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SOME LIFTINGS OF POISSON STRUCTURES TO WEIL BUNDLES

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Abstract. We establish a formula for the Schouten-Nijenhuis bracket of linear liftings of skew-symmetric tensor fields to any Weil bundle. As a result we obtain a construction of some liftings of Poisson structures to Weil bundles.

Keywords: natural operator, product preserving bundle functor, Weil algebra, Poisson structure

MSC 2000: 58A32, 53D17

0. Instroduction

Liftings of skew-symmetric tensor fields of type $(p,0)$ to the tangent bundle were studied with application to the theory of Poisson structures, their symplectic foliations, canonical vector fields and Poisson-Lie groups in [4]. Next, a classification of linear liftings of skew-symmetric tensor fields of type $(2,0)$ to Weil bundles satisfying a special condition was given in [10], and another classification of linear liftings of skew-symmetric tensor fields of type $(p,0)$ for $p \geqslant 1$ to all Weil bundles in [1]. One may ask which of these last liftings for $p=2$ transform all Poisson structures to Poisson structures on Weil bundles. Our purpose is to answer this question.

For the convenience of the reader we summarize in the first two sections without proofs the relevant material on product preserving bundle functors (see [8] for the general theory of natural operations) and linear liftings of skew-symmetric tensor fields to Weil bundles (see [1]).

1. Product preserving bundle functors

In 1986 Eck [2], Kainz and Michor [5] and Luciano [9] proved independently that every product preserving bundle functor is equivalent to a Weil functor. New approach to this matter was presented by Kolář in [7]. In this section we give a brief sketch of the result following this last paper.

Let F be a functor which transforms each manifold M into a locally trivial bundle $\pi_M\colon FM \longrightarrow M$ and each smooth map $f\colon M \longrightarrow N$ into a smooth map $Ff\colon FM \longrightarrow FN$ such that $\pi_N \circ Ff = f \circ \pi_M$. We call F a bundle functor if for every non-negative integer n and every embedding $f\colon M \longrightarrow N$ between n-dimensional manifolds, Ff is an embedding and $Ff(FM) = \pi_N^{-1}(f(M))$. Hence we can identify FU with $\pi_M^{-1}(U)$, if U is an open subset of a manifold M. Such F is said to be product preserving if for all manifolds M and N the map $(Fp_M, Fp_N)\colon F(M \times N) \longrightarrow FM \times FN$, where $p_M\colon M \times N \longrightarrow M$ and $p_N\colon M \times N \longrightarrow N$ stand for the projections, is a diffeomorphism. Hence we can identify $F(M \times N)$ with $FM \times FN$.

A Weil algebra is, by definition, a finite dimensional associative and commutative $\mathbb{R}$-algebra A with unit, in which there exists an ideal N such that the factor algebra A/N is one-dimensional and $N^{r+1} = 0$ for an integer r. The smallest r with this property is called the order of A. The dimension of the $\mathbb{R}$-vector space N/N^2 is called the width of A. Basic examples of Weil algebras are the algebras $\mathbb{D}_k^r$ of r-jets at 0 of smooth functions $\mathbb{R}^k \longrightarrow \mathbb{R}$. If A is an arbitrary Weil algebra of order r and width k, then there is a surjective algebra homomorphism $\mathbb{D}_k^r \longrightarrow A$.

Let F be a product preserving bundle functor. Put $A = F\mathbb{R}$. Applying F to the addition and multiplication $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ on the field $\mathbb{R}$ we obtain an addition and multiplication $A \times A \longrightarrow A$ on A, so A is an algebra. In fact, it is a Weil algebra.

Conversely, let A be a Weil algebra of order r and width k and let $p\colon \mathbb{D}_k^r \longrightarrow A$ be a surjective algebra homomorphism. If M is a manifold then we say that two smooth maps $\gamma, \delta\colon \mathbb{R}^k \longrightarrow M$ determine the same A-jet if $p(j_0^r(\psi \circ \gamma)) = p(j_0^r(\psi \circ \delta))$ for every smooth function $\psi\colon M \longrightarrow \mathbb{R}$. We will denote by $j^A\gamma$ the A-jet of a smooth map $\gamma : \mathbb{R}^k \longrightarrow M$ and by T^AM the set of A-jets of all such maps. Since every chart $\varphi\colon U \longrightarrow \mathbb{R}^n$ on M induces the chart $T^AU \ni j^A\gamma \longrightarrow (p(j_0^r(\varphi^1 \circ \gamma)), \dots, p(j_0^r(\varphi^n \circ \gamma))) \in A^n$ on T^AM, T^AM is a manifold, and so a bundle over M with the projection $T^AM \ni j^A\gamma \longrightarrow \gamma(0) \in M$. If $f\colon M \longrightarrow N$ is a smooth map between manifolds then we define $T^Af\colon T^AM \longrightarrow T^AN$ by $T^Af(j^A\gamma) = j^A(f \circ \gamma)$. We call T^A the Weil functor induced by A. The Weil functor is a product preserving bundle functor. Though the construction of T^A depends on the choice of p, T^A is unique up to an equivalence. The equivalence of two bundle functors F and G means that there is a system of diffeomorphisms $e_M\colon FM \longrightarrow GM$ covering id_M for every manifold M

and satisfying $e_N \circ Ff = Gf \circ e_M$ for every smooth map $f\colon M \longrightarrow N$ between manifolds.

Therefore we can construct a Weil algebra for every product preserving bundle functor and a product preserving bundle functor for every Weil algebra. These constructions are inverse to each other provided isomorphic algebras and equivalent bundle functors are identified. Thus we have a correspondence between product preserving bundle functors and Weil algebras.

Hence basic examples of product preserving bundle functors are the Weil functors $T^{\mathbb{D}^r_k}$. They are the r-tangent bundle functors $T^r = T^{\mathbb{D}^r_1}$ used in higher order mechanics and the usual tangent bundle functor $T = T^{\mathbb{D}^1_1}$ among them.

2. Linear liftings of skew-symmetric tensor fields to Weil bundles

Let p be a positive integer. We will denote by $\operatorname{Te}^p M$ the $\mathbb{R}$-vector space of tensor fields of type $(p,0)$ on a manifold M. Let A be a Weil algebra and n a non-negative integer.

A lifting of tensor fields of type $(p,0)$ to T^A is, by definition, a system of maps $L_M\colon \operatorname{Te}^p M \longrightarrow \operatorname{Te}^p T^A M$ indexed by n-dimensional manifolds and satisfying for every embedding $f\colon M \longrightarrow N$ between such manifolds and all $t \in \operatorname{Te}^p M$, $u \in \operatorname{Te}^p N$ the implication

$$\bigotimes^p Tf \circ t = u \circ f \Longrightarrow \bigotimes^p TT^A f \circ L_M(t) = L_N(u) \circ T^A f.$$

Moreover, a lifting L of tensor fields of type $(p,0)$ to T^A is said to be linear if L_M is linear for every n-dimensional manifold M.

In [1] some liftings of this kind are constructed. We now give a sketch of the construction, as it will play a fundamental role in our paper.

We call a tensor $E \in \bigotimes^p A$, where A is treated as an $\mathbb{R}$-vector space, equivariant if

$$Z_i^C(E) = Z_j^C(E)$$

for every $C \in A$ and all $i, j \in \{1,\dots,p\}$, where $Z_k^C\colon \bigotimes^p A \longrightarrow \bigotimes^p A$ for $C \in A$ and $k \in \{1,\dots,p\}$ is the $\mathbb{R}$-linear map defined by

$$Z_k^C(X_1 \otimes \ldots \otimes X_p) = X_1 \otimes \ldots \otimes X_{k-1} \otimes CX_k \otimes X_{k+1} \otimes \ldots \otimes X_p.$$

Let us denote by $\operatorname{Eq}^p A$ the set of equivariant tensors. Equivariant tensors may be multiplied by elements of A. Indeed, there is $i \in \{1,\dots,p\}$ because $p \geqslant 1$ and it sufficies to put

$$CE = Z_i^C(E)$$

for $C \in A$ and $E \in \operatorname{Eq}^p A$. Since E is equivariant, it is immaterial which i we choose. Note that $\operatorname{Eq}^1 A = A$ and if $p = 1$, then this product coincides with that in the algebra A. It is also evident that $\operatorname{Eq}^p A$ is an A-module. In addition, the sets of symmetric equivariant tensors and skew-symmetric equivariant tensors, which will be denoted by $\operatorname{SyEq}^p A$ and $\operatorname{SkEq}^p A$ respectively, are submodules of $\operatorname{Eq}^p A$.

The map $A^n \ni X \longrightarrow X^i \otimes e_i \in A \otimes \mathbb{R}^n$, where $e_1, \dots, e_n$ stand for the standard basis of $\mathbb{R}^n$, enables us to identify A^n with $A \otimes \mathbb{R}^n$ and consequently $\bigotimes^p A^n$, where A^n is treated as an $\mathbb{R}$-vector space with $(\bigotimes^p A) \otimes (\bigotimes^p \mathbb{R}^n)$. Hence every $X \in \bigotimes^p A^n$ can be written as $X^{i_1 \dots i_p} \otimes e_{i_1} \otimes \dots \otimes e_{i_p}$, where $X^{i_1 \dots i_p} \in \bigotimes^p A$ for $i_1, \dots, i_p \in \{1, \dots, n\}$ are uniquely determined.

Let $E \in \operatorname{Eq}^p A$. Our purpose is to define an induced by E linear lifting $\overline{E}$ of tensor fields of type $(p, 0)$ to T^A. Since we can use charts on n-dimensional manifolds, we only need to define $\overline{E}_U$ for every open subset U of $\mathbb{R}^n$. Taking such U and $t \in \operatorname{Te}^p U$ we get $t^{i_1 \dots i_p} \colon U \longrightarrow \mathbb{R}$ and so $T^A t^{i_1 \dots i_p} \colon T^A U \longrightarrow A$ for $i_1, \dots, i_p \in \{1, \dots, n\}$. In order to complete the construction it sufficies to put

$$\overline{E}_U(t)(X) = (X, (T^A t^{i_1 \dots i_p}(X) E) \otimes e_{i_1} \otimes \dots \otimes e_{i_p})$$

for $X \in T^A U$, because $T^A U$ is an open subset of A^n, so $TT^A U$ can be interpreted as $T^A U \times A^n$ and consequently $\bigotimes^p TT^A U$ as $T^A U \times \bigotimes^p A^n$. A standard computation shows that the definition makes sense and $\overline{E}$ is really a linear lifting of tensor fields of type $(p, 0)$ to T^A.

Note that if $p = 1$, then $\overline{E}$ coincides with the composition of the flow operator lifting vector fields to T^A and the natural affinor on T^A induced by $E \in A$ (see [6]).

For every manifold M we will denote by $\operatorname{SyTe}^p M$ and $\operatorname{SkTe}^p M$ the subspaces of $\operatorname{Te}^p M$ consisting of symmetric tensor fields and skew-symmetric tensor fields respectively. It is easy to see that

$$\begin{aligned} E \in \operatorname{SyEq}^p A \Longrightarrow \overline{E}_M(\operatorname{SyTe}^p M) &\subset \operatorname{SyTe}^p T^A M, \\ \overline{E}_M(\operatorname{SkTe}^p M) &\subset \operatorname{SkTe}^p T^A M, \\ E \in \operatorname{SkEq}^p A \Longrightarrow \overline{E}_M(\operatorname{SyTe}^p M) &\subset \operatorname{SkTe}^p T^A M, \\ \overline{E}_M(\operatorname{SkTe}^p M) &\subset \operatorname{SyTe}^p T^A M. \end{aligned}$$

In our paper we will be concerned only with the case of skew-symmetric tensor fields.

A lifting of skew-symmetric tensor fields of type $(p, 0)$ to T^A is, by definition, a system of maps $L_M \colon \operatorname{SkTe}^p M \longrightarrow \operatorname{SkTe}^p T^A M$ indexed by n-dimensional manifolds and satisfying for every embedding $f \colon M \longrightarrow N$ between such manifolds and all

$t \in \operatorname{SkTe}^p M$, $u \in \operatorname{SkTe}^p N$ the implication

$$\bigwedge^p Tf \circ t = u \circ f \Longrightarrow \bigwedge^p TT^A f \circ L_M(t) = L_N(u) \circ T^A f.$$

Moreover, a lifting L of skew-symmetric tensor fields of type $(p,0)$ to T^A is said to be linear if L_M is linear for every n-dimensional manifold M.

Due to the above remark, if $E \in \operatorname{SyEq}^p A$, then taking $\overline{E}_M|_{\operatorname{SkTe}^p M}$ for every n-dimensional manifold M yields a linear lifting of skew-symmetric tensor fields of type $(p,0)$ to T^A. It will cause no confusion if we denote it also by $\overline{E}$. The crucial fact is that if $p \leqslant n$, then for every linear lifting L of skew-symmetric tensor fields of type $(p,0)$ to T^A there is a uniquely determined $E \in \operatorname{SyEq}^p A$ such that $L = \overline{E}$ (see [1] for a proof). Thus we have a one-to-one correspondence between linear liftings of skew-symmetric tensor fields to Weil bundles and symmetric equivariant tensors.

3. Multiplication of symmetric equivariant tensors

In this section we define multiplication of symmetric equivariant tensors which will be used in the next section for formulating a relation between linear liftings of skew-symmetric tensor fields to Weil bundles and the Schouten-Nijenhuis bracket.

Let p and q be positive integers.

For all $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,q\}$ we define an $\mathbb{R}$-bilinear map $V_{ij}\colon \bigotimes^p A \times \bigotimes^q A \longrightarrow \bigotimes^{p+q-1} A$ requiring

$$\begin{aligned} V_{ij}(X_1 \otimes \ldots \otimes X_p, Y_1 \otimes \ldots \otimes Y_q) = X_i Y_j \otimes X_1 \otimes \ldots \otimes X_{i-1} \otimes X_{i+1} \otimes \ldots \otimes X_p \\ \otimes Y_1 \otimes \ldots \otimes Y_{j-1} \otimes Y_{j+1} \otimes \ldots \otimes Y_q \end{aligned}$$

for all $X_1,\dots,X_p \in A$ and $Y_1,\dots,Y_q \in A$.

We first show that if $E \in \operatorname{SyEq}^p A$ and $F \in \operatorname{SyEq}^q A$, then $V_{ij}(E,F) = V_{kl}(E,F)$ for all $i,k \in \{1,\dots,p\}$ and $j,l \in \{1,\dots,q\}$. Of course, it sufficies to prove that $V_{ij}(E,F) = V_{11}(E,F)$ for all $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,q\}$. Let $P_\sigma\colon \bigotimes^r A \longrightarrow \bigotimes^r A$ for any positive integer r and $\sigma \in S_r$, where S_r is the group of permutations of $\{1,\dots,r\}$, be an $\mathbb{R}$-linear map defined by

$$P_\sigma(X_1 \otimes \ldots \otimes X_r) = X_{\sigma^{-1}(1)} \otimes \ldots \otimes X_{\sigma^{-1}(r)}$$

for $X_1, \ldots, X_r \in A$. Obviously, if $X \in \bigotimes^r A$, then X is symmetric if and only if $P_\sigma(X) = X$ for every $\sigma \in S_r$. Put

$$\tau = \begin{pmatrix} 1 & \ldots & i-1 & i & i+1 & \ldots & p \\ 2 & \ldots & i & 1 & i+1 & \ldots & p \end{pmatrix}$$

and

$$\upsilon = \begin{pmatrix} 1 & \ldots & j-1 & j & j+1 & \ldots & q \\ 2 & \ldots & j & 1 & j+1 & \ldots & q \end{pmatrix}.$$

It is easily seen that $V_{ij} = V_{11} \circ (P_\tau \times P_\upsilon)$. Since E and F are symmetric, we conclude that $V_{ij}(E,F) = V_{11}(P_\tau(E), P_\upsilon(F)) = V_{11}(E,F)$ as desired.

There are $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$, because $p \geqslant 1$ and $q \geqslant 1$, so we may define multiplication of symmetric equivariant tensors as follows.

Definition. Let $EF = V_{ij}(E,F)$ for $E \in \mathrm{SyEq}^p A$ and $F \in \mathrm{SyEq}^q A$.

By the above, it is immaterial which i and j we choose. Note that $\mathrm{SyEq}^1 A = A$ and if $p = 1$, then this product coincides with that in the A-module $\mathrm{SyEq}^q A$, because $V_{11}(X,Y) = Z_1^X(Y)$ for every $X \in A$ and every $Y \in \mathrm{SyEq}^q A$. We also have the following proposition.

Proposition. *If $E \in \mathrm{SyEq}^p A$ and $F \in \mathrm{SyEq}^q A$, then $EF \in \mathrm{SyEq}^{p+q-1} A$.*

Proof. We first show that EF is equivariant. Of course, it suffices to prove that $Z_k^C(V_{11}(E,F)) = Z_1^C(V_{11}(E,F))$ for every $C \in A$ and every $k \in \{2, \ldots, p+q-1\}$. We need to consider two cases.

If $k \in \{2, \ldots, p\}$, then it is easily seen that $Z_k^C \circ V_{11} = V_{11} \circ (Z_k^C \times \mathrm{id}_{\otimes^q A})$ and $Z_1^C \circ V_{11} = V_{11} \circ (Z_1^C \times \mathrm{id}_{\otimes^q A})$. Since E is equivariant, we conclude that $Z_k^C(V_{11}(E,F)) = V_{11}(Z_k^C(E), F) = V_{11}(Z_1^C(E), F) = Z_1^C(V_{11}(E,F))$ as desired.

If $k \in \{p+1, \ldots, p+q-1\}$, then it is easily seen that $Z_k^C \circ V_{11} = V_{11} \circ (\mathrm{id}_{\otimes^p A} \times Z_{k-p+1}^C)$ and $Z_1^C \circ V_{11} = V_{11} \circ (\mathrm{id}_{\otimes^p A} \times Z_1^C)$. Since F is equivariant, we conclude that $Z_k^C(V_{11}(E,F)) = V_{11}(E, Z_{k-p+1}^C(F)) = V_{11}(E, Z_1^C(F)) = Z_1^C(V_{11}(E,F))$ as desired.

We next show that EF is symmetric. Of course, it suffices to prove that $P_{(1,k)}(V_{11}(E,F)) = V_{11}(E,F)$ for every $k \in \{2, \ldots, p+q-1\}$, which is due to the fact that the group S_{p+q-1} is generated by the transpositions $(1,k)$ with $k \in \{2, \ldots, p+q-1\}$. We need to consider two cases.

If $k \in \{2, \ldots, p\}$, then for all $Y_2, \ldots, Y_q \in A$ we define an $\mathbb{R}$-linear map $N^{Y_2 \ldots Y_q} \colon \bigotimes^p A \longrightarrow \bigotimes^{p+q-1} A$ by $N^{Y_2 \ldots Y_q}(X_1 \otimes \ldots \otimes X_p) = X_1 \otimes \ldots \otimes X_p \otimes Y_2 \otimes \ldots \otimes Y_q$ for all $X_1, \ldots, X_p \in A$, and for every $X \in \bigotimes^p A$ two $\mathbb{R}$-linear maps $R^X \colon \bigotimes^q A \longrightarrow \bigotimes^{p+q-1} A$ and $S^X \colon \bigotimes^q A \longrightarrow \bigotimes^{p+q-1} A$ by $R^X(Y_1 \otimes \ldots \otimes Y_q) = N^{Y_2 \ldots Y_q}(Z_k^{Y_1}(X))$

and $S^X(Y_1 \otimes \ldots \otimes Y_q) = N^{Y_2 \ldots Y_q}(Z_1^{Y_1}(X))$ for all $Y_1, \ldots, Y_q \in A$. Clearly, $R^E = S^E$, because E is equivariant. It is easily seen that $R^X(Y) = P_{(1,k)}(V_{11}(P_{(1,k)}(X), Y))$ and $S^X(Y) = V_{11}(X, Y)$ for all $X \in \bigotimes^p A$ and $Y \in \bigotimes^q A$. Since E is symmetric, we conclude that $P_{(1,k)}(V_{11}(E, F)) = P_{(1,k)}(V_{11}(P_{(1,k)}(E), F)) = R^E(F) = S^E(F) = V_{11}(E, F)$ as desired.

If $k \in \{p+1, \ldots, p+q-1\}$, then for all $X_2, \ldots, X_p \in A$ we define an $\mathbb{R}$-linear map $N^{X_2 \ldots X_p} \colon \bigotimes^q A \longrightarrow \bigotimes^{p+q-1} A$ by $N^{X_2 \ldots X_p}(Y_1 \otimes \ldots \otimes Y_q) = Y_1 \otimes X_2 \otimes \ldots \otimes X_p \otimes Y_2 \otimes \ldots \otimes Y_q$ for all $Y_1, \ldots, Y_q \in A$, and for every $Y \in \bigotimes^q A$ two $\mathbb{R}$-linear maps $R^Y \colon \bigotimes^p A \longrightarrow \bigotimes^{p+q-1} A$ and $S^Y \colon \bigotimes^p A \longrightarrow \bigotimes^{p+q-1} A$ by $R^Y(X_1 \otimes \ldots \otimes X_p) = N^{X_2 \ldots X_p}(Z_{k-p+1}^{X_1}(Y))$ and $S^Y(X_1 \otimes \ldots \otimes X_p) = N^{X_2 \ldots X_p}(Z_1^{X_1}(Y))$ for all $X_1, \ldots, X_p \in A$. Clearly, $R^F = S^F$, because F is equivariant. It is easily seen that $R^Y(X) = P_{(1,k)}(V_{11}(X, P_{(1,k-p+1)}(Y)))$ and $S^Y(X) = V_{11}(X, Y)$ for all $X \in \bigotimes^p A$ and $Y \in \bigotimes^q A$. Since F is symmetric, we conclude that $P_{(1,k)}(V_{11}(E, F)) = P_{(1,k)}(V_{11}(E, P_{(1,k-p+1)}(F))) = R^F(E) = S^F(E) = V_{11}(E, F)$ as desired. This proves the proposition. $\square$

Example. We wish to express explicitly the multiplication defined above in the simplest case, namely for the basic Weil algebras $\mathbb{D}_k^r$ of r-jets at 0 of smooth functions $\mathbb{R}^k \longrightarrow \mathbb{R}$.

We first consider the case $k = 1$. It is known that

$$D_i^s = \sum_{\substack{\alpha_1, \ldots, \alpha_s \in \{0, \ldots, r\} \\ \alpha_1 + \ldots + \alpha_s = i}} (j_0^r \operatorname{id}_{\mathbb{R}})^{\alpha_1} \otimes \ldots \otimes (j_0^r \operatorname{id}_{\mathbb{R}})^{\alpha_s}$$

for $i \in \{(s-1)r, \ldots, sr\}$ form a basis of the $\mathbb{R}$-vector space $\operatorname{SyEq}^s \mathbb{D}_1^r$ for every positive integer s (see [1]). A trivial verification shows that

$$D_i^p D_j^q = \begin{cases} D_{i+j}^{p+q-1}, & \text{if } i+j \leqslant (p+q-1)r, \\ 0, & \text{if } i+j > (p+q-1)r \end{cases}$$

for all $i \in \{(p-1)r, \ldots, pr\}$ and $j \in \{(q-1)r, \ldots, qr\}$.

We now turn to the case $k \geqslant 2$ and $r \geqslant 1$. Then $\operatorname{Eq}^s \mathbb{D}_k^r = N^r \otimes \ldots \otimes N^r$ for every integer $s \geqslant 2$, where N is the ideal of $\mathbb{D}_k^r$ such that $\mathbb{D}_k^r / N$ is one-dimensional and $N^{r+1} = 0$ (see [1]). Let $E \in \operatorname{SyEq}^p \mathbb{D}_k^r$ and $F \in \operatorname{SyEq}^q \mathbb{D}_k^r$. It is immediate that if $p \geqslant 2$ and $q \geqslant 2$, then $EF = 0$. If $p = 1$ and $q \geqslant 2$, then $EF = \pi_{\mathbb{R}}(E)F$ and if $p \geqslant 2$ and $q = 1$, then $EF = \pi_{\mathbb{R}}(F)E$, where $\pi_{\mathbb{R}} : \mathbb{D}_k^r = T^{\mathbb{D}_k^r} \mathbb{R} \longrightarrow \mathbb{R}$ is the projection. Finally, if $p = 1$ and $q = 1$, then EF coincides with the usual product in $\mathbb{D}_k^r$.

The remainder of this section is devoted to the study of some properties of multiplication of symmetric equivariant tensors. Though we will not need these properties, they seem quite interesting from the algebraic point of view.

It is evident that $V_{11}(V_{11}(X,Y),Z) = V_{11}(X, V_{11}(Y,Z))$ for all $X \in \bigotimes^i A$, $Y \in \bigotimes^j A$, $Z \in \bigotimes^k A$, where i, j, k are positive integers, which yields $(EF)G = E(FG)$ for all $E \in \operatorname{SyEq}^i A$, $F \in \operatorname{SyEq}^j A$, $G \in \operatorname{SyEq}^k A$.

For any positive integers i and j put

$$\chi = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & i+j-1 \\ 1 & j+1 & \dots & i+j-1 & 2 & \dots & j \end{pmatrix}.$$

It is evident that $P_\chi(V_{11}(X,Y)) = V_{11}(Y,X)$ for all $X \in \bigotimes^i A$ and $Y \in \bigotimes^j A$, which implies $EF = FE$ for all $E \in \operatorname{SyEq}^i A$ and $F \in \operatorname{SyEq}^j A$, because EF is symmetric.

By the above, $(CE)F = C(EF)$ for all $C \in A$, $E \in \operatorname{SyEq}^i A$ and $F \in \operatorname{SyEq}^j A$, where i and j are positive integers. Combining this with $EF = FE$ we see that the map $\operatorname{SyEq}^i A \times \operatorname{SyEq}^j A \ni (E,F) \longrightarrow EF \in \operatorname{SyEq}^{i+j-1} A$ is A-bilinear. Hence if we write $\operatorname{SyEq} A = \bigoplus_{i=1}^{\infty} \operatorname{SyEq}^i A$, then there is a unique A-bilinear extension $\operatorname{SyEq} A \times \operatorname{SyEq} A \longrightarrow \operatorname{SyEq} A$ of all such maps. Taking this extension for a multiplication on $\operatorname{SyEq} A$ and summing up, we have the following proposition.

Proposition. $\operatorname{SyEq} A$ *is an associative and commutative A-algebra with unit.*

4. Linear liftings of skew-symmetric tensor fields to Weil bundles and the Schouten-Nijenhuis bracket

We begin this section by recalling the notion of the Schouten-Nijenhuis bracket which is a generalization of the usual Lie bracket of vector fields. Let p and q be positive integers and M a manifold. Then there is a unique $\mathbb{R}$-bilinear map $\operatorname{SkTe}^p M \times \operatorname{SkTe}^q M \ni (t,u) \longrightarrow [t,u] \in \operatorname{SkTe}^{p+q-1} M$ satisfying

$$\begin{aligned}
&[t_1 \wedge \dots \wedge t_p, u_1 \wedge \dots \wedge u_q] \\
&\qquad = (-1)^{p+1} \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [t_i, u_j] \wedge t_1 \wedge \dots \wedge t_{i-1} \wedge t_{i+1} \wedge \dots \wedge t_p \\
&\qquad\qquad \wedge u_1 \wedge \dots \wedge u_{j-1} \wedge u_{j+1} \wedge \dots \wedge u_q
\end{aligned}$$

for all $t_1, \dots, t_p, u_1, \dots, u_q \in \operatorname{Te}^1 M$ and such that if U is an open subset of M, $t, t' \in \operatorname{SkTe}^p M$ are such that $t|_U = t'|_U$ and $u, u' \in \operatorname{SkTe}^q M$ are such that $u|_U = u'|_U$, then $[t,u]|_U = [t',u']|_U$ (see [11] for a proof of existence of this map). This map is called the Schouten-Nijenhuis bracket.

We can now formulate our main result.

Theorem. *If p and q are positive integers, $E \in \operatorname{SyEq}^p A$ and $F \in \operatorname{SyEq}^q A$, then*

$$[\overline{E}_M(t), \overline{F}_M(u)] = \overline{EF}_M([t,u])$$

for every n-dimensional manifold M, all $t \in \operatorname{SkTe}^p M$ and $u \in \operatorname{SkTe}^q M$.

Proof. It is known that the theorem holds for $p = 1$ and $q = 1$, namely $[\overline{C}_M(v), \overline{D}_M(w)] = \overline{CD}_M([v,w])$ for all $C, D \in A$ and $v, w \in \operatorname{Te}^1 M$ (see [3]). Combining this with the definitions of the Schouten-Nijenhuis bracket and multiplication of symmetric equivariant tensors we see that the proof will be completed as soon as we can show the following lemma. $\square$

Lemma. *If r is a positive integer, B a finite set and $G_{b1}, \ldots, G_{br} \in A$ for $b \in B$ are such that $\sum\limits_{b \in B} G_{b1} \otimes \ldots \otimes G_{br} \in \operatorname{SyEq}^r A$, then*

$$\left(\overline{\sum_{b \in B} G_{b1} \otimes \ldots \otimes G_{br}}\right)_M (s_1 \wedge \ldots \wedge s_r) = \sum_{b \in B} \overline{G_{b1}}_M(s_1) \wedge \ldots \wedge \overline{G_{br}}_M(s_r)$$

for every n-dimensional manifold M and all $s_1, \ldots, s_r \in \operatorname{Te}^1 M$.

Proof. Since we can use charts on n-dimensional manifolds, we only need to prove the lemma for every open subset U of $\mathbb{R}^n$ instead of M. The left-hand side of the required equality at any $X \in T^A U$ is

$$\left(X, \left(T^A\left(\frac{1}{r!}\sum_{\sigma \in S_r} \operatorname{sgn}\sigma s^{i_1}_{\sigma^{-1}(1)} \ldots s^{i_r}_{\sigma^{-1}(r)}\right)(X) \sum_{b \in B} G_{b1} \otimes \ldots \otimes G_{br}\right) \otimes e_{i_1} \otimes \ldots \otimes e_{i_r}\right).$$

Since the addition and multiplication on $A = T^A \mathbb{R}$ are obtained by applying T^A to those on $\mathbb{R}$ and since $\sum\limits_{b \in B} G_{b1} \otimes \ldots \otimes G_{br}$ is symmetric, this may be rewritten as

$$\left(X, \left(\frac{1}{r!}\sum_{\sigma \in S_r} \operatorname{sgn}\sigma T^A s^{i_1}_{\sigma^{-1}(1)}(X) \ldots T^A s^{i_r}_{\sigma^{-1}(r)}(X) \right.\right.$$
$$\left.\left. \times \sum_{b \in B} G_{b\sigma^{-1}(1)} \otimes \ldots \otimes G_{b\sigma^{-1}(r)}\right) \otimes e_{i_1} \otimes \ldots \otimes e_{i_r}\right),$$

then, by the definition of multiplication on the A-module $\operatorname{SyEq}^r A$, as

$$\left(X, \sum_{b \in B} \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}\sigma (T^A s^{i_1}_{\sigma^{-1}(1)}(X) G_{b\sigma^{-1}(1)}) \otimes \ldots \right.$$
$$\left. \otimes (T^A s^{i_r}_{\sigma^{-1}(r)}(X) G_{b\sigma^{-1}(r)}) \otimes e_{i_1} \otimes \ldots \otimes e_{i_r}\right),$$

and finally, by our identification of $(\bigotimes^r A)\otimes(\bigotimes^r \mathbb{R}^n)$ with $\bigotimes^r(A\otimes\mathbb{R}^n)$, as

$$\Big(X, \sum_{b\in B}\frac{1}{r!}\sum_{\sigma\in S_r}\operatorname{sgn}\sigma((T^A s^{i_1}_{\sigma^{-1}(1)}(X)G_{b\sigma^{-1}(1)})\otimes e_{i_1})\otimes\dots \\ \otimes((T^A s^{i_r}_{\sigma^{-1}(r)}(X)G_{b\sigma^{-1}(r)})\otimes e_{i_r})\Big)$$

which is the right-hand side of the required equality at X. □

5. Application to liftings of Poisson structures to Weil bundles

If M is a manifold, then $t\in\operatorname{SkTe}^2 M$ is called a Poisson structure on M if $[t,t]=0$ (see [11]). We will denote by $\operatorname{Po} M$ the set of Poisson structures on M.

A lifting of Poisson structures to T^A is, by definition, a system of maps $L_M\colon \operatorname{Po} M\longrightarrow \operatorname{Po} T^A M$ indexed by n-dimensional manifolds and satisfying for every embedding $f\colon M\longrightarrow N$ between such manifolds and all $t\in\operatorname{Po} M$, $u\in\operatorname{Po} N$ the implication

$$\bigwedge^2 Tf\circ t=u\circ f\Longrightarrow \bigwedge^2 TT^Af\circ L_M(t)=L_N(u)\circ T^Af.$$

Let $E\in\operatorname{SyEq}^2 A$. If M is an n-dimensional manifold and $t\in\operatorname{Po} M$, then our theorem gives $[\overline{E}_M(t),\overline{E}_M(t)]=\overline{E^2}_M([t,t])=0$, so $\overline{E}_M(t)\in\operatorname{Po} T^A M$. Hence taking $\overline{E}_M|_{\operatorname{Po} M}$ for every n-dimensional manifold M yields a lifting of Poisson structures to T^A. It will cause no confusion if we denote it also by $\overline{E}$. We thus have the following corollary.

Corollary. *Each $E\in\operatorname{SyEq}^2 A$ induces a lifting $\overline{E}$ of Poisson structures to T^A.*

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