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ON THE EXISTENCE OF MULTIPLE SOLUTIONS FOR A NONLOCAL BVP WITH VECTOR-VALUED RESPONSE

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Abstract. The existence of positive solutions for a nonlocal boundary-value problem with vector-valued response is investigated. We develop duality and variational principles for this problem. Our variational approach enables us to approximate solutions and give a measure of a duality gap between the primal and dual functional for minimizing sequences.

Keywords: nonlocal boundary-value problems, positive solutions, duality method, variational method

MSC 2000: 34B18

1. Introduction

We are dealing with the nonlinear problem of the form

$$x''(t) + G_x(t, x(t)) = 0 \quad \text{a.e. in } (0,1). \tag{1}$$

Our aim is to answer the question when the above differential equation possesses a positive solution $x\colon [0,1] \to \mathbb{R}^n$ such that

$$x(0) = 0 \tag{2}$$

and the non-local boundary condition

$$x(1) = \int_\alpha^\beta x(s)\,\mathrm{d}g(s) \tag{3}$$

hold, where $\alpha,\ \beta \in (0,1)$, $g = (g_1,\dots,g_n)\colon\ [0,1] \longrightarrow \mathbb{R}^n$ and $\int_\alpha^\beta x(s)\,\mathrm{d}g(s) = [\int_\alpha^\beta x_i(s)\,\mathrm{d}g_i(s)]_{i\in\{1,\dots,n\}}$. Speaking precisely, (1)–(2)–(3) is the system of n BVPs

$$\begin{cases} x_i''(t) + G_{x_i}(t,x(t)) = 0 & \text{a.e. in } [0,1],\\ x_i(0) = 0 & \text{for } i = 1,\dots,n,\\ x_i(1) = \int_\alpha^\beta x_i(s)\,\mathrm{d}g_i(s) \end{cases}$$

with the integrals $\int_\alpha^\beta x_i(s)\,\mathrm{d}g_i(s)$ meant in the sense of Riemann-Stieltjes.

Investigation of similar problems was initiated by V. Il'in and E. Moiseev (see [12], [13]) who studied the existence of solutions for (1) being an equation of Sturm-Liouville type with (2) and the multi-point condition

$$x'(1) = \sum_{i=1}^{m} a_i x'(\zeta_i) \tag{4}$$

where $a_1,\dots,a_m \in \mathbb{R}$ have the same sign. These results were motivated by [2] and [3] (due to Bitsadze and Bitsadze and Samarskii). Since then more general multi-point BVPs have been studied, among others by C. Gupta ([7], [9], [10]), C. Gupta, S. K. Ntouyas and P. Ch. Tsamatos ([8]), R. Ma ([19]) and R. Ma and N. Castaneda ([20]). In [21] R. Ma presents the extension of Erbe's and Wang's results for two-point BVPs and his own results for three-point BVPs. That paper is devoted to the existence of positive solutions for the m-point boundary value problem

$$\begin{cases} u'' + h(t)f(u) = 0 \quad \text{for } 0 < t < 1,\\ \alpha u(0) - \beta u'(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i),\\ \gamma u(1) - \delta u'(1) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where the constants $\alpha,\beta,\gamma,\delta \geqslant 0$ and functions $\varphi(t) = \beta + \alpha t$, $\varphi(t) = \gamma + \delta - \gamma t$ for $t \in [0,1]$, $\xi_i \in (0,1)$, $a_i, b_i \in (0,\infty)$ for $i \in \{1,\dots,m-2\}$ satisfy some additional assumptions.

The problem like (1)–(2) with the non-local boundary condition

$$x'(1) = \int_{t_0}^{1} x'(s)\,\mathrm{d}g(s) \tag{5}$$

is widely discussed, among others, in [15], where the one-dimensional case is considered, G has the special form $G_x(t,x) = q(t)f(x)$ for some functions $q\colon\ [0,1] \to [0,\infty)$, $f\colon\ \mathbb{R} \to \mathbb{R}$, with q, f being continuous, f—nonnegative for $x \geqslant 0$ and such that

$\sup\limits_{x\in[0,v]} \{f(x)\} \leqslant \theta v$ for some positive v and θ. The last inequality is the continuous version of (4). Moreover, (5) becomes (4) when g is a piece-wise constant, increasing function having a finitely many jumps. The above works present topological approach and use methods associated with the fixed point theorem in cones due to Krasnoselskii (see [18]). Many of these papers are devoted to the problem containing restrictions on the slope of the solution (see, e.g. [14]–[16]), where rather mild conditions lead to a positive integral operator. We have also discussed (1)–(2) with a nonlocal condition similar to (3) with x' instead of x (see [24]). Now we want to study a more difficult situation concerning the case when the restrictions are made on the solutions themselves. Then the integral condition (3) does not give, in general, the positivity of the corresponding integral operator. Our investigations are justified by the large number of papers associated with similar problems, among others, [1], [4], [5], [6], [11], [15], [16], [21], [27]. This work is motivated mainly by [17]. The approach presented here is based on methods of calculus of variations which is of great importance in many disciplines of science and the starting point for various approximate numerical schemes such as Ritz, finite difference, and finite element methods. In this article some numerical results are also presented. Speaking precisely, our approach enables us to numerically characterize approximate solutions and give, also in the superlinear case, a measure of a duality gap between the primal and dual functional for minimizing sequences.

We shall use the following notation:

(n1): *for all* $x, y \in \mathbb{R}^n$ *we say that* $x \geqslant y$ *if* $x_i - y_i > 0$ *for* $i = 1, \ldots, n$; $n > 0$ *is a given integer number*;

(n2): *for all* $x, y \in \mathbb{R}^n$ *by* xy *we mean the vector* $[x_i y_i]_{i=1,\ldots,n}$ *and by* $\langle x, y\rangle$ *the scalar product of* x *and* y: $\langle x, y\rangle := \sum\limits_{i=1}^{n} x_i y_i$;

(n3): *for all* $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ *and* $0 < w$ *we define the sets*

$$P_{0,w} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ 0 < x_i < w_i \textit{ for all } i = 1, \ldots, n\}$$

and

$$\overline{P}_{0,w} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ 0 \leqslant x_i \leqslant w_i \text{ for all } i = 1, \ldots, n\};$$

(n4): A *denotes the space of absolutely continuous functions* $x\colon\ [0, T] \to \mathbb{R}^n$ *with* $x' \in L^2([0, T], \mathbb{R}^n)$;

(n5): A_0 *is the subspace of* A *consisting of all* $x \in A$ *such that* $x(0) = 0$ *with the norm* $\|x\|_{A_0} = (\int_0^1 |x'(t)|^2\, \mathrm{d}t)^{1/2}$;

(n6): $A_{0b} := \{x \in A_0$, *such that* x *satisfies* (3)$\}$;

and assumptions:

(g): $g=(g_1,\ldots,g_n)\colon [0,1]\longrightarrow\mathbb{R}^n$, *for all* $i=1,\ldots,n$ g_i *are increasing functions such that* $g_i(\alpha)=0$ *and* $1-\beta g_i(\beta)>0$, $\bigl[1-\int_\alpha^\beta t\,\mathrm{d}g_i(t)\bigr]>0$*;*

(G1): $G(t,\cdot)$ *is convex and continuously differentiable in* $\overline{P}_{0,w}$ *for a certain* w *given as in* (n3) *and for a.a.* $t\in[0,1]$, $G(\cdot,x)$ *is measurable in* $[0,1]$ *for all* $x\in\overline{P}_{0,w}$*;*

(G2): $G_x(t,\cdot)$ *is nonnegative in* $\overline{P}_{0,w}$ *for a.a.* $t\in[0,1]$*;*

(G3): $\int_0^1 G_x(t,0)\,\mathrm{d}t\neq 0$, $-\infty<\int_0^1 G(t,0)\,\mathrm{d}t$, $\int_0^1 G(t,w)\,\mathrm{d}t<+\infty$.

Throughout the paper we shall assume conditions (g), (G1)–(G3). We would like to stress that because of (G1) and (G2), $x_j\mapsto G(t,(x_1,\ldots,x_j,\ldots x_n))$ and $x_j\mapsto G_{x_i}(t,(x_1,\ldots,x_j,\ldots x_n))$ for each $i=1,\ldots,n$ and $j=1,\ldots,n$, $t\in[0,1]$, are increasing functions if $(x_1,\ldots,x_j,\ldots x_n)$ lies in $\overline{P}_{0,w}$.

We consider the general case when G satisfies hypotheses (G1)–(G3), so that our assumptions are not strong enough to use the results discussed above: $n\geqslant 1$, t and x are not separated in the nonlinearity, $G_x(\cdot,x)$ is measurable only, $G_x(t,\cdot)$ is not, in general, quiet at infinity. We are also able to omit the condition $g(t_0+)>0$. It turns out that these weaker assumptions are still sufficient to conclude the existence of solutions for (1).

Since we will propose an approach based on variational methods, we treat our equation as the Euler-Lagrange equation for the integral functional $J\colon A_0\to\mathbb{R}$ of the form

$$J(x)=\int_0^T\Bigl(-G(t,x(t))+\frac{1}{2}\left|x'(t)\right|^2\Bigr)\,\mathrm{d}t-\langle x(1),x'(1)\rangle\,. \tag{6}$$

However, we believe that our paper may contribute some new look at this problem. This is because we propose to study (1)–(2)–(3) by duality methods in a way, to some extent, analogous to the methods developed in sublinear cases [22], [23]. Since the functional (6) is, in general, unbounded in A_0 (especially in the superlinear case), it is obvious that we must look for critical points of J of "*minmax*" type. The main difficulties which appear here are: what kind of sets we should choose over which we wish to calculate "*minmax*" of J and then to link this value with critical points of J. The methods based on the mountain pass theorems, the saddle point theorems (see e.g. [22], [25], [28]), the Morse theory, cannot be applied directly to find critical points of J, because of the boundary condition (3). So we develop a duality theory which relates the infimum of the energy functional associated with the problem on a special set X, to the infimum of the dual functional on a corresponding set X^d. The links between the minimizers of the two functionals give a variational principle and, in consequence, their relation to our BVP. We also present the numerical version of the variational principle.

Remark 1.1. We are interested in the solvability of (1)–(2)–(3) in the set

$$\overline{X} = \{x \in A_{0b} \colon x' \text{ belongs to } A, x(t) \geqslant 0 \\ \text{for all } t \in [0,1] \text{ and } x''(t) \leqslant 0 \text{ a.e. on } (0,1)\}.$$

Before we use the variational approach, we have to prove some auxiliary results which come from the topological methods based on any fixed point theorems. Namely, we are looking for an operator $\mathbf{A}$ defined on $\overline{X}$ with the property that $\mathbf{A}$ maps $\overline{X}$ into $\overline{X}$ and (1)–(2)–(3) is equivalent to the problem of existence of a fixed point of $\mathbf{A}$ in $\overline{X}$. We will construct the operator $\mathbf{A}$ in a way similar to that used in [17]. Thus we calculate from (1)

$$-x'(t) = \int_0^t G_x(s, x(s))\,\mathrm{d}s - x'(0)$$

and further

$$x(t) = x(1) + \int_t^1 \int_0^r G_x(s, x(s))\,\mathrm{d}s\,\mathrm{d}r - (d_x + x(1))(1-t),$$

where $d_x := \int_0^1 (1-s) G_x(s, x(s))\,\mathrm{d}s$ and $x'(0) = d_x + x(1)$. Integrating by parts, we arrive at

$$\begin{aligned}\int_t^1 \int_0^r G_x(s, x(s))\,\mathrm{d}s\,\mathrm{d}r &= \Big[r \int_0^r G_x(s, x(s))\,\mathrm{d}s\Big]_t^1 - \int_t^1 rG_x(r, x(r))\,\mathrm{d}r \\ &= \int_0^1 G_x(s, x(s))\,\mathrm{d}s - t\int_0^t G_x(s, x(s))\,\mathrm{d}s \\ &\quad - \int_0^1 rG_x(r, x(r))\,\mathrm{d}r + \int_0^t rG_x(r, x(r))\,\mathrm{d}r \\ &= \int_0^1 (1-s) G_x(s, x(s))\,\mathrm{d}s - \int_0^t (t-s) G_x(s, x(s))\,\mathrm{d}s \\ &= d_x - \int_0^t (t-s) G_x(s, x(s))\,\mathrm{d}s,\end{aligned}$$

so that

$$x(t) = tx(1) + d_x t - \int_0^t (t-s) G_x(s, x(s))\,\mathrm{d}s. \tag{7}$$

Taking into account (3) we obtain

$$x(1) = x(1)c + cd_x - \int_\alpha^\beta \int_0^t (t-s) G_x(s, x(s))\,\mathrm{d}s\,\mathrm{d}g(t)$$

and

$$x(1) = bcd_x - b\int_\alpha^\beta\int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s\,\mathrm{d}g(t) \tag{8}$$

with $c := \left[\int_\alpha^\beta t\,\mathrm{d}g_i(t)\right]_{i=1,\dots,n}$ and $b := [1-c_i]^{-1}_{i=1,\dots,n}$. Substituting (8) into (7) yields

$$\begin{aligned}
x(t) &= tbcd_x - tb\int_\alpha^\beta\int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s\,\mathrm{d}g(t)\\
&\quad + t\int_0^1 (1-s)G_x(s,x(s))\,\mathrm{d}s - \int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s\\
&= tb\int_0^1 (1-s)G_x(s,x(s))\,\mathrm{d}s - tb\int_\alpha^\beta \int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s\,\mathrm{d}g(t)\\
&\quad - \int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s.
\end{aligned}$$

On account of the above consideration it is easy to show that the existence of the fixed point of $\mathbf{A}$ defined in the set $\overline{X}$ by

$$\begin{aligned}
\mathbf{A}x(t) &= tb\int_0^1 (1-s)G_x(s,x(s))\,\mathrm{d}s - tb\int_\alpha^\beta\int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s\,\mathrm{d}g(t)\\
&\quad - \int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s
\end{aligned}$$

is equivalent to the solvability of our problem.

Definition 1.2. *We say that a nonempty set $X \subset \overline{X}$ has property* (P) *if $\mathbf{A}X \subset X$, namely for each $x \in X$ there exists $w \in X$ such that $w = Ax$.*

We need the following auxiliary lemma, which is a direct consequence of Lemma 2.3 from [17]:

Lemma 1.3. *If $x \in C^0(I,\mathbb{R}^n)$, $x(0) = 0$ and each component $x_i\colon [0,1] \to \mathbb{R}$, $i = \{1,\dots,n\}$, is a concave function satisfying the condition*

$$x_i(1) = \int_\alpha^\beta x_i(s)\,\mathrm{d}g_i(s),$$

where $\alpha,\beta \in (0,1)$, $g_i\colon [0,1] \longrightarrow \mathbb{R}$ are increasing functions with $1-\beta g_i(\beta) > 0$, then there exists $\mu > 0$ with the following property: for all $i = 1,\dots,n$ we have

$$x_i(t) \geqslant \mu\|x_i\|_{C^0([0,1],\mathbb{R})}$$

for $t \in [\alpha, 1]$ *with* $\mu := \min\{\gamma, 1-\beta, (\beta-\alpha)\gamma g_1(\beta), \ldots, (\beta-\alpha)\gamma g_n(\beta)\}$, $\gamma := \min\{\alpha, 1-\beta, (1-\beta)(1-\alpha)^{-1}\}$, *and*

$$x_i(t) \geqslant 0$$

for all $t \in [0, 1]$.

Lemma 1.4. *If* $x \in \overline{X}$ *and* $x(t) \in [0, w]$ *for* $t \in [0, 1]$, *then* $\mathbf{A}x \in \overline{X}$.

Proof. Indeed, fix $x \in \overline{X}$ such that $x(t) \in [0, w]$ for $t \in [0, 1]$. Thus, by the definition of operator $\mathbf{A}$, we obtain that $\mathbf{A}x$ belongs to A and satisfies conditions (2) and (3), so that $x \in A_{0b}$. Moreover,

$$(\mathbf{A}x)''(t) = -G_x(t, x(t)) \leqslant 0$$

for a.a. $t \in [0, 1]$, which means that each component of $\mathbf{A}x$ is a concave function. Now applying Lemma 1.3 to each component of $\mathbf{A}x$ we infer $\mathbf{A}x(t) \geqslant 0$ for all $t \in [0, 1]$. Moreover, the definition of $\mathbf{A}$ gives

$$\begin{aligned}(\mathbf{A}x)'(t) = b\int_0^1 (1-s)G_x(s, x(s))\,\mathrm{d}s - b\int_\alpha^\beta \int_0^t (t-s)G_x(s, x(s))\,\mathrm{d}s\,\mathrm{d}g(t)\\ - \int_0^t G_x(s, x(s))\,\mathrm{d}s\end{aligned}$$

which implies (together with the assumption $x(t) \in [0, w]$) that $(\mathbf{A}x)'$ belongs to A. Summarizing: $\mathbf{A}x \in A_{0b}$, $(\mathbf{A}x)' \in A$, $\mathbf{A}x(t) \geqslant 0$ and $(\mathbf{A}x)''(t) \leqslant 0$ for a.a. $t \in [0, 1]$, so that $\mathbf{A}x \in \overline{X}$. □

Remark 1.5. Let us note that, in the general case described by hypotheses (G1)–(G3), J is not necessarily bounded in $\overline{X}$.

Our plans are to prove that in each set $X \subset \overline{X}$ satisfying the conditions

MH1 : X *has the property* (P);

MH2 : *for each* $x \in X$: $x(t) \in \overline{P}_{0,w}$ for all $t \in [0, 1]$;

the functional J possesses at least one minimizer which is a solution of our problem. We shall describe a sequence of disjoint subsets of $\overline{X}$ for which, as we will show in the last section, conditions (MH1)–(MH2) are valid. This will imply the existence of multiple distinguished solutions. To this effect let us assume in Sections 2, 3, 4 and 5 that

MH: *there exists a nonempty subset* X *of* $\overline{X}$ *for which* (MH1)–(MH2) hold.

In the last part of this paper we impose some additional conditions on G, which guarantee the existence of such X.

In the sequel X is a fixed set satisfying (MH1)–(MH2).

2. Duality results

In this section we shall develop the duality which describes the relations between the critical value of J and the infimum of the dual functional $J_D\colon X^d \longrightarrow \mathbb{R}$ given by

$$J_D(p) = -\int_0^1 \frac{1}{2}\left|p(t)\right|^2 + \int_0^1 G^*(t, -p'(t))\,\mathrm{d}t, \tag{9}$$

where G^* is the Fenchel conjugate of G with respect to the second variable and

$$X^d = \{p \in A\colon \text{ there exists } x \in X \text{ such that } p(t) = x'(t) \text{ for } t \in [0,1]\}.$$

Now we formulate a remark which describes the links between the elements of the sets X and X^d and follows directly from the definition of $\mathbf{A}$ and X:

Remark 2.1. Under condition (MH1), for each $x \in X$ there exists $p \in X^d$ such that $p'(\cdot) = -G_x(\cdot, x(\cdot))$ and therefore

$$\int_0^1 \langle -p'(t), x(t)\rangle\,\mathrm{d}t - \int_0^1 G^*(t, -p'(t))\,\mathrm{d}t = \int_0^1 G(t, x(t))\,\mathrm{d}t.$$

To describe the duality we need a kind of perturbation $J_x\colon L^2([0,1],\mathbb{R}^n)\times\mathbb{R}^n \to \mathbb{R}$ of J and convexity of the function considered on a whole space. Thus let us define for each $x \in X$ the perturbation of J by

$$\begin{aligned} J_x(y,a) &= \int_0^1 \left(\widetilde{G}(t, x(t)+y(t)) - \frac{1}{2}\left|x'(t)\right|^2\right)\mathrm{d}t + \langle x(1)-a, x'(1)\rangle \\ &= \int_0^1 \left(\widetilde{G}(t, x(t)+y(t)) - \frac{1}{2}\left|x'(t)\right|^2\right)\mathrm{d}t + \langle x(1), x'(1)\rangle - \langle a, x'(1)\rangle, \end{aligned}$$

with

$$\widetilde{G}(t,x) = \begin{cases} G(t,x) & \text{if } x \in \overline{P}_{0,w},\ t \in [0,T], \\ +\infty & \text{if } x \notin \overline{P}_{0,w},\ t \in [0,T]. \end{cases}$$

We need this notation only for the purpose of duality and we will not change the notation for the functional J containing G or $\widetilde{G}$ for $y \in L^2([0,1],\mathbb{R}^n)$, $a \in \mathbb{R}^n$. This follows from the fact that our investigation reduces to the set X and for all $x \in X$ we have $x(t) \in \overline{P}_{0,w}$ on $[0,1]$.

For $x \in X$ and $p \in X^d$ we define a type of a conjugate of J_x by

$$\begin{aligned} J_x^{\#}(p) &= \sup_{y\in L^2, a\in\mathbb{R}^n}\left(\int_0^1 \langle y(t), p'(t)\rangle\,\mathrm{d}t + \langle p(1), a\rangle - J_x(y,a)\right) \\ &= \sup_{y\in L^2}\left\{\int_0^1 \langle y(t), p'(t)\rangle\,\mathrm{d}t - \int_0^1 \widetilde{G}(t, x(t)+y(t))\,\mathrm{d}t\right\} \\ &\quad + \sup_{a\in\mathbb{R}^n}\{\langle a, p(1)\rangle + \langle a, x'(1)\rangle\} + \int_0^1 \frac{1}{2}\left|x'(t)\right|^2\,\mathrm{d}t - \langle x(1), x'(1)\rangle. \end{aligned} \tag{10}$$

From the definition of Fenchel's conjugate G^* of $\widetilde{G}$ with respect to the second variable, we get

$$
\begin{aligned}
(11)\qquad J_x^{\#}(p) &= -\int_0^1 \langle x(t), p'(t)\rangle \,\mathrm{d}t + \frac{1}{2}\int_0^1 |x'(t)|^2 \,\mathrm{d}t \\
&\quad + \int_0^1 G^*(t, p'(t))\,\mathrm{d}t - \langle x(1), x'(1)\rangle + l(x'(1) + p(1)) \\
&= \int_0^1 \langle x'(t), p(t)\rangle \,\mathrm{d}t - \langle x(1), p(1)\rangle + \frac{1}{2}\int_0^1 |x'(t)|^2 \,\mathrm{d}t \\
&\quad + \int_0^1 G^*(t, p'(t))\,\mathrm{d}t - \langle x(1), x'(1)\rangle + l(x'(1) + p(1)),
\end{aligned}
$$

where $l\colon \mathbb{R}^n \to \{0, +\infty\}$

$$
l(b) = \begin{cases} 0 & \text{for } b = 0, \\ +\infty & \text{for } b \neq 0. \end{cases}
$$

Now we prove two auxiliary equalities. First of them is obtained as "min" from $J_x^{\#}(p)$ with respect to $x \in X$. The other one is a consequence of calculation of "min" of $J_x^{\#}(p)$ over the set X^d. To this effect we use Fenchel's conjugate, but the main difficulty which appears here is: how to calculate the conjugate with respect to nonlinear spaces X and X^d. We need some trick based upon the special structure of the sets X and X^d. First we observe that for each $p \in X^d$ there exists $x_p \in X$ such that

$$
\int_0^1 \langle x_p'(t), p(t)\rangle \,\mathrm{d}t - \int_0^1 \frac{1}{2}\left|x_p'(t)\right|^2 \,\mathrm{d}t = \int_0^1 \frac{1}{2k(t)} |p(t)|^2 \,\mathrm{d}t
$$

and further

$$
\begin{aligned}
&\int_0^1 \langle x_p'(t), p(t)\rangle \,\mathrm{d}t - \int_0^1 \frac{1}{2}\left|x_p'(t)\right|^2 \,\mathrm{d}t \\
&\quad \leqslant \sup_{x \in \{z \in X, p(1) = z'(1)\}} \left\{ \int_0^1 \langle x'(t), p(t)\rangle \,\mathrm{d}t - \int_0^1 \frac{1}{2} |x(t)'|^2 \,\mathrm{d}t \right\} \\
&\quad \leqslant \sup_{x \in X} \left\{ \int_0^1 \langle x'(t), p(t)\rangle \,\mathrm{d}t - \int_0^1 \frac{1}{2} |x(t)'|^2 \,\mathrm{d}t \right\} \\
&\quad \leqslant \sup_{x' \in L^2} \left\{ \int_0^1 \langle x'(t), p(t)\rangle \,\mathrm{d}t - \int_0^1 \frac{1}{2} |x'(t)|^2 \,\mathrm{d}t \right\} = \int_0^1 \frac{1}{2} |p(t)|^2 \,\mathrm{d}t.
\end{aligned}
$$

So all inequalities above are equalities. Finally, we infer for all $p \in X^d$ that

$$
\begin{aligned}
\sup_{x \in X} \left(-J_x^{\#}(-p)\right) &= \sup_{x \in X} \left\{ \int_0^1 \langle x'(t), p(t)\rangle \,\mathrm{d}t - \frac{1}{2}\int_0^1 |x'(t)|^2 \,\mathrm{d}t \right. \\
&\quad \left. - \int_0^1 G^*(t, -p'(t))\,\mathrm{d}t - \langle x(1), p(1)\rangle + \langle x(1), x'(1)\rangle - l(x'(1) - p(1)) \right\}.
\end{aligned}
$$

Therefore

$$\sup_{x\in X}\left(-J_x^{\#}(-p)\right) = -\int_0^1 G^*(t,-p'(t))\,\mathrm{d}t + \int_0^1 \frac{1}{2}\,|p(t)|^2\;\mathrm{d}t = -J_D(p). \tag{12}$$

The other assertion, which is necessary in the proof of the duality principle, has the following form: for all $x \in X$,

$$\sup_{p\in X^d}\left(-J_x^{\#}(-p)\right) \leqslant -J(x). \tag{13}$$

To obtain this fix $x \in X$. Then Remark 2.1 leads to the existence of $\bar{p} \in X^d$ such that

$$\int_0^1 \langle -\bar{p}'(t), x(t)\rangle\;\mathrm{d}t - \int_0^1 G^*(t,-\bar{p}'(t))\,\mathrm{d}t = \int_0^1 \widetilde{G}(t,x(t))\,\mathrm{d}t.$$

Because of the definition of G^* the chain of relations

$$\begin{aligned}
&\int_0^1 \langle -\bar{p}'(t), x(t)\rangle\;\mathrm{d}t - \int_0^1 G^*(t,-\bar{p}'(t))\,\mathrm{d}t\\
&\quad\leqslant \sup_{p\in X^d}\left\{\int_0^1 \langle -p'(t), x(t)\rangle\;\mathrm{d}t - \int_0^1 G^*(t,-p'(t))\,\mathrm{d}t\right\}\\
&\quad= \sup_{p'\in L^2}\left\{\int_0^1 \langle -p'(t), x(t)\rangle\;\mathrm{d}t - \int_0^1 G^*(t,-p'(t))\,\mathrm{d}t\right\} = \int_0^1 \widetilde{G}(t,x(t))\,\mathrm{d}t,
\end{aligned}$$

holds and finally

$$\begin{aligned}
\sup_{p\in X^d}\left(-J_x^{\#}(-p)\right) &= \sup_{p\in X^d}\left\{\int_0^1 \langle x(t), -p'(t)\rangle\;\mathrm{d}t - \int_0^1 G^*(t,-p'(t))\,\mathrm{d}t\right.\\
&\quad\left. - l(x'(1)-p(1))\right\} + \langle x(1), x'(1)\rangle\\
&\quad - \frac{1}{2}\int_0^1 |x'(t)|^2\;\mathrm{d}t \leqslant \sup_{p\in X^d}\{-l(x'(1)-p(1))\}\\
&\quad + \sup_{p\in X^d}\left\{\int_0^1 \langle x(t), -p'(t)\rangle\;\mathrm{d}t - \int_0^1 G^*(t,-p'(t))\,\mathrm{d}t\right\}\\
&\quad + \langle x(1), x'(1)\rangle - \frac{1}{2}\int_0^1 |x'(t)|^2\;\mathrm{d}t = -\int_0^1 (-\widetilde{G}(t,x(t))\\
&\quad + \frac{1}{2}\,|x'(t)|^2)\,\mathrm{d}t + \langle x(1), x'(1)\rangle = -J(x),
\end{aligned}$$

which is our claim.

As a consequence of assertions (13) and (12) we infer

$$\begin{aligned}\inf_{x\in X} J(x) &= -\sup_{x\in X}\left(-J(x)\right) \leqslant -\sup_{x\in X}\sup_{p\in X^d}\left(-J_x^{\#}(-p)\right) = -\sup_{p\in X^d}\sup_{x\in X}\left(-J_x^{\#}(-p)\right)\\ &= -\sup_{p\in X^d}\left(-J_D(p)\right) = \inf_{p\in X^d} J_D(p).\end{aligned}$$

So we obtain the duality principle in the form of an inequality:

Theorem 2.2. *For functionals J and J_D we have the duality relation*

$$\inf_{x\in X} J(x) \leqslant \inf_{p\in X^d} J_D(p). \tag{14}$$

3. Necessary conditions

Denote by $\partial J_x(y)$ the subdifferential of J_x. In particular,

$$\begin{aligned}\partial J_x(0) = \Big\{p' \in L^2\left([0,1],\mathbb{R}^n\right)\colon \int_0^1 G^*(t,p'(t))\,\mathrm{d}t + \int_0^1 \widetilde{G}(t,x(t))\,\mathrm{d}t\\ = \int_0^1 \langle p'(t),x(t)\rangle\ \mathrm{d}t\Big\}.\end{aligned} \tag{15}$$

The next result formulates a variational principle for “min” arguments.

Theorem 3.1. *Let $\bar{x} \in X$ be such that $J(\bar{x}) = \inf_{x\in X} J(x)$. Then there exists $\bar{p} \in X^d$ with $\bar{p}(t) = \bar{p}(1) - \int_t^1 \bar{p}'(s)\,\mathrm{d}s$, where $\bar{p}(1) = \bar{x}'(1)$ and $-\bar{p}' \in \partial J_{\bar{x}}(0)$, such that $\bar{p}$ satisfies*

$$J_D(\bar{p}) = \inf_{p\in X^d} J_D(p).$$

Furthermore,

$$J_{\bar{x}}(0) + J_{\bar{x}}^{\#}(-\bar{p}) = 0, \tag{16}$$

$$J_D(\bar{p}) - J_{\bar{x}}^{\#}(-\bar{p}) = 0. \tag{17}$$

P r o o f. By Theorem 2.2, to prove the first assertion it suffices to show that $J(\bar{x}) \geqslant J_D(\bar{p})$. Using Remark 2.1 we can observe that for $\bar{x}$ there exists $\bar{p} \in X^d$ such that $\bar{p}'(t) = -G_x(t,\bar{x}(t))$ a.e. on $(0,1)$, which implies

$$\int_0^1 \langle -\bar{p}'(t),\bar{x}(t)\rangle\ \mathrm{d}t - \int_0^1 G^*(t,-\bar{p}'(t))\,\mathrm{d}t = \int_0^1 \widetilde{G}(t,\bar{x}(t))\,\mathrm{d}t. \tag{18}$$

Combining (18) and (15) we get the inclusion $-\bar{p}' \in \partial J_{\bar{x}}(0)$. Thus $J^*_{\bar{x}}(-\bar{p}') = J^{\#}_{\bar{x}}(-\bar{p})$ (where $J^*_{\bar{x}}(-\bar{p}')$ denotes the Fenchel transform of $J_{\bar{x}}$ at $-\bar{p}'$) gives (16). It follows that

$$-J(\bar{x}) = -J^{\#}_{\bar{x}}(-\bar{p}) \leqslant \sup_{x \in X}(-J^{\#}_{x}(-\bar{p})) = -J_D(\bar{p}),$$

where the last equality is due to (12). Hence $J(\bar{x}) \geqslant J_D(\bar{p})$ and so $J_D(\bar{p}) = J(\bar{x}) = \inf\limits_{x \in X} J(x) = \inf\limits_{p \in X^d} J_D(p)$. (17) is a simple consequence of (16) and the chain of equalities $J_{\bar{x}}(0) = -J(\bar{x}) = -J_D(\bar{p})$. □

Corollary 3.2. *Let $\bar{x} \in X$ be a minimizer of J on X: $J(\bar{x}) = \inf\limits_{x \in X} J(x)$. Then there exists $\bar{p} \in X^d$ such that the pair $(\bar{x}, \bar{p})$ satisfies the relations*

$$-\bar{p}'(t) = G_x(t, \bar{x}(t)), \tag{19}$$

$$\bar{p}(t) = \bar{x}'(t), \tag{20}$$

$$J_D(\bar{p}) = \inf_{p \in X^d} J_D(p) = \inf_{x \in X} J(x) = J(\bar{x}). \tag{21}$$

P r o o f. Equalities (16) and (17) imply

$$\int_0^1 G(t, \bar{x}(t))\,\mathrm{d}t + \int_0^1 G^*(t, -\bar{p}'(t))\,\mathrm{d}t - \int_0^1 \langle \bar{x}(t), -\bar{p}'(t)\rangle\,\mathrm{d}t = 0,$$
$$\int_0^1 \frac{1}{2}\left|\bar{p}(t)\right|^2\,\mathrm{d}t + \int_0^1 \frac{1}{2}\left|\bar{x}'(t)\right|^2\,\mathrm{d}t - \int_0^1 \langle \bar{x}'(t), \bar{p}(t)\rangle\,\mathrm{d}t = 0$$

and further (19) and (20). Relations (21) are a direct consequence of Theorem 3.1 and Theorem 2.2. □

4. Variational principle for minimizing sequences

In this section we show that a statement similar to Theorem 3.1 is true for minimizing sequences of J and J_D.

Theorem 4.1. *Let $\{x_m\}_{m \in \mathbb{N}}$, $x_m \in X$, $m = 1, 2, \ldots$, be a minimizing sequence of J and let*

$$+\infty > \inf_{m \in \mathbb{N}} J(x_m) = a > -\infty.$$

Then there exists $\{p_m\}_{m \in \mathbb{N}} \subset X^d$ with $p_m(t) = p_m(1) - \int_t^1 p'_m(s)\,\mathrm{d}s$, $p_m(1) = x'_m(1)$ and $-p'_m \in \partial J_{x_m}(0)$ such that $\{p_m\}_{m \in \mathbb{N}}$ is a minimizing sequence for J_D, i.e.

$$\inf_{x \in X} J(x) = \inf_{m \in \mathbb{N}} J(x_m) = \inf_{m \in \mathbb{N}} J_D(p_m) = \inf_{p \in X^d} J_D(p). \tag{22}$$

Furthermore,

$$J_{x_m}(0) + J^{\#}_{x_m}(-p_m) = 0,$$

$$J_D(p_m) - J^{\#}_{x_m}(-p_m) \leqslant \varepsilon, \tag{23}$$

$$0 \leqslant J(x_m) - J_D(p_m) \leqslant \varepsilon \tag{24}$$

for a given $\varepsilon > 0$ and sufficiently large m.

Proof. We have that $\infty > \inf_{m\in\mathbb{N}} J(x_m) = a > -\infty$, and therefore for a given $\varepsilon > 0$ there exists m_0 such that $J(x_m) - a < \varepsilon$ for all $m \geqslant m_0$. Further the proof is similar to that of Theorem 2.2, so we only sketch it. First we observe that there exists $p_m \in X^d$ such that $p'_m(t) = -G_x(t, x_m(t))$ a.e. on $(0,1)$, which implies for all $m \in \mathbb{N}$

$$\int_0^1 \widetilde{G}(t, x_m(t))\,\mathrm{d}t = \int_0^1 -G^*(t, -p'_m(t))\,\mathrm{d}t - \int_0^1 < x_m(t), p'_m(t) > \,\mathrm{d}t.$$

Hence, we get the inclusion $-p'_m \in \partial J_{x_m}(0)$ and the inequality

$$-J(x_m) \leqslant -J_D(p_m)$$

which together with Theorem 2.2 implies (22). Again by Theorem 2.2

$$J_D(p_m) + \varepsilon \geqslant J(x_m) \quad \text{for} \quad m \geqslant m_0,$$

so that (24) holds.

Since $-p'_m \in \partial J_{x_m}(0)$ we infer $J_{x_m}(0) + J^{\#}_{x_m}(-p_m) = 0$ for all $m \in \mathbb{N}$. (23) follows from the following two facts: $J_{x_m}(0) = -J(x_m) = -J^{\#}_{x_m}(-p_m)$ and $\inf_{x\in X} J(x) = \inf_{m\in\mathbb{N}} J_D(p_m) = a$. $\square$

A direct consequence of this theorem is the following corollary.

Corollary 4.2. *Let $\{x_m\}_{m\in\mathbb{N}} \subset X$ be a minimizing sequence for J and let*

$$+\infty > \inf_{m\in\mathbb{N}} J(x_m) = a > -\infty.$$

Then there exists a minimizing sequence $\{p_m\}_{m\in\mathbb{N}} \subset X^d$ for J_D such that

$$-p'_m(t) = G_x(t, x_m(t)) \tag{25}$$

and $p_m(t) = p_m(1) - \int_t^1 p'_m(s)\,\mathrm{d}s$, $p_m(1) = x'_m(1)$ for all $m \in \mathbb{N}$. Moreover,

$$\lim_{m\to\infty} \int_0^1 \Big(\frac{1}{2}\left|p_m(t)\right|^2 + \frac{1}{2}\left|x'_m(t)\right|^2 - \langle x'_m(t), p_m(t)\rangle\Big)\,\mathrm{d}t = 0. \tag{26}$$

5. Existence of a solution for the non-local BVP

The last problem which we have to solve is to prove the existence of $\bar{x} \in \overline{X}$ satisfying (1)–(2)–(3).

Theorem 5.1. *Under hypotheses* (G1)–(G3) *and* (MH1)–(MH2) *there exists a solution* $\bar{x} \in \overline{X}$ *of* (1)–(2)–(3) *such that*

$$\inf_{x \in X} J(x) \geqslant J(\bar{x}). \tag{27}$$

Proof. We start with the observation that (G1), (G3) and (MH2) yield the following estimate: for all $x \in X$ we have

$$J(x) \geqslant \frac{1}{2}\|x'\|^2_{L^2([0,1],\mathbb{R}^n)} \,\mathrm{d}t - \int_0^1 G(t,w)\,\mathrm{d}t - \|w\|_{\mathbb{R}^n}\,\|x'\|_{L^2([0,1],\mathbb{R}^n)}, \tag{28}$$

which leads to the boundedness of J on X. Let us consider the sets $S_z = \{x \in X, J(x) \leqslant z\}$ with $z \in \mathbb{R}$.It is clear that S_z is nonempty for sufficiently large z, so let us choose a minimizing sequence $\{x_m\}_{m\in\mathbb{N}}$ for J from S_z. Taking into account (28) we obtain the boundedness of $\{x_m\}_{m\in\mathbb{N}}$ with respect to the norm $\|x'\|_{L^2([0,1],\mathbb{R}^n)}$ and, consequently, by the definition of $\|\cdot\|_{A_0}$, in A_{0b}. This implies that (going if necessary to a subsequence) $\{x_m\}_{m\in\mathbb{N}}$ converges weakly to a certain element $\bar{x} \in A_{0b}$ and further, $x_m \underset{m\to\infty}{\to} \bar{x}$ (uniformly). Thus $0 \leqslant \bar{x}$ on $[0,1]$. Our task is now to show that $\bar{x}' \in A$ and $\bar{x}'' \leqslant 0$ a.e. on $(0,1)$. By (28) we know that $\inf\limits_{x\in X} J(x) > \infty$, so we can apply Corollary 4.2 which gives the existence of $\{p_m\}_{m\in\mathbb{N}} \subset X^d$ such that

$$p'_m(t) = -G_x(t, x_m(t)), \text{ for a.e. } t \in (0,1) \tag{29}$$

and

$$\lim_{m\to\infty} \int_0^1 \Big(\frac{1}{2}\,|p_m(t)|^2 + \frac{1}{2}\,|x'_m(t)|^2 - \langle x'_m(t), p_m(t)\rangle\Big)\,\mathrm{d}t = 0. \tag{30}$$

The first assertion leads to the pointwise convergence of $\{p'_m\}_{m\in\mathbb{N}}$:

$$\lim_{n\to\infty} p'_m(t) = \lim_{n\to\infty} -G_x(t, x_m(t)) = -\lim_{n\to\infty} G_x(t, \bar{x}(t)) \tag{31}$$

and the boundedness of $\{p'_m\}_{m\in\mathbb{N}}$ in the $L^1([0,1],\mathbb{R}^n)$ norm. Moreover, by (28) and the fact that $\{x_m\}_{m\in\mathbb{N}} \subset S_z$ we know that $\{x'_m(1)\}_{m\in\mathbb{N}}$ is bounded and therefore we may assume (up to a subsequence) that it is convergent. On account of the above reasoning the sequence $\{p_m\}_{m\in\mathbb{N}} \subset A$ given by $p_m(t) = p_m(1) - \int_t^1 p'_m(s)\,\mathrm{d}s$, where $p_m(1) = x'_m(1)$, tends uniformly to a certain $\bar{p}(t) = \bar{p}(1) - \int_t^1 \bar{p}'(s)\,\mathrm{d}s$ such that

$$\bar{p}'(t) = -G_x(t, \bar{x}(t)).$$

Due to (30) and the facts that $p_m \underset{m\to\infty}{\to} \bar{p}$ uniformly and $x'_m \underset{m\to\infty}{\rightharpoonup} \bar{x}'$ in $L^2([0,1],\mathbb{R}^n)$ we have

$$\begin{aligned}
0 &= \lim_{m\to\infty} \left[\int_0^1 \left(\frac{1}{2}\left|p_m(t)\right|^2 + \frac{1}{2}\left|x'_m(t)\right|^2 \right) \mathrm{d}t - \int_0^1 \langle x'_m(t), p_m(t)\rangle\, \mathrm{d}t \right] \\
&= \int_0^1 \frac{1}{2}\left|\bar{p}(t)\right|^2 + \lim_{m\to\infty} \left[\int_0^1 \frac{1}{2}\left|x'_m(t)\right|^2 \,\mathrm{d}t \right] - \int_0^1 \langle \bar{x}'(t), \bar{p}(t)\rangle\, \mathrm{d}t \\
&\geqslant \int_0^T \frac{1}{2}\left|\bar{p}(t)\right|^2 \,\mathrm{d}t + \int_0^T \frac{1}{2}\left|\bar{x}'(t)\right|^2 \,\mathrm{d}t - \int_0^T \langle \bar{x}'(t), \bar{p}(t)\rangle\, \mathrm{d}t.
\end{aligned}$$

In view of the property of the Fenchel inequality all inequalities above are equalities. Finally, we infer

$$\bar{p}(t) = \bar{x}'(t).$$

Thus, $\bar{x}' \in A$ and $\bar{x}'' = \bar{p}' = -G_x(\cdot, \bar{x}(\cdot)) \leqslant 0$ a.e. on $(0,1)$, which we have claimed.

The last assertion is a simple consequence of the fact that the functional J is weakly lower semicontinuous in A_{0b} and thus also in $\overline{X}$. Therefore

$$\inf_{x\in X} J(x) = \lim_{m\to\infty} J(x_m) = \liminf m \to \infty J(x_m) \geqslant J(\bar{x}).$$

□

6. Existence of a sequence of solutions

6.1. Existence of a solution in a bounded set. In this section we present the existence result for $P_{0,w}$, where w is a certain element of $\mathbb{R}^n$ such that $w > 0$, and for G satisfying the system of conditions

(G1′)–(G3′): (G1–G3) *are valid for* $P_{0,w}$;

(G4) *there exist* $d, e \in \mathbb{R}^n$ such that $d < w$ and $eb < \mathbf{2}$, *(b was given in Section 1)* $\mathbf{2} = (2,\ldots,2) \in \mathbb{R}^n$ *and*

$$\int_0^1 G_x(s,d)\,\mathrm{d}s \leqslant ed.$$

As a consequence of (G4) we infer

Lemma 6.1. *If $x \in P_{0,w}$ and $x \leqslant d$ then $\mathbf{A}x \leqslant d$.*

P r o o f. Taking into account the monotonicity of G_{x_i} and condition (G4) we have

$$(Ax(t)) \leqslant b \int_0^1 (1-s) G_x(s, x(s))\,\mathrm{d}s \leqslant b \int_0^1 (1-s) G_x(s,d)\,\mathrm{d}s$$
$$\leqslant bed \int_0^1 (1-s)\,\mathrm{d}s \leqslant \frac{1}{2} bed \leqslant d,$$

so that $Ax(t) \leqslant d$. □

We define X_1 to be

$$X_1 = \left\{ x \in \overline{X}, x(t) \leqslant d \text{ for all } t \in [0,1] \right\}.$$

Lemmas 1.4 and 6.1 lead to the following results

Lemma 6.2. *For X_1 given above* (MH1) *and* (MH2) *hold.*

Theorem 6.3. *Under assumptions* (g), (G1′)–(G3′), (G4) *there exists a solution of our problem belonging to X_1 and being a minimizer of J on X_1.*

P r o o f. Application of Theorem 4.1 to the set X_1 yields the existence of $\bar{x} \in \overline{X}$ satisfying (1)–(2)–(3) and the inequality

$$\inf_{x \in X_1} J(x) \geqslant J(\bar{x}).$$

So to prove our claim it suffices to show that $\bar{x} \in X_1$. Taking into account the definition of $\bar{x}$ we have that $x_m \underset{m\to\infty}{\longrightarrow} \bar{x}$ (uniformly), where $\{x_m\}_{m\in\mathbb{N}} \subset X_1$ is described in the proof of Theorem 5.1, so that $\bar{x}(t) \leqslant d$ for all $t \in [0,1]$ and, consequently, $\bar{x} \in X_1$. □

6.2. Existence of multiple solutions.

Our task is now to show the existence of other solutions of (1)–(2)–(3). Therefore for each $m \in K$, where K is a certain subset of $\mathbb{N}$, we shall construct a set denoted by X^m such that $X^m \cap X^l = \emptyset$ for $m \neq l$, $m, l \in K$, including a solution to (1)–(2)–(3). It is clear that we need some additional information concerning the properties of G on a sequence of polyhedra $\left\{\overline{P}_{0,w^m}\right\}_{m\in K}$, so that we impose

(G1″)–(G3″): (G1)–(G3) *are valid for* $\overline{P}_{0,w^m}$ for each $m \in K$;

(G5) *there exist* $\{\lambda^m\}_{m\in K} \subset \mathbb{R}_+$, $\{e^m\}_{m\in K}, \{\underline{d}^m\}_{m\in K}, \{\bar{d}^m\}_{m\in K} \subset \mathbb{R}^n$ *such that for each* $m \in K$

a:

$$0 \leqslant \underline{d}^m < \bar{d}^m < \underline{d}^{m+1} \leqslant w^m,$$

b:

$$be^m \leqslant \mathbf{2},$$

c:

$$\int_0^1 G_x(s, \bar{d}^m)\, \mathrm{d}s \leqslant e\bar{d}^m,$$

d:

$$\frac{(1-\beta)^2 \alpha}{2\,(1-\beta g_i(\beta))} \lambda_i^m > 1 \quad \textit{for each } i = 1, \ldots, n,$$

e:

$$G_x(t, \mu \underline{d}^m) \geqslant \lambda^m \underline{d}^m \quad \textit{for all } t \in [\alpha, 1].$$

Now we shall construct a sequence of disjoint sets X^m satisfying assumptions (MH1) and (MH2). Let us put for all $m \in K$

$$\begin{aligned} X^m := \{x \in \overline{X},\ \text{there exists } t_0 \in (0,1)\ &\text{ such that } \underline{d}^m \leqslant x(t_0) \\ &\text{ and } x(t) \leqslant \bar{d}^m \text{ for all } t \in [0,1]\}. \end{aligned}$$

We will show that $\mathbf{A}X^m \subset X^m$.

Lemma 6.4. $\mathbf{A}X^m \subset X^m$ *for all* $m \in K$.

P r o o f. For a given $m \in K$ fix $x \in X^m$. By virtue of conditions (G5b–c), an analysis similar to that in the proof of Lemma 6.1 leads to the conclusion that

$$\mathbf{A}x \leqslant \bar{d}^m \text{ on } [0,1].$$

In the proof of the lower estimate we will employ the scheme used by Karakostas and Tsamatos in [17]. Put $u(t) := \mathbf{A}x(t)$ for $t \in [0,1]$. By the definition of $\mathbf{A}$ we infer that u is a solution of the problem

$$\begin{cases} u''(t) + G_x(t, x(t)) = 0 \quad \text{a.e. in } [0,1], \\ u(0) = 0, \\ u(1) = \int_\alpha^\beta u(s)\, \mathrm{d}g(s). \end{cases}$$

Let us consider the compact set

$$E(u) := \left\{k \in [\alpha, \beta], u(1) = \int_\alpha^\beta u(s)\, \mathrm{d}g(s) = u(k) \int_\alpha^\beta \mathrm{d}g(s) = u(k) g(\beta)\right\}.$$

From the above we conclude that u is a solution of the BVP of the form

$$\begin{cases} y''(t) + G_x(t, x(t)) = 0 \quad \text{a.e. on } (0,1), \\ y(0) = 0, \\ y(1) = y(k_u)g(\beta), \end{cases}$$

where $k_u := \min_{u \in E} E(u)$. It is clear that u can be written as

$$\begin{aligned} u(t) = l_u t \int_0^1 (1-s)G_x(s,x(s))\,\mathrm{d}s - t l_u g(\beta) \int_0^{k_u} (k_u - s)G_x(s,x(s))\,\mathrm{d}s \\ - \int_0^t (t-s)G_x(s,x(s))\,\mathrm{d}s \end{aligned}$$

in $[0,1]$, with $l_u := [(1-k_u g_i(\beta))^{-1}]_{i=1,\ldots,n} \geqslant [(1-\beta g_i(\beta))^{-1}]_{i=1,\ldots,n} > 0$. The following chain of equalities holds:

$$\begin{aligned} (32)\quad (\mathbf{A}x)(k_u) = u(k_u) &= l_u k_u \int_0^1 (1-s)G_x(s,x(s))\,\mathrm{d}s \\ &- k_u l_u g(\beta) \int_0^{k_u} (k_u - s)G_x(s,x(s))\,\mathrm{d}s - \int_0^{k_u} (k_u - s)G_x(s,x(s))\,\mathrm{d}s \\ &= l_u k_u \int_0^1 (1-s)G_x(s,x(s))\,\mathrm{d}s - l_u \int_0^{k_u} (k_u - s)G_x(s,x(s))\,\mathrm{d}s \\ &= l_u k_u \int_0^1 G_x(s,x(s))\,\mathrm{d}s - l_u k_u \int_0^1 sG_x(s,x(s))\,\mathrm{d}s \\ &- l_u k_u \int_0^{k_u} G_x(s,x(s))\,\mathrm{d}s + l_u \int_0^{k_u} sG_x(s,x(s))\,\mathrm{d}s \\ &= l_u k_u \int_0^{k_u} G_x(s,x(s))\,\mathrm{d}s + l_u k_u \int_{k_u}^1 G_x(s,x(s))\,\mathrm{d}s \\ &- l_u k_u \int_0^{k_u} sG_x(s,x(s))\,\mathrm{d}s - l_u k_u \int_{k_u}^1 sG_x(s,x(s))\,\mathrm{d}s \\ &- l_u k_u \int_0^{k_u} G_x(s,x(s))\,\mathrm{d}s + l_u \int_0^{k_u} sG_x(s,x(s))\,\mathrm{d}s \\ &l_u k_u \int_{k_u}^1 (1-s)G_x(s,x(s))\,\mathrm{d}s + l_u(1-k_u) \int_0^{k_u} sG_x(s,x(s))\,\mathrm{d}s. \end{aligned}$$

Application of Lemma 1.3 yields for all $i = 1, \ldots, n$ and $t \in [\alpha, 1]$

$$x_i(t) \geqslant \mu \|x_i\|_{C([0,1],\mathbb{R})} \geqslant \mu \underline{d}^m.$$

Combining the last inequality and assertion (32) we can derive

$$\begin{aligned}(\mathbf{A}x)(k_u) &\geqslant l_u k_u \int_{k_u}^{1} (1-s) G_x(s, x(s)) \, \mathrm{d}s \geqslant l_u k_u \int_{\beta}^{1} (1-s) G_x(s, \mu \underline{d}^m) \, \mathrm{d}s \\ &\geqslant \Big[\frac{(1-\beta)^2 \alpha}{2(1-\beta g_i(\beta))} \lambda_i^m\Big]_{i=1,\dots,n} \underline{d}^m \geqslant \underline{d}^m,\end{aligned}$$

as we have claimed. □

Now Theorem 5.1 and arguments similar to those in the proof of Theorem 6.3 give

Theorem 6.5. *Under assumptions* (g), (G1″)–(G3″) *and* (G5) *there exists a sequence* $\{x^m\}_{m\in K}$ *of distinguished solutions of* (1)–(2)–(3) *such that* x^m *belongs to* X^m, $m \in K$.

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