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R_0-ALGEBRAS AND WEAK DUALLY RESIDUATED LATTICE ORDERED SEMIGROUPS

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Abstract. We introduce the notion of weak dually residuated lattice ordered semigroups (WDRL-semigroups) and investigate the relation between R_0-algebras and WDRL-semigroups. We prove that the category of R_0-algebras is equivalent to the category of some bounded WDRL-semigroups. Moreover, the connection between WDRL-semigroups and DRL-semigroups is studied.

Keywords: R_0-algebra, DRL-semigroup, WDRL-semigroup

MSC 2000: 06F05, 03G25

1. Introduction

The notion of dually residuated lattice ordered semigroups (in short DRL-semigroups) was introduced by K. L. N. Swamy in [9] as a common generalization of Brouwerian algebras and commutative lattice ordered groups. In [3]–[4], T. Kovář have made an intensive study of the DRL-semigroups. In 1998, J. Rachůnek investigated the relation between MV-algebras [1] and DRL-semigroups and proved that MV-algebras are categorically equivalent to $DRL_{1(i)}$-semigroups [5]–[6].

R_0-algebras were introduced by Wang [8] as an algebraic counterpart of Formal System $\pounds^*$ [10]. It is worth noting that R_0-algebras are different from MV-algebras because the identity $(x \to y) \to y = (y \to x) \to x$ holds in MV-algebras [2], but it does not hold in R_0-algebras. In fact, R_0-algebra is an algebra induced by a left continuous t-norm and its corresponding residuum, but MV-algebra is an algebra induced by a continuous t-norm and its corresponding residuum. From this point of view, it is meaningful to study R_0-algebras.

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In this paper, we introduce the notion of WDRL-semigroups and investigate the relation between R_0-algebras and WDRL-semigroups. We prove that R_0-algebras are categorically equivalent to some WDRL-semigroups. Moreover, we discuss the connection between WDRL-semigroups and DRL-semigroups and prove that each DRL-semigroup is a WDRL-semigroup, but the converse may not be true. The condition under which a WDRL-semigroup is a DRL-semigroup is established.

Let us introduce the notions of R_0-algebras and WDRL-semigroups.

Definition 1.1 ([10]). An R_0-algebra is an algebra $L = (L, \wedge, \vee, 0, 1, \neg, \rightarrow)$ of type $(2, 2, 0, 0, 1, 2)$ such that

(i) $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice,

(ii) $\neg$ is an order-reversing involution operation on L,

(iii) $\rightarrow$ is a binary operation on L which satisfies the following:

(R1) $x \rightarrow y = \neg y \rightarrow \neg x$,

(R2) $1 \rightarrow x = x$,

(R3) $(y \rightarrow z) \vee ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(R4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(R5) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$,

(R6) $(x \rightarrow y) \vee ((x \rightarrow y) \rightarrow (\neg x \vee y)) = 1$.

Example 1.2. Let $L = [0, 1]$. For any $x, y \in L$, we define

$$x \wedge y = \min\{x, y\},\ x \vee y = \max\{x, y\},\ \neg x = 1 - x,\ x \rightarrow y = \begin{cases} 1, & x \leqslant y, \\ \neg x \vee y, & x > y. \end{cases}$$

Then $(L, \wedge, \vee, \neg, \rightarrow, 0, 1)$ is an R_0 algebra. But it is not an MV-algebra because $(0.4 \rightarrow 0.6) \rightarrow 0.6 = 0.6 \neq (0.6 \rightarrow 0.4) \rightarrow 0.4 = 1$.

Remark 1.3. In [7], the authors have proved that the requirement of distributivity in Definition 1.1 is redundant. That is, if L is a bounded lattice with order-reversing involution $\neg$ and satisfies (R1)–(R5), then L is a bounded distributive lattice.

Definition 1.4. A WDRL-semigroup is an algebra $L = (L, +, 0, \vee, \wedge, -)$ of type (2, 0, 2, 2, 2) such that

(DRL1) $(L, +, 0)$ is a commutative monoid,

(DRL2) $(L, \vee, \wedge)$ is a lattice,

(DRL3) $x + (y \vee z) = (x + y) \vee (x + z), x + (y \wedge z) = (x + y) \wedge (x + z)$ for any $x, y, z \in L$,

(DRL4) if $\leqslant$ denotes the order on L induced by the lattice $(L, \vee, \wedge)$, then for each $x, y \in L$, the element $x - y$ is the smallest $z \in L$ such that $y + z \geqslant x$, (DRL5) L satisfies the identity

$$(((x-y)\vee 0)+y)\wedge(((y-x)\vee 0)+x)\leqslant x\vee y,$$

(DRL6) $x-x\geqslant 0$ for each $x\in L$.

Remark 1.5. If the condition (DRL5) of Definition 1.4 is replaced by (DRL5′), then $L=(L,+,0,\vee,\wedge,-)$ is called a DRL-semigroup defined by K. L. N. Swamy in [9], where

(DRL5′) L satisfies the identity $((x-y)\vee 0)+y\leqslant x\vee y$.

Obviously, each DRL-semigroup is a WDRL-semigroup, but the converse may not be true. This is showed by the following example.

Example 1.6. Suppose $0<a<b<c<1$ and let $L=\{0,a,b,c,1\}$. For all $x,y\in L$, we define $x\wedge y=\min\{x,y\}$, $x\vee y=\max\{x,y\}$. Define $+$ and $-$ on L as follows:

$+$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	b	1	1
b	b	b	1	1	1
c	c	1	1	1	1
1	1	1	1	1	1

$-$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	b	0	0
1	1	c	b	a	0

Then $(L,\wedge,\vee,+,-,0)$ is a WDRL-semigroup. But it is not a DRL-semigroup because $((c-a)\vee 0)+a=c+a=1\nleqslant c\vee a=c$. This shows that WDRL-semigroup is a generalization of DRL-semigroup.

The following proposition shows the relation between WDRL-semigroups and DRL-semigroups.

Proposition 1.7. *A* WDRL-*semigroup L is a* DRL-*semigroup if and only if $((x-y)\vee 0)+y=((y-x)\vee 0)+x$ for all $x,y\in L$.*

P r o o f. Suppose that L is a WDRL-semigroup and satisfies $((x-y)\vee 0)+y=((y-x)\vee 0)+x$ for all $x,y\in L$. Then $(((x-y)\vee 0)+y)\wedge(((y-x)\vee 0)+x)=((x-y)\vee 0)+y$. From (DRL5) it follows that $((x-y)\vee 0)+y\leqslant x\vee y$, i.e. (DRL5′) holds. This together with (DRL1–DRL4, DRL6) implies that L is a DRL-semigroup. The converse is obvious. □

The following example shows that the condition (DRL5) is independent of all the remaining conditions.

Example 1.8. Let $L = \{0, a, b, c, d, 1\}$. For any $x, y \in L$, we define $\vee, \wedge, +$ and—as follows:

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	b	c	d	1
b	b	b	b	b	b	1
c	c	c	b	c	b	1
d	d	d	b	b	d	1
1	1	1	1	1	1	1

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	b	c	d	b
c	0	a	c	c	a	c
d	0	a	d	a	d	d
1	0	a	b	c	d	1

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	1	1	1	1
c	c	1	1	1	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

$-$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	0	0	0
b	b	a	0	a	a	0
c	c	a	0	0	a	0
d	d	a	0	a	0	0
1	1	a	a	a	a	0

Obviously, $(L, \wedge, \vee, +, -, 0)$ satisfies conditions (DRL1)–(DRL4) and (DRL6). But it does not satisfy (DRL5) because $((c-d)\vee 0+d)\wedge((d-c)\vee 0+c) = (a+d)\wedge(a+c) = 1\wedge 1 = 1 \nleqslant c\vee d = b$.

2. Some properties of R_0-algebras and WDRL-semigroups

In this section, we study the properties of R_0-algebras and WDRL-semigroups.

Lemma 2.1 ([8]). *The following properties hold in R_0-algebras:*

(1) $\neg x = x \to 0$,

(2) $x \leqslant y$ *if and only if* $x \to y = 1$,

(3) $\neg x \leqslant x \to y$,

(4) $x \leqslant (x \to y) \to y$,

(5) $\neg(x \vee y) = \neg x \wedge \neg y, \neg(x \wedge y) = \neg x \vee \neg y$,

(6) *if* $x \leqslant y$, *then* $z \to x \leqslant z \to y, y \to z \leqslant x \to z$,

(7) $(x \to y) \vee (y \to x) = 1$,

(8) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(9) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

(10) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Let L be an R_0-algebra. For any $x, y \in L$, we define

$$x + y = \neg x \to y.$$

Proposition 2.2. *If L is an R_0-algebra, then $(L,+,0)$ is a commutative monoid.*

Proof. It suffices to show that $+$ is commutative, associative and $x+0=x$ for any $x\in L$.

Indeed, $x+y=\neg x\to y=\neg y\to\neg(\neg x)=\neg y\to x=y+x$ by (R1), that is, $+$ is commutative.

It follows from (R4) and the commutativity of $+$ that $x+(y+z)=x+(z+y)=\neg x\to(\neg z\to y)=\neg z\to(\neg x\to y)=z+(x+y)=(x+y)+z$. This shows that $+$ is associative.

$x+0=\neg x\to 0=\neg(\neg x)=x$ follows from Lemma 2.1(1) and involution of $\neg$. Therefore $(L,+,0)$ is a commutative monoid. □

Proposition 2.3. *Let L be an R_0-algebra. The following properties hold:*

(1) $x+1=1$,

(2) $x+\neg x=1$,

(3) $x\vee y\leqslant x+y$,

(4) $x\leqslant y$ *if and only if* $\neg x+y=1$,

(5) *if* $x\leqslant y$, *then* $x+z\leqslant y+z$,

(6) $x+(y\vee z)=(x+y)\vee(x+z)$,

(7) $x+(y\wedge z)=(x+y)\wedge(x+z)$.

Proof. (1) $x+1=\neg x\to 1=1$ follows from Lemma 2.1(2).

(2) By Lemma 2.1(2) we have $x+\neg x=\neg x\to\neg x=1$.

(3) Since $\neg x\leqslant x\to y$, it follows that $\neg x\to(x\to y)=1$. This together with (R4) implies that $x\to(x+y)=x\to(\neg x\to y)=\neg x\to(x\to y)=1$. Using Lemma 2.1(2) we get $x\leqslant x+y$. Similarly, $y\leqslant x+y$. Hence $x\vee y\leqslant x+y$.

(4) $x\leqslant y$ if and only if $x\to y=1$ if and only if $\neg(\neg x)\to y=1$ if and only if $\neg x+y=1$ by the involution of $\neg$ and Lemma 2.1(2).

(5) If $x\leqslant y$, then $\neg z\to x\leqslant\neg z\to y$ by Lemma 2.1(6), i.e., $z+x\leqslant z+y$.

(6) By (R5) we obtain that $x+(y\vee z)=\neg x\to(y\vee z)=(\neg x\to y)\vee(\neg x\to z)=(x+y)\vee(x+z)$.

(7) $x+(y\wedge z)=\neg x\to(y\wedge z)=(\neg x\to y)\wedge(\neg x\to z)=(x+y)\wedge(x+z)$ follows from Lemma 2.1(10). □

Proposition 2.4. *If L is an R_0-algebra, then for any $x,y\in L$, there exists the smallest element $z\in L$ such that $y+z\geqslant x$. We denote z by $x-y$, that is,*

(i) $y+(x-y)\geqslant x$,

(ii) *if* $y+z\geqslant x$, *then* $x-y\leqslant z$.

Proof. Let

$$P(x,y)=\{z\in L\colon\ y+z\geqslant x, x,y\in L\}.$$

Since $x+y \geqslant x$ by Proposition 2.3(3), then $x \in P(x,y)$, which implies that $P(x,y) \neq \emptyset$. Next we prove $x - y = \neg(x \to y)$. Since $y + \neg(x \to y) = \neg y \to \neg(x \to y) = (x \to y) \to y$, it follows from Lemma 2.1(4) that $(x \to y) \to y \geqslant x$, i.e., $y + \neg(x \to y) \geqslant x$. This shows that $\neg(x \to y) \in P(x,y)$.

Let $z \in P(x,y)$, i.e., $y + z \geqslant x$, then $x \to (y+z) = 1$ by Lemma 2.1(2), and so $x \to (z+y) = 1$ by Proposition 2.2. On the other hand, from (R1) and (R4), we have $\neg(x \to y) \to z = \neg z \to \neg(\neg(x \to y)) = \neg z \to (x \to y) = x \to (\neg z \to y) = x \to (z+y)$. This leads to $\neg(x \to y) \to z = 1$. By Lemma 2.1(2) we have $\neg(x \to y) \leqslant z$. Hence $x - y = \neg(x \to y)$. □

Remark 2.5. Proposition 2.4 shows that $x - y = \neg(x \to y)$ in R_0-algebras.

Proposition 2.6. *Let L be an R_0-algebra. The following properties hold:*
(1) $x - y \leqslant z$ *if and only if* $x \leqslant y + z$,
(2) $x - y \leqslant x, x - y \leqslant \neg y$,
(3) $x - x = 0, x - 0 = x$,
(4) $(x+y) - y \leqslant x$,
(5) *if* $x \leqslant y$, *then* $x - z \leqslant y - z, z - y \leqslant z - x$,
(6) $x - (y \wedge z) = (x - y) \vee (x - z)$,
(7) $(x - y) \wedge (y - x) = 0$.

Proof. (1) If $x - y \leqslant z$, then $(x-y)+y \leqslant y+z$ by Proposition 2.3(5). In view of Proposition 2.4 we have $(x-y)+y \geqslant x$, and so $x \leqslant y+z$. Conversely, if $x \leqslant y+z$, from Proposition 2.4 it follows that $x - y \leqslant z$.

(2) Since $x+y \geqslant x$, we have $x-y \leqslant x$ by (1). Similarly, from $y + \neg y = 1 \geqslant x$ and (1) we get $x - y \leqslant \neg y$.

(3) From $x = x + 0$ and (1) it follows that $x - x \leqslant 0$, thus $x - x = 0$. Next we prove $x - 0 = x$. Obviously, by (2) we obtain $x - 0 \leqslant x$. On the other hand, from Proposition 2.4 we have $x \leqslant (x-0)+0 = x - 0$. Consequently, $x - 0 = x$.

(4) Since $x+y \leqslant x+y$, we deduce $(x+y) - y \leqslant x$ from (1).

(5) From Proposition 2.4 it follows that $y \leqslant (y-z)+z$. If $x \leqslant y$, then $x \leqslant (y-z)+z$, thus $x - z \leqslant y - z$ by (1). On the other hand, $z \leqslant (z-x)+x$ follows from Proposition 2.4. If $x \leqslant y$, then $(z-x)+x \leqslant (z-x)+y$, and so $z \leqslant (z-x)+y$. Hence $z - y \leqslant z - x$ by (1).

(6) $x - (y \wedge z) \leqslant t$ if and only if $x \leqslant t + (y \wedge z) = (t+y) \wedge (t+z)$ if and only if $x \leqslant t+y, x \leqslant t+z$ if and only if $x-y \leqslant t, x-z \leqslant t$ if and only if $(x-y) \vee (x-z) \leqslant t$ by repeatedly using (1) and Lemma 2.3(7). Hence $x - (y \wedge z) = (x-y) \vee (x-z)$.

(7) From Lemma 2.1(7), we have $(x \to y) \vee (y \to x) = 1$, then $\neg(x \to y) \wedge \neg(y \to x) = 0$. By Proposition 2.4 we obtain $x - y = \neg(x \to y)$, thus $(x-y) \wedge (y-x) = 0$. □

Proposition 2.7. *Let L be an R_0-algebra. Then for any $x, y \in L$,*

$$((x-y)+y) \wedge ((y-x)+x) = x \vee y.$$

P r o o f. From Propositions 2.3(3) and 2.4 we have $(x-y)+y \geqslant y$ and $(x-y)+y \geqslant x$, respectively. Hence $(x-y)+y \geqslant x \vee y$. Similarly, $(y-x)+x \geqslant x \vee y$. This leads to $((x-y)+y) \wedge ((y-x)+x) \geqslant x \vee y$. Conversely, $((x-y)+y) \wedge ((y-x)+x) = (((x-y)+y) \wedge ((y-x)+x)) - 0 = (((x-y)+y) \wedge ((y-x)+x)) - ((x-y) \wedge (y-x)) = ((((x-y)+y) \wedge ((y-x)+x)) - (x-y)) \vee ((((x-y)+y) \wedge ((y-x)+x)) - (y-x)) \leqslant (((x-y)+y)-(x-y)) \vee (((y-x)+x)-(y-x)) \leqslant y \vee x = x \vee y$ by using Proposition 2.6(3, 7, 6, 5, 4). Therefore $((x-y)+y) \wedge ((y-x)+x) = x \vee y$. $\square$

Lemma 2.8. *The following properties hold in* WDRL*-semigroups:*

(1) *if $x \leqslant y$, then $x+z \leqslant y+z$,*
(2) *if $x \leqslant y$, then $x-z \leqslant y-z, z-y \leqslant z-x$,*
(3) *$x-y \leqslant z$ if and only if $x \leqslant y+z$,*
(4) $(x-y)-z = (x-z)-y$,
(5) $x-(y+z) = (x-y)-z$,
(6) $(x-y)+y \geqslant x$,
(7) $(x+y)-y \leqslant x$,
(8) $(((x-y) \vee 0)+y) \wedge (((y-x) \vee 0)+x) = x \vee y$,
(9) $x-x=0$.

P r o o f. The proof is similar to that in [9]. $\square$

3. Main results

In this section, the relation between R_0-algebras and WDRL-semigroups is discussed, and it will be proved that the category of R_0-algebras is equivalent to the category of some WDRL-semigroups.

Theorem 3.1. *Let $(L, \vee, \wedge, \neg, \rightarrow, 0, 1)$ be an R_0-algebra. Define*

$$x+y = \neg x \rightarrow y, \ x-y = \neg(x \rightarrow y),$$

then $(L, \vee, \wedge, +, -, 0)$ is a bounded WDRL*-semigroup, and satisfies*

(DRL7) $$1-(1-x) = x,$$

and

$$\text{(DRL8)} \qquad (x-y)\wedge((x\wedge\neg y)-(x-y))=0.$$

P r o o f. From Propositions 2.2, 2.3(6, 7), 2.4, 2.7 and Definition 1.1, we see that $(L,\vee,\wedge,+,-,0)$ is a bounded WDRL-semigroup with the greatest element 1. Now we prove that (DRL7) and (DRL8) hold. Indeed, $1-(1-x)=\neg(1\to\neg(1\to x))=\neg\neg x=x$. Thus (DRL7) holds. By (R6) we have $(x\to y)\vee((x\to y)\to(\neg x\vee y))=1$. Thus $\neg(x\to y)\wedge\neg((x\to y)\to(\neg x\vee y))=0$. Since $x-y=\neg(x\to y)$, then $x-y=\neg(x\to y)=\neg(\neg y\to\neg x)=\neg y-\neg x$. Hence $\neg((x\to y)\to(\neg x\vee y))=(x\to y)-(\neg x\vee y)=\neg(\neg x\vee y)-\neg(x\to y)=(x\wedge\neg y)-(x-y)$. Therefore $(x-y)\wedge((x\wedge\neg y)-(x-y))=\neg(x\to y)\wedge\neg((x\to y)\to(\neg x\vee y))=0$. This shows that (DRL8) holds. $\square$

Theorem 3.2. *Let* $(L,+,0,\vee,\wedge,-)$ *be a* WDRL-*semigroup with the greatest element* 1 *and satisfy the identities* (DRL7) *and* (DRL8). *Define*

$$\neg x=1-x,\ \ x\to y=\neg x+y,$$

then $(L,\vee,\wedge,\neg,\to,0,1)$ *is an* R_0*-algebra.*

P r o o f. (i) Firstly, we prove that $\neg$ is an order-reversing involution mapping.

If $x\leqslant y$, from Lemma 2.8 (2) we have $1-y\leqslant 1-x$, i.e., $\neg y\leqslant\neg x$. This shows that $\neg$ is an order-reversing mapping. Since $\neg\neg x=1-\neg x=1-(1-x)$, it follows from (DRL7) that $\neg\neg x=x$. Hence $\neg$ is an order-reversing involution mapping.

(ii) Now we prove that if a WDRL-semigroup L has the greatest element 1 and satisfies (DRL7), then L is a bounded lattice and 0 is the smallest element of L.

Indeed, by (DRL4) we have $(1-x)+x\geqslant 1$. Since 1 is the largest element of L, it follows that $(1-x)+x=1$, which implies that $1-0=(1-0)+0=1$. By (DRL7) we have $1-(1-0)=0$, and so $1-1=0$. On the other hand, since 1 is the largest element of L, we have $1-x\leqslant 1$, and so $1-1\leqslant 1-(1-x)$. By (DRL7) we obtain $0\leqslant x$. This shows that 0 is the smallest element of L. Hence $(L,\wedge,\vee,0,1)$ is a bounded lattice.

From (i) and (ii), we have $(L,\wedge,\vee,\neg,0,1)$ is a bounded lattice with the order-reversing involution $\neg$. Now we prove that (R1)–(R6) hold.

(R1) By (i) we have $\neg y\to\neg x=\neg(\neg y)+\neg x=y+\neg x=x\to y$. Thus (R1) holds.

(R2) $1\to x=\neg 1+x=(1-1)+x=0+x=x$ follows from (ii) and (DRL1).

(R3) Since $(x\to y)\to(x\to z)=(\neg x+y)\to(\neg x+z)=\neg(\neg x+y)+(\neg x+z)=(\neg y-\neg x)+(\neg x+z)=((\neg y-\neg x)+\neg x)+z\geqslant\neg y+z=y\to z$ by Lemma 2.8(5, 6), we have $(y\to z)\vee((x\to y)\to(x\to z))=(x\to y)\to(x\to z)$.

(R4) $x \to (y \to z) = \neg x + (\neg y + z) = \neg y + (\neg x + z) = y \to (x \to z)$ by (DRL1).

(R5) $x \to (y \vee z) = \neg x + (y \vee z) = (\neg x + y) \vee (\neg x + z) = (x \to y) \vee (x \to z)$ by (DRL3).

(R6) From (i), we know that $\neg$ is an order-reversing involution mapping, which implies that $\neg(x \wedge y) = \neg x \vee \neg y$ for any $x, y \in L$. Thus $\neg(x-y) \vee \neg((x \wedge \neg y)-(x-y)) = \neg 0 = 1-0 = 1$ by (DRL8) and (ii). Since $\neg(\neg x+y) = 1-(\neg x+y) = (1-\neg x)-y = (1-(1-x))-y = x-y$ by Lemma 2.8(5) and (DRL7), we have $\neg(x-y) = \neg x+y = x \to y$, and $\neg((x \wedge \neg y)-(x-y)) = \neg(x \wedge \neg y)+(x-y) = (\neg x \vee y)+(x-y) = (x-y)+(\neg x \vee y) = \neg(\neg(x-y))+(\neg x \vee y) = (\neg(x-y)) \to (\neg x \vee y) = (x \to y) \to (\neg x \vee y)$. Consequently, $(x \to y) \vee ((x \to y) \to (\neg x \vee y)) = \neg(x-y) \vee \neg((x \wedge \neg y)-(x-y)) = 1$. This shows that (R6) holds.

From the above and Remark 1.3, we see that $(L, \wedge, \vee, \neg, \to, 0, 1)$ is an R_0-algebra. $\square$

From Theorems 3.1 and 3.2, we can easily verify the following theorems.

Theorem 3.3. *Let* $(L_i, \vee_i, \wedge_i, \neg_i, \to_i, 0_i, 1_i)$ $(i = 1, 2)$ *be* R_0*-algebras and* $f\colon L_1 \to L_2$ *a homomorphism of* R_0*-algebras. Then* f *is also a homomorphism of the induced* WDRL*-semigroups* $(L_1, +_1, 0_1, \wedge_1, \vee_1, -_1)$ *and* $(L_2, +_2, 0_2, \wedge_2, \vee_2, -_2)$.

Theorem 3.4. *Let* $i = 1, 2$ *and* $(L_i, +_i, 0_i, \vee_i, \wedge_i, -_i)$ *be* WDRL*-semigroups with the greatest elements* 1_i*, respectively, and satisfy the identities* (DRL7) *and* (DRL8). *Let* $f\colon L_1 \to L_2$ *be a homomorphism of* WDRL*-semigroups such that* $f(1_1) = 1_2$. *Then* f *is also a homomorphism of the induced* R_0*-algebras* $(L_1, \wedge_1, \vee_1, \neg_1, \to_1, 0_1, 1_1)$ *and* $(L_2, \wedge_2, \vee_2, \neg_2, \to_2, 0_2, 1_2)$.

Theorem 3.5. R_0*-algebras are categorically equivalent to bounded* WDRL*-semigroups satisfying the identities* (DRL7) *and* (DRL8).

Proof. If $(L, \wedge, \vee, \neg, \to, 0, 1)$ is an R_0-algebra, let $\Gamma(L) = (L, +, 0, \wedge, \vee, -, 1)$. For any R_0-algebras L_1, L_2 and R_0-algebra homomorphism $f\colon L_1 \to L_2$, we define $\Gamma(f)\colon \Gamma(L_1) \to \Gamma(L_2)$ by $\Gamma(f) = f$. If we denote by $\Re_0$ the category of all R_0-algebras and by $WDRL$ the category of all bounded WDRL-semigroups satisfying (DRL7) and (DRL8), then Theorems 3.3 and 3.4 imply that $\Gamma\colon \Re_0 \to WDRL$ is a functor which is an equivalence. $\square$

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