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RIEMANN TYPE INTEGRALS FOR FUNCTIONS TAKING VALUES IN A LOCALLY CONVEX SPACE

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Abstract. The McShane and Kurzweil-Henstock integrals for functions taking values in a locally convex space are defined and the relations with other integrals are studied. A characterization of locally convex spaces in which Henstock Lemma holds is given.

Keywords: Pettis integral, McShane integral, Kurzweil-Henstock integral, locally convex spaces

MSC 2000: 28B05, 46G10

1. Introduction

The aim of this paper is to study the McShane and the Kurzweil-Henstock integrals for functions defined on a bounded interval of the real line and taking values in a locally convex space. It is known that for Banach valued functions, Henstock Lemma may fail to be true. Banach spaces for which this lemma holds have been characterized by Skvortsov and Solodov in [14] and by Di Piazza and Musiał in [3]. Nakanishi in [10] and Sakurada and Nakanishi in [11] extended the McShane and the Kurzweil-Henstock integrals to functions taking values in vector spaces called (UCs-N) spaces. The Banach spaces, the Fréchet spaces and the strict inductive limits of Fréchet spaces can be defined as complete (UCs-N) spaces. In particular, they proved that if $f\colon [a,b] \to X$ is a McShane or Kurzweil-Henstock integrable function and X is a Hilbertian (UCs-N) space endowed with nuclearity, Henstock Lemma holds. It is natural to ask for which locally convex spaces the lemma is true.

We define (see § 3) the McShane and the Kurzweil-Henstock integrals for functions taking values in a locally convex space and study their properties.

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In § 4 we establish that the McShane integral lies between the Bochner and the Pettis integral.

The main result of the paper is that in a Fréchet space Henstock Lemma holds true if and only if the space is endowed with nuclearity (Theorem 4).

2. Definitions and notation

Let X be a Hausdorff locally convex topological vector space (briefly a locally convex space) with its topology $\mathscr{T}$ and topological dual X^*. $\mathscr{P}(X)$ denotes a family of $\mathscr{T}$-continuous seminorms on X so that the topology is generated by $\mathscr{P}(X)$. For $p \in \mathscr{P}(X)$, let $V_p = \{x \in X \colon p(x) \leqslant 1\}$, so that V_p^0, the polar of V_p in X^*, is a *weak**-closed, absolutely convex equicontinuous set in X^*.

For a set E of the real numbers $|E|$, χ_E and $\partial(E)$ denote respectively the Lebesgue outer measure, the characteristic function and the boundary of E. A set $E \subset \mathbb{R}$ is called *negligible* if $|E| = 0$. $\mathscr{F}$ denotes the family of all Lebesgue measurable subsets of $[0,1]$. The word "measurable" as well as the expression "almost everywhere" (abbreviated as a.e.) always refer to the Lebesgue measure. An *interval* is a compact subinterval of $\mathbb{R}$. A collection of intervals is called *nonoverlapping* if their interiors are disjoint. The symbol $\mathscr{I}$ denotes the family of all subintervals of $[0,1]$. A *partition* $\mathscr{P}$ in $[0,1]$ is a collection $\{(I_i, t_i) \colon i = 1, \dots, s\}$, where $I_1, \dots, I_s$ are nonoverlapping subintervals of $[0,1]$ and $t_1, \dots, t_s \in [0,1]$. Given a set $E \subset \mathbb{R}$, we say that $\mathscr{P}$ is

(i) a partition *in* E if $\bigcup\limits_{i=1}^{s} I_i \subset E$;

(ii) a partition *of* E if $\bigcup\limits_{i=1}^{s} I_i = E$;

(iii) a *Perron* partition if $t_i \in I_i$, $i = 1, \dots, s$.

Given $f \colon [0,1] \to X$ and a partition $\mathscr{P} = \{(I_1, t_1), \dots, (I_s, t_s)\}$ in $[0,1]$, we set

$$\sigma(f, \mathscr{P}) = \sum_{i=1}^{s} |I_i| f(t_i).$$

A *gauge* δ on $E \subset [0,1]$ is a positive function on E. For a given gauge δ on E a partition $\mathscr{P} = \{(I_i, t_i) \colon i = 1, \dots, s\}$ in $[0,1]$ is called δ-*fine* if $I_i \subset \bigl(t_i - \delta(t_i), t_i + \delta(t_i)\bigr)$.

A function $f \colon [0,1] \to X$ is called *weakly-measurable* if the function $x^* f$ is measurable for every $x^* \in X^*$.

We recall the following definitions (see [2, Definition 2.4]).

Definition 1. A function $f \colon [0,1] \to X$ is said to be strongly (or Bochner) integrable if there exists a sequence $(f_n)_n$ of simple functions such that

(i) $f_n(t) \to f(t)$ a.e., i.e. f is strongly measurable;
(ii) $p\big(f(t) - f_n(t)\big) \in L^1([0,1])$ for each $n \in \mathbb{N}$ and $p \in \mathscr{P}(X)$, and for all $p \in \mathscr{P}(X)$ $\lim\limits_{n\to\infty} \int_0^1 p\big(f(t) - f_n(t)\big)\,\mathrm{d}t = 0$;
(iii) $\int_A f_n$ converges in X for each measurable subset A of $[0,1]$.

In this case we put $(\mathrm{B})\int_A f = \lim\limits_{n\to\infty} \int_A f_n$.

Definition 2. A function $f\colon [0,1] \to X$ is said to be integrable by seminorm if for any $p \in \mathscr{P}(X)$ there exist a sequence $(f_n^p)_n$ of simple functions and a subset $X_0^p \subset [0,1]$ with $|X_0^p| = 0$, such that
(i) $\lim\limits_{n\to\infty} p\big(f_n^p(t) - f(t)\big) = 0$ for all $t \in [0,1] \setminus X_0^p$, i.e. f is measurable by seminorms;
(ii) $p\big(f(t) - f_n^p(t)\big) \in L^1([0,1])$ for each $n \in \mathbb{N}$ and $p \in \mathscr{P}(X)$, and for all $p \in \mathscr{P}(X)$ $\lim\limits_{n\to\infty} \int_0^1 p\big(f(t) - f_n^p(t)\big)\,\mathrm{d}t = 0$;
(iii) for each measurable subset A of $[0,1]$ there exists an element $y_A \in X$ such that $\lim\limits_{n\to\infty} p\big(\int_A f_n^p(t) - y_A\big) = 0$ for every $p \in \mathscr{P}(X)$.

We then put $\int_A f = y_A$.

Clearly a Bochner integrable function is integrable by seminorms, and the two definitions coincide in a Banach space.

Definition 3. A function $f\colon [0,1] \to X$ is said to be Pettis integrable if x^*f is Lebesgue integrable on $[0,1]$ for each $x^* \in X^*$ and if for every measurable set $E \subset [0,1]$ there is a vector $\nu(E) \in X$ such that $x^*\big(\nu(E)\big) = \int_E x^*f(t)\,\mathrm{d}t$ for all $x^* \in X^*$.

The set function $\nu\colon \mathscr{F} \to X$ is called the indefinite Pettis integral of f. It is known that ν is a countably additive vector measure, continuous with respect to the Lebesgue measure (in the sense that if $|E| = 0$ then $\nu(E) = 0$).

3. McShane and Kurzweil-Henstock integrals

From now on X will be a complete locally convex space.

Definition 4. A function $f\colon [0,1] \to X$ is said to be McShane integrable (respectively, Kurzweil-Henstock integrable) (briefly McS-integrable (KH-integrable)) on $[0,1]$, if there exists a vector $w \in X$ with the following property: given $\varepsilon > 0$ and $p \in \mathscr{P}(X)$ there exists a gauge δ_p on $[0,1]$ such that for each δ_p-fine partition (Perron partition) $\mathscr{P} = \{(I_i, t_i)\colon i = 1, \ldots, s\}$ of $[0,1]$, we have

$$p\big(\sigma(f, \mathscr{P}) - w\big) < \varepsilon.$$

We denote by $\mathrm{McS}([0,1], X)$ (respectively, $\mathrm{KH}([0,1], X)$) the family of all McS-integrable (KH-integrable) functions on $[0,1]$ and we set $w = (\mathrm{McS})\int_0^1 f$ ($w =$

(KH) $\int_0^1 f$). Clearly a McS-integrable function is KH-integrable. The vector w in Definition 4 is uniquely determined by $f \in \mathrm{McS}([0,1],X)$ ($\mathrm{KH}([0,1],X)$). Indeed, let $f \in \mathrm{McS}([0,1],X)$ (respectively, $\mathrm{KH}([0,1],X)$) and let w_1 and w_2 be two vectors satisfying Definition 4. Choose $\varepsilon > 0$, $p \in \mathscr{P}(X)$ and find gauges δ_p^i, $i = 1,2$, such that $p\bigl(\sigma(f,\mathscr{P}) - w_i\bigr) < \frac{1}{2}\varepsilon$ for each δ_p^i-fine partition of $[0,1]$. If $\delta_p = \min\{\delta_p^1, \delta_p^2\}$, for all δ_p-fine partitions of $[0,1]$, we get

$$p(w_1 - w_2) \leqslant p\bigl(\sigma(f,\mathscr{P}) - w_1\bigr) + p\bigl(\sigma(f,\mathscr{P}) - w_2\bigr) < \varepsilon.$$

From the arbitrariness of ε we get that $p(w_1 - w_2) = 0$ for all $p \in \mathscr{P}(X)$. Since the space X is Hausdorff it is separated, hence $w_1 = w_2$.

The following proposition can be proved in a standard way.

Proposition 1. *Let $f\colon [0,1] \to X$ and $g\colon [0,1] \to X$ be two* McS-*integrable (respectively,* KH-*integrable) functions, then:*

(i) *the function $f + g$ is* McS-*integrable (*KH-*integrable);*

(ii) *for each $\alpha \in \mathbb{R}$ the function αf is* McS-*integrable (*KH-*integrable);*

(iii) *if $x^* \in X^*$ the real valued function $x^* f$ is Lebesgue integrable (Kurzweil-Henstock integrable).*

Proposition 2. *A function $f\colon [0,1] \to X$ is* McS-*integrable (respectively, KH-integrable) on $[0,1]$, if and only if for each $\varepsilon > 0$ and $p \in \mathscr{P}(X)$ there exists a gauge δ_p on $[0,1]$ such that*

$$p\bigl(\sigma(f,\mathscr{P}_1) - \sigma(f,\mathscr{P}_2)\bigr) < \varepsilon$$

whenever $\mathscr{P}_1$ and $\mathscr{P}_2$ are δ_p-fine partitions (Perron partitions) of $[0,1]$.

P r o o f. Necessity can be proved in a standard way. To prove sufficiency, given $\varepsilon > 0$ and $p \in \mathscr{P}(X)$ there exists a gauge δ_p on $[0,1]$ such that

$$p\bigl(\sigma(f,\mathscr{P}_1) - \sigma(f,\mathscr{P}_2)\bigr) < \frac{\varepsilon}{2}$$

whenever $\mathscr{P}_1$ and $\mathscr{P}_2$ are δ_p-fine partitions of $[0,1]$. Let $\mathscr{D} = \{(\mathscr{L}, m)\colon \mathscr{L}$ is a finite subset of $\mathscr{P}(X)$ and $m \in \mathbb{N}\}$ and define for $(\mathscr{L}, m)$ and $(\mathscr{K}, n) \in \mathscr{D}$, $(\mathscr{L}, m) \ll (\mathscr{K}, n)$ if and only if $\mathscr{L} \subset \mathscr{K}$ and $m \leqslant n$. Now let $\mathscr{L} = \{p_1, \ldots, p_k\}$ be a finite subset of $\mathscr{P}(X)$. Since $\mathscr{P}(X)$ is filtering, we can find a seminorm $p_{\mathscr{L}} \in \mathscr{P}(X)$ and a constant $c_{\mathscr{L}} \geqslant 1$ such that for every $i \in \{1, \ldots, k\}$ and $x \in X$,

$$p_i(x) \leqslant c_{\mathscr{L}} p_{\mathscr{L}}(x). \tag{1}$$

For each $(\mathscr{L}, m) \in \mathscr{D}$ choose a positive function $\delta_m^{\mathscr{L}}$ such that, if $\mathscr{P}_1$ and $\mathscr{P}_2$ are two $\delta_m^{\mathscr{L}}$-fine partitions of $[0,1]$, then

$$p_{\mathscr{L}}\big(\sigma(f, \mathscr{P}_1) - \sigma(f, \mathscr{P}_2)\big) < \frac{1}{mc_{\mathscr{L}}}. \tag{2}$$

Without loss of generality we can set $\delta_m^{\mathscr{L}} \leqslant \min\{\delta_m^{p_1}, \dots, \delta_m^{p_k}\}$, where $\delta_m^{p_i}$, $i = 1, \dots, k$ is the gauge corresponding to the seminorm p_i and to $\varepsilon = 1/m$. Moreover, we can assume that, for each fixed $\mathscr{L}$, the sequence $(\delta_m^{\mathscr{L}})_m$ is non-decreasing. For any $(\mathscr{L}, m) \in \mathscr{D}$, let $\mathscr{P}_m^{\mathscr{L}}$ be a $\delta_m^{\mathscr{L}}$-fine partition of $[0,1]$. The family $\{\sigma(f, \mathscr{P}_m^{\mathscr{L}})\colon (\mathscr{L}, m) \in \mathscr{D}\}$ is a Cauchy net. Indeed, let $p \in \mathscr{P}(X)$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/N < \frac{1}{2}\varepsilon$. If $(\mathscr{L}, m) \gg (\{p\}, N)$, the partition $\mathscr{P}_m^{\mathscr{L}}$ is $\delta_m^{\mathscr{L}}$-fine and hence also $\delta_m^{\{p\}}$-fine, and analogously if $(\mathscr{K}, n) \gg (\{p\}, N)$, the partition $\mathscr{P}_n^{\mathscr{K}}$ is $\delta_n^{\mathscr{K}}$-fine and hence also $\delta_n^{\{p\}}$-fine. So considering a $\delta_N^{\{p\}}$-fine partition $\mathscr{P}_N^{\{p\}}$, since if $N \leqslant m < n$ then $\delta_N^{\{p\}} \leqslant \delta_m^{\{p\}} \leqslant \delta_n^{\{p\}}$, we obtain by (1) and (2):

$$\begin{aligned} &p\big(\sigma(f, \mathscr{P}_m^{\mathscr{L}}) - \sigma(f, \mathscr{P}_n^{\mathscr{K}})\big) \\ &\quad\leqslant p\big(\sigma(f, \mathscr{P}_m^{\mathscr{L}}) - \sigma(f, \mathscr{P}_N^{\{p\}})\big) + p\big(\sigma(f, \mathscr{P}_N^{\{p\}}) - \sigma(f, \mathscr{P}_n^{\mathscr{K}})\big) \\ &\quad\leqslant \frac{1}{m} + \frac{1}{n} \leqslant \frac{1}{N} + \frac{1}{N} < \varepsilon. \end{aligned} \tag{3}$$

As the space X is complete there exists a vector w such that the net $\{\sigma(f, \mathscr{P}_m^{\mathscr{L}})\colon (\mathscr{L}, m) \in \mathscr{D}\}$ converges to w. We want to prove that w is the integral of f. To do this fix $\varepsilon > 0$ and let $q \in \mathscr{P}(X)$. Since $\{\sigma(f, \mathscr{P}_m^{\mathscr{L}})\colon (\mathscr{L}, m) \in \mathscr{D}\}$ converges to w there exists a natural number N with $1/N < \frac{1}{2}\varepsilon$ such that

$$q\big(\sigma(f, \mathscr{P}_n^{\mathscr{L}}) - w\big) < \frac{\varepsilon}{2} \tag{4}$$

whenever $(\{q\}, N) \ll (\mathscr{L}, n)$. Now let $\mathscr{P}$ be a $\delta_N^{\{q\}}$-fine partition of $[0,1]$. By (4) and (3) we obtain

$$\begin{aligned} q\big(\sigma(f, \mathscr{P}) - w\big) &\leqslant q\big(\sigma(f, \mathscr{P}) - \sigma(f, \mathscr{P}_n^{\mathscr{L}})\big) + q\big(\sigma(f, \mathscr{P}_n^{\mathscr{L}}) - w\big) \\ &\leqslant \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

and the assertion follows. □

The following lemma can be proved, as in the real case, by the Cauchy test.

Lemma 1. *Let $f\colon [0,1] \to X$ be a* McS-*integrable (respectively,* KH-*integrable) function. Then, if $0 \leqslant a \leqslant b \leqslant 1$, $f\chi_{[a,b]}$ is* McS-*integrable (*KH-*integrable).*

For each $f \in \mathrm{McS}([0,1],X)$ (respectively, $f \in \mathrm{KH}([0,1],X)$), the interval function $F(I) = (\mathrm{McS})\int_I f$ $(F(I) = (\mathrm{KH})\int_I f)$ is called the *primitive* of f. For a McS-integrable (KH-integrable) function the following version of Henstock Lemma holds and can be proved as in the real case.

Lemma 2. *Let $f\colon [0,1] \to X$ be a* McS-*integrable (respectively,* KH-*integrable) function. Then for each $\varepsilon > 0$ and each $p \in \mathscr{P}(X)$ there corresponds a gauge δ_p such that*

$$p\biggl(\sum_{i=1}^{s}\bigl(|I_i|f(t_i) - F(I_i)\bigr)\biggr) < \varepsilon$$

for each δ_p-fine partition (Perron partition) $\mathscr{P} = \{(I_i,t_i)\colon i = 1,\ldots,s\}$ in $[0,1]$.

Remark 1. If the previous inequality holds when the seminorm p is inside the summation sign Σ, the function f is said to satisfy Henstock Lemma. For real valued functions Henstock Lemma is satisfied by every McS-integrable (or KH-integrable) function. For vector valued functions this is no longer true.

4. Relation between the McShane and the Kurzweil-Henstock integrals and other types of integral

In this section we will establish some relations between the integrals defined by means of Riemann sums and the Bochner and the Pettis integral.

We recall that a function $f\colon [0,1] \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n \in X$ and $E_1, E_2, \ldots, E_n \in \mathscr{F}$ such that $f = \sum\limits_{i=1}^{n} x_i\chi_{E_i}$.

Following an idea of Fremlin ([5]), we will prove that each integrable by seminorm function is McS-integrable. First we need the following lemmas.

Lemma 3. *If $f\colon [0,1] \to X$ is a simple function then $f \in \mathrm{McS}([0,1],X)$.*

Proof. Since the McS-integral is linear, it is sufficient to consider the case $f(x) = \chi_E(x)\cdot w$ where E is a measurable set in $[0,1]$ and w is a non null vector in X. For each $I \in \mathscr{I}$ put $F(I) = |E \cap I|\cdot w$. Choose an open set G and a closed set F such that $F \subset E \subset G$. Define a gauge δ_p on $[0,1]$ in the following way:

$$\delta_p(x) = \begin{cases} \operatorname{dist}(x,G) & \text{if } x \in F,\\ \inf\{\operatorname{dist}(x,\partial(G));\ \operatorname{dist}(x,F)\} & \text{if } x \in G \setminus F,\\ \operatorname{dist}(x,F) & \text{if } x \in [0,1] \setminus G.\end{cases}$$

Let $\mathscr{P} = \{(I_i, x_i)\colon\ i = 1, \ldots, s\}$ be a δ_p-fine partition of $[0,1]$; it follows that

$$\begin{aligned}
p\Big(\sum_{i=1}^{s} |I_i| f(x_i) - |E| w\Big) &= p\Big(\sum_{i=1}^{s} \big(|I_i| f(x_i) - F(I_i)\big)\Big) \\
&\leqslant p\Big(\sum_{x_i \in E} \big(|I_i| f(x_i) - F(I_i)\big)\Big) + p\Big(\sum_{x_i \notin E} \big(F(I_i)\big)\Big) \\
&= p\Big(\sum_{x_i \in E} \big(|I_i| \cdot w - |E \cap I_i| \cdot w\big)\Big) + p\Big(\sum_{x_i \notin E} \big(|E \cap I_i| \cdot w\big)\Big) \\
&\leqslant p(w) \sum_{x_i \in E} \big||I_i| - |E \cap I_i|\big| + p(w) \sum_{x_i \notin E} |E \cap I_i| \\
&\leqslant 2p(w) \cdot |G \setminus F|.
\end{aligned}$$

If $p(w) = 0$ the assertion follows trivially, otherwise we choose F and G such that $|G \setminus F| < \varepsilon / 2p(w)$. Therefore $f \in \mathrm{McS}([0,1], X)$ and $F(I) = |E \cap I| \cdot w$. $\square$

Lemma 4. *Let $f\colon\ [0,1] \to X$ be a function. Given $\varepsilon > 0$ and $p \in \mathscr{P}(X)$, there is a gauge δ_p such that*

$$\sum_{i=1}^{s} p\big(f(x_i)\big) |I_i| \leqslant \overline{\int_0^1} p\big(f(t)\big)\,\mathrm{d}t + \varepsilon$$

for each δ_p-fine partition $\mathscr{P} = \{(I_i, t_i)\colon\ i = 1, \ldots, s\}$ of $[0,1]$, where the integral in the last inequality is the upper Lebesgue integral.

P r o o f. Let $p \in \mathscr{P}(X)$. We can consider only the case $\overline{\int_0^1} p\big(f(t)\big)\,\mathrm{d}t < \infty$, otherwise the inequality is obvious. Choose a real-valued function g on $[0,1]$ such that $g(t) \geqslant p\big(f(t)\big)$ for all t and $\int_0^1 g(t)\,\mathrm{d}t = \overline{\int_0^1} p\big(f(t)\big)\,\mathrm{d}t$. Given $\varepsilon > 0$, find $\delta_p(x)$ such that if $\mathscr{P} = \{(I_i, t_i)\colon\ i = 1, \ldots, s\}$ is a δ_p-fine partition of $[0,1]$ we have

$$\Big|\sum_{i=1}^{s} g(t_i) |I_i| - \int_0^1 g(t)\,\mathrm{d}t\Big| < \varepsilon. \tag{5}$$

Hence from (5)

$$\sum_{i=1}^{s} p\big(f(t_i)\big) |I_i| \leqslant \sum_{i=1}^{s} g(t_i) |I_i| \leqslant \overline{\int_0^1} p\big(f(t)\big)\,\mathrm{d}t + \varepsilon.$$

$\square$

Proposition 3. *Let $f\colon [0,1]\to X$ be a function which is integrable by seminorm. Then it is* McS-*integrable (and also* KH-*integrable) and the two integrals coincide.*

P r o o f. Choose $p\in\mathscr{P}(X)$ an fix $\varepsilon>0$. Let $\varphi_p\colon [0,1]\to X$ be a simple function such that

$$\int_0^1 p\big(f(t)-\varphi_p(t)\big)\,\mathrm{d}t<\frac{\varepsilon}{4}. \tag{6}$$

The function φ_p is McS-integrable as we have already proved, thus there is a gauge δ_p^1 such that

$$p\Big(\sum_{i=1}^s |I_i|\varphi_p(t_i)-\int_0^1\varphi_p\Big)<\frac{\varepsilon}{4} \tag{7}$$

for each δ_p^1-fine partition $\mathscr{P}=\{(I_i,t_i)\colon\ i=1,\dots,s\}$ of $[0,1]$. By Lemma 4 there is a gauge δ_p^2 such that

$$\sum_{i=1}^s p\big(f(t_i)-\varphi_p(t_i)\big)|I_i|\leqslant\int_0^1 p\big(f(t)-\varphi_p(t)\big)\,\mathrm{d}t+\frac{\varepsilon}{4}. \tag{8}$$

Let $\delta_p=\min\{\delta_p^1,\delta_p^2\}$ and take a δ_p-fine partition $\mathscr{P}=\{(I_i,t_i)\colon\ i=1,\dots,s\}$ of $[0,1]$. Then by (8), (7) and (6) we have

$$\begin{aligned}
&p\Big(\sum_{i=1}^s |I_i|f(t_i)-\int_0^1 f\Big)\\
&\qquad\leqslant p\Big(\sum_{i=1}^s\big(|I_i|f(t_i)-|I_i|\varphi_p(t_i)\big)\Big)+p\Big(\sum_{i=1}^s |I_i|\varphi_p(t_i)-\int_0^1\varphi_p\Big)\\
&\qquad\quad+p\Big(\int_0^1\varphi_p-\int_0^1 f\Big)\\
&\qquad\leqslant\int_0^1 p\big(f(t)-\varphi_p(t)\big)\,\mathrm{d}t+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\int_0^1 p\big(f(t)-\varphi_p(t)\big)\\
&\qquad<\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon,
\end{aligned}$$

which implies the McS-integrability of f. □

Since a Bochner integrable function is integrable by seminorm we get

Corollary 1. *If $f\colon [0,1] \to X$ is a Bochner-integrable function then it is* McS-*integrable (and* KH-*integrable) and the two integrals coincide.*

Remark 2. If $f\colon [0,1] \to X$, $h\colon [0,1] \to X$ are two functions such that $f = h$ a.e., then $f \in \mathrm{McS}([0,1],X)$ (respectively, $\mathrm{KH}([0,1],X)$) if and only if $h \in \mathrm{McS}([0,1],X)$ ($\mathrm{KH}([0,1],X)$). In this case we have $\int_0^1 f = \int_0^1 h$.

Now we are going to see when a KH-integrable function is integrable by seminorm.

Theorem 1. *Let $f\colon [0,1] \to X$ be a measurable by seminorm function. Then f is integrable by seminorm if and only if f is* KH-*integrable and for each $p \in \mathscr{P}(X)$, the real valued function $p\bigl(f(x)\bigr)$ is* KH-*integrable.*

P r o o f. The necessity has been proved in Proposition 3. For the converse implication, we observe that if $f\colon [0,1] \to X$ is a measurable by seminorm function such that for each $p \in \mathscr{P}(X)$, the real valued function $p\bigl(f(x)\bigr)$ is KH-integrable, then $p\bigl(f(x)\bigr)$ is integrable. So the assertion follows by ([2, Theorem 2.10]). $\square$

We investigate now the relationship between the Pettis and the McShane integral.

For each $p \in \mathscr{P}(X)$, $p^{-1}(0)$ is a vector subspace and p defines a norm on $X/p^{-1}(0)$. Let X_p be the associated Banach space, namely the completion of the normed linear space $X/p^{-1}(0)$, and π_p the canonical mapping of X into X_p (see [12, 0.11.1]). Given a function $f\colon [0,1] \to X$ and a seminorm $p \in \mathscr{P}(X)$, define a function $f_p\colon [0,1] \to X_p$ by

$$f_p(t) = (\pi_p \circ f)(t) = \pi_p\bigl(f(t)\bigr)$$

for $t \in [0,1]$.

Remark 3. We note that if $f\colon [0,1] \to X$ is McS-integrable (respectively, KH-integrable) then also $f_p\colon [0,1] \to X_p$ is McS-integrable (KH-integrable). Indeed, let w denote the McS-integral (KH-integral) of f. Choose $\varepsilon > 0$, $p \in \mathscr{P}(X)$ and find a gauge δ_p such that

$$p\bigl(\sigma(f,\mathscr{P}) - w\bigr) < \varepsilon \tag{9}$$

for each δ_p-fine partition (Perron partition) $\mathscr{P} = \{(I_i,t_i)\colon i = 1,\ldots,s\}$ in $[0,1]$. Since

$$p\bigl(\sigma(\pi_p \circ f,\mathscr{P}) - \pi_p(w)\bigr) = p\bigl(\sigma(f,\mathscr{P}) - w\bigr),$$

from (9) we get

$$p\bigl(\sigma(\pi_p \circ f,\mathscr{P}) - \pi_p(w)\bigr) < \varepsilon.$$

Theorem 2. *Let $f\colon [0,1] \to X$ be a* McS*-integrable function. Then the function f is Pettis integrable.*

Proof. By the previous remark we get that for each $p \in \mathscr{P}(X)$ the function $f_p\colon [0,1] \to X_p$ is McS-integrable. Then by ([6, Theorem 2C]) f_p is Pettis integrable. Since X is complete, it follows from [1, Lemma 2.9] that the function f is Pettis integrable. □

When the range space is a Banach space it is known that a measurable Pettis integrable function is McS-integrable ([7, Theorem 17]). For functions taking values in a locally convex space the problem is investigated in [8].

Theorem 3. *Let $f\colon [0,1] \to X$ be a function which is measurable by seminorm. Then f is integrable by seminorm if and only if f is* McS*-integrable and for each $p \in \mathscr{P}(X)$, the real valued function $p\big(f(x)\big)$ is* McS*-integrable.*

Proof. Necessity has been proved in Proposition 3. To prove the converse, let $f\colon [0,1] \to X$ be a measurable by seminorm function such that for each $p \in \mathscr{P}(X)$, the real valued function $p\big(f(x)\big)$ is McS-integrable. Then $p\big(f(x)\big)$ is integrable. Moreover, by Theorem 2 the function f is Pettis integrable, and the assertion follows by ([2, Theorem 2.6]). □

5. Integrals of functions with values in nuclear locally convex spaces

Nakanishi in [10] and Sakurada and Nakanishi in [11] extended the definition of the McShane and the Kurzweil-Henstock integral to functions taking values in a ranked vector space ((UCs-N) space) defined as the ranked union space of a certain partially ordered class of countably pseudo-metric spaces. Moreover, they showed that the Henstock Lemma holds true for the case when the (UCs-N) space is a Hilbertian (UCs-N) space endowed with nuclearity. A Fréchet space is a particular case of the complete (UCs-N) space satisfying the separation property. The main result of this paper is Theorem 4, in which we prove that in a Fréchet space the Henstock Lemma holds if and only if the space is nuclear. The necessary part was proved in [10, Theorem 1] and [11, Theorem 1]. Our proof is different, moreover, we give a complete characterization of Fréchet spaces in which Henstock Lemma holds true.

For Banach valued functions this problem was investigated by Skvortsov and Solodov in ([14, Theorem 3]) for functions defined on an interval of the real line and by Di Piazza and Musiał in ([3, Theorem 3]) for functions whose domain is an outer regular σ-finite quasi-Radon measure space.

The symbol φ denotes the null vector of X. If V is a convex, balanced neighbourhood of φ, denote by $M_V(x) = \inf\{\lambda\colon\ x/\lambda \in V,\ \lambda > 0\}$ the Minkowski functional of V. Then M_V is a continuous seminorm on X and $X_V = X/M_V^{-1}(0)$ is a normed linear space.

We recall the following definitions (see [12]):

Definition 5. Let X be a locally convex space and Y a Banach space. A linear operator $T\colon\ X \to Y$ is called a nuclear operator if

$$T(x) = \sum_{n=1}^{\infty} c_n f_n(x) y_n$$

where $\{f_n\}_n$ is an equi-continuous sequence of continuous linear functionals on X, $\{y_n\}_n$ is a bounded sequence of elements in Y and $\{c_n\}_n$ is a sequence of nonnegative numbers with $\sum\limits_{n=1}^{\infty} c_n < \infty$.

Definition 6. A locally convex space X is called a nuclear space if, for any convex balanced neighborhood V of φ, there exists another convex balanced neighborhood $U \subseteq V$ of φ such that the canonical mapping

$$T\colon\ X_U \to \hat{X}_V$$

where $\hat{X}_V$ is the completion of X_V, is nuclear.

We recall that a locally convex space X is a *Fréchet space* or simply an (F)-*space* if it is a complete space in which the topology is induced by a sequence of pseudonorms. A series $\sum\limits_i x_i$ in X is *unconditionally convergent* if for each permutation $n(i)$ of positive integers the series $\sum\limits_i x_{n(i)}$ converges ([9, p. 116]). Our main result follows:

Theorem 4. *Let X be an* (F)*-space. Then Henstock Lemma holds true for each* McS*-integrable* (*or* KH*-integrable*) *function $f\colon\ [0,1] \to X$ if and only if X is nuclear.*

P r o o f. Assume that X is a nuclear (F)-space and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of seminorms determining the topology of X. For each n, X_n is the completion of the quotient space $X/p_n^{-1}(0)$. We can suppose that the sequence $\{p_n\}_{n=1}^{\infty}$ is increasing, then there is a natural continuous embedding

$$\pi_n\colon\ X_{n+1} \to X_n$$

of the space X_{n+1} into the space X_n. Since the space X is nuclear, for each n, π_n is an absolutely summing operator (see [12, 4.4.6]), that is, there exists a positive constant C_n such that, for arbitrary $x_1, \ldots, x_l \in X$, $\sum\limits_{i=1}^{l} p_n\bigl(\pi_n(x_i)\bigr) \leqslant C_n p_{n+1}\Bigl(\sum\limits_{i=1}^{l} x_i\Bigr)$.

Moreover, f is McShane integrable, so for a fixed $\varepsilon > 0$ and for $n = 1, 2, \dots$ it is possible to find gauges δ_n, $\delta_1 \geqslant \dots \geqslant \delta_n \geqslant \dots$ such that

$$p_n\biggl(\sum_{i=1}^{s}\bigl(|I_i|f(t_i) - F(I_i)\bigr)\biggr) < \frac{\varepsilon}{C_{n-1}}$$

for each δ_n-fine partition $\{(I_i, t_i),\ i = 1, \dots, s\}$ of $[0,1]$. Since π_n is absolutely summing we get

$$\begin{aligned}\sum_{i=1}^{s} p_{n-1}\bigl(|I_i|f(t_i) - F(I_i)\bigr) &= \sum_{i=1}^{s} p_{n-1}\bigl(\pi_{n-1}\bigl(|I_i|f(t_i) - F(I_i)\bigr)\bigr)\\ &\leqslant C_{n-1}p_n\biggl(\sum_{i=1}^{s}\bigl(|I_i|f(t_i) - F(I_i)\bigr)\biggr) < \varepsilon,\end{aligned}$$

and this is true for every natural number n. Therefore Henstock Lemma holds.

To prove the converse implication assume that Henstock Lemma holds and suppose that the space X is not nuclear. By the Grothendieck version of Dvoretzky Rogers Theorem in (F)-spaces ([13, Theorem 7.3.2]), let $\sum\limits_n x_n$ be an unconditionally convergent series which is not absolutely convergent. Let C be the Cantor set in $[0,1]$ and denote by (a_i^r, b_i^r), $r \geqslant 0$, $1 \leqslant i \leqslant 2^r$ the contiguous intervals of length $1/3^{r+1}$ adjacent to C. Denote by d_i^r the center of (a_i^r, b_i^r).

Define

$$f(t) = \begin{cases} \varphi & \text{if } t \in C \text{ or } t = d_i^r,\ i = 1, \dots, 2^r,\ r = 0, 1, \dots,\\[4pt] \dfrac{3^r}{2^r}x_r & \text{if } t \in (a_i^r, d_i^r),\ i = 1, \dots, 2^r,\ r = 0, 1, \dots,\\[4pt] -\dfrac{3^r}{2^r}x_r & \text{if } t \in (d_i^r, b_i^r),\ i = 1, \dots, 2^r,\ r = 0, 1, \dots. \end{cases}$$

We want to prove that f is McShane integrable to φ. Indeed, fix $\varepsilon > 0$ and let $p \in \mathscr{P}(X)$. By ([9, Theorem 4]), there is a natural number R such that

$$\sup_{\varepsilon_i = 1 \text{ or } 0} p\biggl(\sum_{R+1}^{\infty}\varepsilon_i x_i\biggr) < \frac{\varepsilon}{4}.$$

For any sequence of real numbers $(\theta_i)_{R+1}^{\infty}$ satisfying the condition $|\theta_i| \leqslant 1$ for $i = R+1, R+2, \dots$ we have

$$p\biggl(\sum_{R+1}^{\infty}\theta_i x_i\biggr) < \frac{\varepsilon}{2}.$$

In fact, let us expand θ_i into the dyadic form

$$\theta_i = \sum_{j=0}^{\infty} \frac{\varepsilon_{i,j}}{2^j}, \quad \varepsilon_{i,j} = 0 \text{ or } 1.$$

Then

$$(10) \qquad p\Big(\sum_{R+1}^{\infty} \theta_i x_i\Big) = p\Big(\sum_{R+1}^{\infty}\sum_{j=0}^{\infty} \frac{\varepsilon_{i,j} x_i}{2^j}\Big) = p\Big(\sum_{j=0}^{\infty} \frac{1}{2^j} \sum_{R+1}^{\infty} \varepsilon_{i,j} x_i\Big)$$
$$\leqslant \sum_{j=0}^{\infty} \frac{1}{2^j} \sup_{\varepsilon_i = 1 \text{ or } 0} p\Big(\sum_{R+1}^{\infty} \varepsilon_i x_i\Big) \leqslant 2 \sup_{\varepsilon_i = 1 \text{ or } 0} p\Big(\sum_{R+1}^{\infty} \varepsilon_i x_i\Big).$$

For $i = 1, \ldots, 2^r$ let $U_i^r = (a_i^r, b_i^r) \setminus \{d_i^r\}$, $U_R = \bigcup_{r=0}^{R} \bigcup_{i=1}^{2^r} U_i^r$, $U = \bigcup_{R=0}^{\infty} U_R$ and $V = C \bigcup \Big(\bigcup_{r=0}^{\infty} \bigcup_{i=1}^{2^r} \{d_i^r\}\Big)$. Moreover, let $K = \max\{p(x_1), \ldots, p(x_R)\}$ and choose a positive real number ϱ such that $\varrho K 3^{R+1} < \frac{1}{2}\varepsilon$. Define $\delta_p(\xi)$ as follows:

$$\delta_p(\xi) = \begin{cases} \min\{|\xi - a_i^r|, |\xi - b_i^r|, |\xi - d_i^r|\} & \text{if } \xi \in U_i^r,\ i = 1, \ldots, 2^r,\ r = 0, 1, \ldots, \\ \varrho & \text{if } \xi \in V. \end{cases}$$

Let $\mathscr{P} = \{(I_i, t_i)\colon\ i = 1, \ldots, s\}$ be any δ_p-fine partition of $[0,1]$.

Now $p\Big(\sum_{i=1}^{s} |I_i| f(t_i)\Big) = p\Big(\sum_{t_i \in U} |I_i| f(t_i)\Big)$. For $i = 1, \ldots, s$ each interval I_i is either entirely in U_R or it is disjoint from U_R, thus

$$(11) \qquad p\Big(\sum_{t_i \in U} |I_i| f(t_i)\Big) \leqslant p\Big(\sum_{I_i \subset U_R} |I_i| f(t_i)\Big) + p\Big(\sum_{I_i \bigcap U_R = \Phi} |I_i| f(t_i)\Big).$$

Let us estimate the two sums separately. From the definition of f and of $\delta_p(\xi)$ for $\xi \in U_i^r$, it follows that there are numbers θ_i^r, $0 \leqslant r \leqslant R$, $i = 1, \ldots, 2^r$, such that $|\theta_i^r| < 2\varrho$ and

$$(12) \quad p\Big(\sum_{I_i \subset U_R} |I_i| f(t_i)\Big) \leqslant \sum_{r=0}^{R} \sum_{i=1}^{2^r} \frac{3^r}{2^r} p(x_r) \theta_i^r$$
$$\leqslant \sum_{r=0}^{R} \frac{3^r}{2^r} p(x_r) \sum_{i=1}^{2^r} \theta_i^r \leqslant \sum_{r=0}^{R} \frac{3^r}{2^r} p(x_r) \sum_{i=1}^{2^r} 2\varrho$$
$$= 2\varrho \sum_{r=0}^{R} \frac{3^r}{2^r} 2^r p(x_r) \leqslant 2\varrho K \frac{3^{R+1} - 1}{2} = K\varrho(3^{R+1}) < \frac{\varepsilon}{2}.$$

For $r \geqslant R$ we can find numbers θ_r for which $|\theta_r| < 1$ and

$$p\Big(\sum_{I_i \bigcap U_R = \Phi} f(t_i)|I_i|\Big) \leqslant p\Big(\sum_{r=R+1}^{\infty} x_r \theta_r\Big) < \frac{\varepsilon}{2}. \tag{13}$$

From (11), (12) and (13) we obtain

$$p\Big(\sum_{t_i \in U} |I_i| f(t_i)\Big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore the function f is McS-integrable with the integral equal to φ. Now let

$$F(t) = \begin{cases} \varphi & \text{if } t \in C, \\ \dfrac{3^r}{2^r}(t - a_i^r)x_r & \text{if } t \in (a_i^r, d_i^r],\ i = 1, \dots, 2^r,\ r = 0, 1, \dots, \\ -\dfrac{3^r}{2^r}(t - b_i^r)x_r & \text{if } t \in (d_i^r, b_i^r),\ i = 1, \dots, 2^r,\ r = 0, 1, \dots, \end{cases}$$

be the primitive of f. Since the series $\sum\limits_{i=0}^{\infty} p(x_i)$ is not absolutely convergent there is ε_0 such that for all N there are $m, n > N$ satisfying

$$\sum_{i=m}^{n} p(x_i) \geqslant 6\varepsilon_0. \tag{14}$$

Define a gauge δ_p relatively to ε_0 as before. Let N be a natural number for which $\frac{1}{2} \cdot 3^{-N} < \varrho$. Then for $n, m > N$, $\mathscr{P} = \big\{\big(d_i^r, (a_i^r, d_i^r)\big),\ r = m, \dots, n,\ i = 1, \dots, 2^r\big\}$ is a δ_p-fine partition in $[0, 1]$. By (14) we have

$$\begin{aligned} \sum_{r=m}^{n} \sum_{i=1}^{2^r} p\big(|I_i| f(t_i) - F(I_i)\big) &= \sum_{r=m}^{n} \sum_{i=1}^{2^r} p\big(F(I_i)\big) \\ &= \sum_{r=m}^{n} \sum_{i=1}^{2^r} \frac{3^r}{2^r} \frac{1}{2 \cdot 3^{r+1}} p(x_r) = \frac{1}{6} \sum_{r=m}^{n} p(x_r) \geqslant \varepsilon_0, \end{aligned}$$

which is in contradiction with the hypothesis that Henstock Lemma is valid. Therefore the assertion holds. $\square$

Corollary 2. *Let X be an* (F)*-space. Then the McShane integrability is equivalent to the Bochner integrability if and only if X is nuclear.*

Proof. Assume that X is a nuclear (F)-space and let $f\colon [0,1] \to X$ be a McS-integrable function. By Proposition 1, for each $x^* \in X^*$ the real valued function $x^* f$ is Lebesgue integrable. Since the space X is nuclear it follows that for each $p \in \mathscr{P}(X)$ the function $p\bigl(f(x)\bigr)$ is integrable and therefore f is strongly integrable (see [4, p. 257]). The converse follows from Theorem 4. The function f constructed in the proof of the previous theorem is a McS-integrable function such that the real valued function pf is not McS-integrable, therefore f cannot be Bochner integrable ([2, Proposition 2.5]). $\square$

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