Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 2, 507–513

Persistent URL: http://dml.cz/dmlcz/128081

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A NOTE ON EMBEDDING INTO PRODUCT SPACES

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(Received September 8, 2003)

Abstract. Using factorization properties of an operator ideal over a Banach space, it is shown how to embed a locally convex space from the corresponding Grothendieck space ideal into a suitable power of E, thus achieving a unified treatment of several embedding theorems involving certain classes of locally convex spaces.

Keywords: factorization, embedding, opertator ideal

MSC 2000: 46A11, 47A68

Introduction

An old and well-known result of Saxon [10] and Valdivia [15] states that every nuclear space embeds as a subspace of a suitable product of an arbitrarily given infinite-dimensional Banach space. On the other hand, Bellenot [1] showed that not all Schwartz spaces can be embedded into powers of l_p for $p > 1$. We show that the possibility of embedding locally convex spaces from a given Grothendieck space ideal is closely related to the factorisability of an operator ideal (generating the given space ideal) over a suitable class of Banach spaces. The class of these Banach spaces then turns out to be universal for the given space ideal in the sense that every space in the latter class is included in a suitable power of each Banach space in the former class. Moreover, under certain mild restrictions on the operator ideals $\mathscr{A}$, it is further shown that each such Banach space when equipped with a (natural) locally convex topology generated by the ideal $\mathscr{A}$ even acts as a universal generator for the associated Grothendieck space ideal. With this formulation, it is possible to achieve a simple and unified approach to several results of Valdivia, Saxon, Jarchow and Randtke on the embedding of nuclear and Schwartz spaces into product spaces. The aim of the present note is to give a proof of the following theorem:

Theorem. *Let $\mathscr{A}$ be an operator ideal and E a Banach space. Consider the following statements:*

i) *Every operator in $\mathscr{A}$ factors through a subspace of E.*
ii) *Every $\mathscr{A}$-space is linearly homeomorphic to a subspace of an $\mathscr{A}$-projective limit of closed linear subspaces of E.*
iii) *Every $\mathscr{A}$-space is linearly homeomorphic to a subspace of a sufficiently high power of E.*
iv) *E equipped with the associated $\mathscr{A}$-topology is a universal generator for* Groth($\mathscr{A}$).

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *Further, if the operator ideal $\mathscr{A}$ is idempotent and has the Hahn-Banach extension property (see Definition below), then* (ii) $\Rightarrow$ (iv).

Corollaries

(a) *For $\mathscr{A} = K$, the ideal of compact operators, the Banach spaces satisfying* (i) *are precisely those which 'contain l_∞ finitely'. In particular, every Schwartz space embeds into a sufficiently high power of c_0 or l_∞—an old result of Randtke* [9].

(b) *c_0 equipped with the associated Schwartz topology is a universal generator for the class of all Schwartz spaces.*
This follows because K is idempotent and has the Hahn-Banach Extension property. The latter statement is a consequence of a well-known theorem of Terzioglu [14] on the characterisation of compact operators on Banach spaces. The above corollary, therefore, yields a simpler and a more general approach to the construction of universal generators for Schwartz spaces. The results of Randtke [8] and Jarchow [3] on these constructions, therefore, follow as special cases of this more general approach.

(c) *Every strongly nuclear space is a subspace of a sufficiently high power of an arbitrarily given infinite-dimensional Banach space E and each E equipped with the strongest nuclear topology is a universal generator for the class of strongly nuclear spaces.*
(Take $\mathscr{A} = N_0$, the ideal of strongly nuclear operators, which satisfies the conditions of the theorem). (This result is due to Junek [6] see also [12]).

Before we prove the above theorem, we include the necessary background material on locally convex spaces and operator ideals, for which one may look up [11] and [7], respectively. In what follows, X, Y shall stand for a Hansdorff locally convex space (lcs for short) whereas $E, F, \ldots$ will denote Banach spaces. The dual of a space X shall be designated by X^* whereas for an operator ideal $\mathscr{A}$, the symbol $\mathscr{A}(E, F)$

shall stand for the component of all (bounded linear) operators from E into F which belong to $\mathscr{A}$. Further, given $\mathscr{A}$ and a (directed) family $\{X_\alpha;\ \alpha \in \Lambda\}$ of locally convex spaces together with linear maps $f_{\alpha\beta} \in \mathscr{A}(X_\beta, X_\alpha)$ for $\alpha < \beta$, the associated $\mathscr{A}$-projective limit defined by:

$$\operatorname{Projlim} f_{\alpha\beta}(X_\beta) = \left\{ \bar{x} = (x_\alpha) \in \prod_{\alpha\in\Lambda} X_\alpha; f_{\alpha\beta}(x_\beta) = x_\alpha,\ \forall\, \alpha < \beta \right\}$$

shall be considered equipped with the (relative) product topology. Corresponding to a given operator ideal $\mathscr{A}$, there exists on each Banach space E a locally convex topology—the *associated $\mathscr{A}$-topology*—which is determined by the family of semi-norms $\{p_{T,F}\,;\ T \in \mathscr{A}(E, F), F$ a Banach space$\}$ where

$$p_{T,F}(x) = \|T(x)\|, \quad x \in E.$$

We denote by $E_{\tau_{\mathscr{A}}}$ the Banach space E equipped with the associated locally convex $\mathscr{A}$ topology $\tau_{\mathscr{A}}$. For $F = l_2$ and $\mathscr{A} = L$, the ideal of all bounded linear operators, we get the associated *Hilbertisable topology* on E whereas the associated *Schwartz topology* on E is obtained by taking $\mathscr{A} = K$, the ideal of compact operators on E. These are, respectively, the finest Hilbertisable and Schwartz topologies on E weaker than the norm topology, which are consistent with the duality $\langle E, E^*\rangle$. We also remark that for a given operator ideal $\mathscr{A}$, the associated $\mathscr{A}$-topology on E may not be an $\mathscr{A}$-space in the sense of Pietsch ([7], § 29.6). This can be seen, for example, by considering the ideal N of nuclear operators for which the associated N-topology can never be nuclear on any infinite-dimensional Banach space X which is not isomorphic to a Hilbert space. In fact, the nuclearity of τ_N—the associated N-topology —entails that $N(X, Y) = N_{1,2,2}(X, Y)$ for each Banach space Y. In particular, for $Y = X$, the above equality yields that X is Hilbertian. The last statement follows from a famous theorem of Johnson, Konig, Retherford and T. Jaegerman [5] which says that Hilbert spaces are precisely those Banach spaces where nuclear operators have absolutely summable eigenvalues. The argument for a general Banach space is completed by observing that for each Hilbert space X there exists a Banach space Y and a nuclear operator from X into Y which is not in $N_{1,2,2}$.

Given an operator ideal $\mathscr{A}$, we shall denote by $\operatorname{Groth}(\mathscr{A})$ the Grothendieck space ideal generated by $\mathscr{A}$. In other words, a locally convex space $X \in \operatorname{Groth}(\mathscr{A})$ if for each $U \in U_0(X)$ (an X-neighbourhood base in X) there exists $V \in U_0(X)$ such that the canonical (linear) map $\varphi_U^V\colon X_V \to X_U$ belongs to $\mathscr{A}$. Here X_V denotes the completion of $X/\operatorname{Ker} p_V$ with respect to the norm induced by p_V, the Minkowski functional of V. Members of $\operatorname{Groth}(\mathscr{A})$ shall also be called *$\mathscr{A}$-spaces.*

Finally, given an operator ideal $\mathscr{A}$, we say that $\mathscr{A}$

a) is *idempotent* if $\mathscr{A} = \mathscr{A}^2$, i.e., for each pair of Banach spaces E and F and $T \in \mathscr{A}(E,F)$, there exists a Banach space G and $T_1 \in \mathscr{A}(E,G)$, $T_2 \in \mathscr{A}(G,F)$ such that $T = T_2T_1$.
b) has *Hahn-Banach Extension property* (HBEP) if for any Banach spaces E, F and G, isomorphic embedding $I\colon E \to F$ and $T \in \mathscr{A}(E,G)$, there exists $S \in \mathscr{A}(F, G_\infty)$ such that $J_G \circ T = S \circ I$. Here $G_\infty = l_\infty(B_{G^*})$, the space of all bounded functions on the (dual) unit ball of G^*, and $J_G\colon G \to G_\infty$ is the canonical embedding. For each idempotent operator ideal $\mathscr{A}, \tau_{\mathscr{A}}$ is always an $\mathscr{A}$-space topology. The ideals of compact operators K, the strongly nuclear operators N_0 and HK, the ideal of operators factoring over a Hilbert space compact operator are all idempotent and, in addition, have the HBEP. However, operator ideals N and $N_{1,2,2}$ consisting, respectively of nuclear operators and operators factoring over trace class operators do not have these properties. The importance of HBEP arises from the easily-checked fact that for such operator ideals $\mathscr{A}, E_{\tau_{\mathscr{A}}}$ (Sec [6], 7.4.8) is a subspace of $F_{\tau_{\mathscr{A}}}$ whenever E is a subspace of F (and conversely).

Proof of the theorem

(i) $\Rightarrow$ (ii). Let X be an $\mathscr{A}$-space. By Schaefer [10, p. 53], X is linearly homeomorphic to a subspace of a projective limit:

$$\operatorname{Projlim} f_\mu \nu(X_\nu)$$

of Banach spaces $\{X_\nu;\ \nu \in \Lambda\}$ where for each pair ν, μ of indices in the directed set Λ with $\nu > \mu$, there exists a linear map $f_{\mu\nu} \in \mathscr{A}(X_\nu, X_\mu)$. By (i), for each such pair, there exists a closed linear subspace $E_{\mu\nu}$ of E and continuous linear operators $P_{\mu\nu}\colon X_\nu \to E_{\mu\nu}$ and $Q_{\mu\nu}\colon E_{\mu\nu} \to X_\mu$ such that

$$(*) \qquad f_{\mu\nu} = Q_{\mu\nu} P_{\mu\nu}.$$

Further, we consider on the set $J = \{(\mu,\nu);\ \mu,\nu \in \Lambda,\ \nu > \mu\}$ the relation:

$$(\mu,\nu) > (\lambda,\delta) \Leftrightarrow \mu > \delta$$

and define, for each such pair, the mapping

$$g_{(\lambda,\delta)(\mu,\nu)} = P_{\lambda\delta} f_{\delta\mu} Q_{\mu\nu}\colon E_{\mu\nu} \to E_{\lambda\delta}.$$

Then the map

$$T\colon \operatorname{Projlim} f_{\mu\nu}(X_\mu) \to \operatorname{Projlim} g_{(\lambda,\delta)(\mu,\nu)}(E_{\mu\nu})$$

defined by

$$T(\{x_\nu\}) = \{P_{\mu\nu}(x_\nu);\ (\mu,\nu) \in J\}$$

is easily seen to be linear, continuous and injective and, in fact, a homeomorphism by virtue of $(*)$.

(ii) $\Rightarrow$ (iii). Trivial.

(ii) $\Rightarrow$ (iv). Let $X \in \mathrm{Groth}(\mathscr{A})$. By (ii), X is a subspace of a projective limit

$$Y = \mathrm{Projlim}\, f_{\mu\nu}(E_\nu)$$

where E_ν is a closed subspace of E for each $\nu \in \Lambda$. Let $T\colon Y \to E^\Lambda$ be the inclusion map and $S\colon E^\Lambda \to (E_{\tau_\mathscr{A}})^\Lambda$ denote the identity map which is obvioulsy continuous by virtue of $\tau_\mathscr{A}$ being weaker than the norm topology. We complete the proof by showing that ST is an open map onto its image and hence a linear (into) homeomorphism. Thus let U be a basic 0-neighbourhood in Y. Then we can write $U = \{\overline{x} = (x_\mu);\ \|x_\mu\| < 1, \forall\, \mu \in J\}$ for some finite subset $J \subset \Lambda$. For μ, choose $\nu = \nu(\mu) \in \Lambda$, $\nu > \mu$ and a linear map $f_{\mu\nu} : E_\nu \to E_\mu$ where $f_{\mu\nu} \in \mathscr{A}(E_\nu, E_\mu)$. Since $\mathscr{A}$ enjoys the Hahn-Banach Extension property, there exist linear maps $\hat{f}_{\mu\nu}\colon E \to E_\mu^\infty$ such that $\hat{f}_{\mu\nu}$ extends $f_{\mu\nu}$ and $\hat{f}_{\mu\nu} \in \mathscr{A}$. Thus the set

$$V = \{\bar{x} = (x_\mu) \in (E_{\tau_\mathscr{A}})^\Lambda\colon\ \|\hat{f}_{\mu\nu}(x_\nu)\| < 1,\ \mu \in J\}$$

is a 0-neighbourhood in $(E_{\tau_\mathscr{A}})^\Lambda$, by virtue of the continuity of $\hat{f}_{\mu\nu}$ on $E_{\tau_\mathscr{A}}$. In other words, $V \cap ST(Y)$ is a 0-neighbourhood in $ST(Y)$. Finally for $\bar{x} \in V \cap ST(Y)$, we see that for $\mu \in J$, $\|x_\mu\| = \|\hat{f}_{\mu\nu}(x_\nu)\| < 1$, so that $\bar{x} \in ST(U)$ and, therefore, $ST(U)$ is a 0-neighbourhood in $(E_{\tau_\mathscr{A}})^\Lambda$. This completes the proof of the theorem.

Remarks

(a) For $\mathscr{A} = HK$, the ideal of (compact) operators which factorise over a compact operator between Hilbert spaces, we recover a deep result of Bellenot [2] on the Schwartz-Hilbert variety Groth (HK) which asserts that l_2 equipped with the associated Schwartz topology is a universal generator for the Grothendieck space ideal generated by the ideal HK. This follows as an important consequence of Dvoretzky's spherical sections theorem ([3, Chap. 19]), which says that every compact map between Hilbert spaces factorises over a subspace of an arbitrarily given infinite dimensional Banach space. Also, by (i) $\Rightarrow$ (ii), every Schwartz-Hilbert space embeds as a subspace of a sufficiently high power of each given infinite-dimensional Banach space.

(b) The Saxon-Valdivia theorem as referred to in the Introduction follows from the main theorem by taking $\mathscr{A} = N_{1,2,2}$, the ideal of nuclear operators factoring over a trace class operator between Hilbert spaces. Since Groth $(N_{1,2,2})$ gives all nuclear spaces and a trace-class operator (between Hilbert spaces) factors over an arbitrarily given infinite dimensional Banach space, it follows from (i) $\Rightarrow$ (iii) that a nuclear space embeds into a suitable product of an arbitrarily given infinite-dimensional Banach space.

(c) It is known from [6] (see also [13] for a more general version) that $\tau_{N_{1,2,2}}$ is even the finest strongly nuclear topology on each Banach space E which means that E_τ where $\tau = \tau_{N_{1,2,2}}$ can never be a universal generator for the class of all nuclear spaces. This may be explained by the fact that $N_{1,2,2}$ does not have the Hahn-Banach Extension property, so that the implication (ii) $\Rightarrow$ (iv) cannot be applied in this case. However, an operator ideal which admits L_∞ as a right factor can be shown to possess HBEP. In particular, the operator ideal $N_{1,2,2} \circ L_\infty$ has HBEP. Further, noting that $\tau_{N_{1,2,2}} = \tau_{N_{1,2,2}} \circ L_\infty$, we get

Corollary 2

Corollary 2. *Each locally convex space in Groth* $(N_{1,2,2} \circ L_\infty)$ *is strongly nuclear but the coverse is not true.*

Proof. The first part of the corollary follows from the main theorem applied to the operator ideal $\mathscr{A} = N_{1,2,2} \circ L_\infty$, for which $\tau_{\mathscr{A}}$ is strongly nuclear on each Banach space, whereas the assertion regarding the converse follows from the well-known fact ([7, Theorem 29.8.2] and [12, Theorem 6]) that the ideal of strongly nuclear operators is the only operator ideal generating the class of strongly nuclear spaces as a Grothendieck space ideal. □

Acknowledgements. The author would like to thank the referee whose comments have resulted in an improved presentation of the paper. He would also like to thank Dr. Waseem Bari of the Department of Phyhsics for his help in typing the manuscript.

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