1 \to 2 \to 3 \to 4$. Let $P = {}^{3}_{2} \oplus {}^{4}_{\substack{3\\2}}$. Then $\operatorname{pd}_R P = 1$ and P is (2,1)-quasi-projective. $\operatorname{Gen}(P) = \left\{ {}^{4}_{\substack{3\\2}}, {}^{4}_{3}, {}^{3}_{2}, 3, 4 \right\}$ while $\operatorname{Pres}(P) = \left\{ {}^{4}_{\substack{3\\2}}, {}^{3}_{2}, 4 \right\} = \operatorname{Pres}^3(P) \subseteq \operatorname{Ker} E_P^{i\geqslant 1}$. Thus, P is an almost 2-tilting module. Since P is also semi-Σ-quasi-projective, P is a 2-quasi-progenerator which is not projective.

For an almost n-tilting module, we have the following.

Theorem 2.6. *Let P be an almost n-tilting R-module with $A = \operatorname{End}_R P$ such that P_A has flat dimension $\leqslant t$. Then $\operatorname{gd} A \leqslant \operatorname{gd} R + t$.*

Proof. Clearly we may assume that $d = \operatorname{pd}_R P < \infty$. By Theorem 2.3 it's sufficient to show that $D = \max\Big\{\Big(\sup\limits_{M \in A_P} P\text{-}\operatorname{res.dim}(M) - d\Big), 0\Big\} = 0$, where A_P is defined as in Theorem 2.3.

Take any $M \in A_P$. By Lemma 2.1 we have an exact sequence

$$\ldots \to^{f_{d+1}} P_{d+1} \to^{f_d} \ldots \to^{f_1} P_1 \to M \to 0$$

such that $M_i = \Im f_i \in \operatorname{Pres}^n(P)$ for $1 \leqslant i \leqslant d+1$, where $P_i \in \operatorname{Add}_R P$. Since P is almost n-tilting, we have $\operatorname{Pres}^n(P) \subseteq \operatorname{Ker} E_P^{i\geqslant 1}$ by Definition 2.4. Hence, we obtain that $\operatorname{Ext}^1_R(M_d, M_{d+1}) \cong \operatorname{Ext}^{d+1}_R(M, M_{d+1}) = 0$ by dimension shifting. Therefore, the exact sequence $0 \to M_{d+1} \to P_{d+1} \to M_d \to 0$ splits. It follows that $M_d \in \operatorname{Add}_R P$. By Lemma 2.2 we have $P\text{-}\operatorname{res.dim}(M) \leqslant d$. Thus $D = \max\Big\{\Big(\sup\limits_{M \in A_P} P\text{-}\operatorname{res.dim}(M) - d\Big), 0\Big\} = 0$. □

Corollary 2.7 [10]**.** *Let P be an almost tilting module. Then* $\operatorname{gd} A \leqslant \operatorname{gd} R + 1$.

Proof. Since P is $*$-module, we know that P_A is of flat dimension not more than 1, by [10]. Now apply Theorem 2.5. □

As mentioned before, every finitely generated projective module whose flat dimension is finite over its endomorphism ring is almost n-tilting for some n, so we have the following corollary.

Corollary 2.8. *Assume P is selfsmall and projective with $A = \operatorname{End}_R P$ such that P_A has flat dimension $\leqslant t$, then $\operatorname{gd} A \leqslant \operatorname{gd} R + t$. In particular, if P is a projective 2-quasi-progenerator, then $\operatorname{gd} A \leqslant \operatorname{gd} R$.*

Since endomorphism rings of finitely generated projective modules over VN regular rings are likewise VN regular, we have also the following corollary as a special case of the above result.

Corollary 2.9. *Let R be a* VN *regular ring and P be a finitely generated projective R-module with $A = \operatorname{End}_R P$. Then $\operatorname{gd} A \leqslant \operatorname{gd} R$.*

The following is another corollary of 2.8.

Corollary 2.10. *Let R be a commutative ring and P be a finitely generated projective R-module with $A = \operatorname{End}_R P$. Then $\operatorname{gd} A \leqslant \operatorname{gd} R$.*

Proof. If R is commutative and P is finitely generated projective, then P is a self-generator by [15, Theorem 3.1]. Thus P is a quasi-progenerator. Now the result follows from Corollary 2.8 or [10]. □

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