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FINITE RANK OPERATORS IN JACOBSON RADICAL $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$

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Abstract. In this paper we investigate finite rank operators in the Jacobson radical $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ of $\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$, where $\mathcal{N}$, $\mathcal{M}$ are nests. Based on the concrete characterizations of rank one operators in $\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$ and $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$, we obtain that each finite rank operator in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ can be written as a finite sum of rank one operators in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and the weak closure of $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ equals $\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$ if and only if at least one of $\mathcal{N}$, $\mathcal{M}$ is continuous.

Keywords: Jacobson radical, finite rank operator

MSC 2000: 47L75

1. Introduction

Finite rank operators and rank one operators have played a central role in the theory of nest algebras since the inception of that theory. For example, Ringrose make very effective use of the rank one operators in a nest algebra in his characterization of the radical of a nest algebra [10] and in his theorem that algebraic isomorphisms of nest algebras are necessarily spatial [11]. In a nest algebra, any finite rank operator is a finite sum of rank one operators from the nest algebra [2]. The theorem has been verified for special cases of reflexive algebras, namely algebras whose subspace lattice $\mathcal{L}$ forms an atomic Boolean algebra [9] or $\mathcal{L}$ is commutative and has finite width [6].

Recall that the Jacobson radical of a Banach algebra coincides with the elements T such that AT is quasinilpotent for every A in the algebra. The Jacobson radical of a Banach algebra is a structural object that has been frequently studied over the years. In [10], Ringrose characterized the Jacobson radical of a nest algebra. In [1], Davidson and Orr pushed the characterization further to the case of all width two

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CSL algebras. The result is essential to our paper. For a subspace lattice $\mathcal{L}$, we denote by $\mathcal{R}_{\mathcal{L}}$ the Jacobson radical of $\operatorname{Alg}\mathcal{L}$.

The main purpose of this paper is to study finite rank operators in the radical $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ of $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$. As we know, each finite rank operator in the radical of a nest algebra can be written as a finite sum of rank one operators in this radical. This result owes much to the total order of $\mathcal{N}$. In the case of $\mathcal{N}\otimes\mathcal{M}$, the key to the main result is Lemma 4 which gives a concrete description of rank one operators in $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$. As an application of Lemma 4, we give a simple proof of the tensor product formula in [3]. At last, we compute the weak closure of the radical $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and show that $\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} = \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$ if and only if at least one of $\mathcal{N}$, $\mathcal{M}$ is continuous.

Let us introduce some notation and terminology. $\mathcal{H}$ represents a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ the set of finite-rank operators on $\mathcal{H}$. A sublattice $\mathcal{L}$ of the projection lattice of $\mathcal{B}(\mathcal{H})$ is said to be a subspace lattice if it contains 0 and I and is strongly closed, where we identify projections with their ranges. If the elements of $\mathcal{L}$ pairwise commute, $\mathcal{L}$ is a commutative subspace lattice (CSL). A subspace lattice is completely distributive if distributive laws are valid for families of arbitrary cardinality (see [8]). A nest $\mathcal{N}$ is a totally ordered subspace lattice. For $L\in\mathcal{L}$, we define

$$L_- = \bigvee\{E\in\mathcal{L}\colon\ L\nleq E\}.$$

In the case of nests, either N_- is the immediate predecessor of N or $N = N_-$. If $N = N_-$ for any $N\in\mathcal{N}$, $\mathcal{N}$ is called a continuous nest. If $\mathcal{L}$ is a subspace lattice, $\operatorname{Alg}\mathcal{L}$ denotes the set of operators in $\mathcal{B}(\mathcal{H})$ that leave the elements of $\mathcal{L}$ invariant. If $\mathcal{L}$ is a CSL, $\operatorname{Alg}\mathcal{L}$ is said to be a CSL algebra. If $\mathcal{L}$ is a nest, $\operatorname{Alg}\mathcal{L}$ is said to be a nest algebra.

Let $\mathcal{H}_i$ $(i=1,2)$ be complex Hilbert spaces. If $\mathcal{L}_i\subset\mathcal{B}(\mathcal{H}_i)$ $(i=1,2)$ are subspace lattices, $\mathcal{L}_1\otimes\mathcal{L}_2$ is the subspace lattice in $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$ generated by $\{L_1\otimes L_2\colon\ L_i\in\mathcal{L}_i,\ i=1,2\}$. If $\mathcal{S}_i\subset\mathcal{B}(\mathcal{H}_i)$ $(i=1,2)$ are subspaces, then $\mathcal{S}_1\otimes\mathcal{S}_2$ denotes the linear span of $\{S_1\otimes S_2\colon\ S_i\in\mathcal{S}_i\}$; $\mathcal{S}_1\otimes_w\mathcal{S}_2$ denotes the weak closure of $\mathcal{S}_1\otimes\mathcal{S}_2$ in $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$.

2. Finite rank operators

In the sequel we suppose that $\mathcal{N}$ and $\mathcal{M}$ are nests on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively; and that $\mathcal{N}\otimes\mathcal{M}$ is the tensor product of $\mathcal{N}$ and $\mathcal{M}$. $\mathcal{R}_{\mathcal{N}}$, $\mathcal{R}_{\mathcal{M}}$ and $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ denote Jacobson radicals of $\operatorname{Alg}\mathcal{N}$, $\operatorname{Alg}\mathcal{M}$ and $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$ respectively.

For $x, y \in \mathcal{H}$, the rank-one operator xy^* is defined by the equation

$$(xy^*)(z) = \langle z, y\rangle\, x \quad \forall\, z \in \mathcal{H}.$$

Lemma 1. *Let $\mathcal{L}$ be a subspace lattice and let $\{N_\alpha\colon \alpha \in \Lambda\}$ be a family of elements in $\mathcal{L}$. Then $\Big(\bigvee\limits_{\alpha\in\Lambda} N_\alpha\Big)_- = \bigvee\limits_{\alpha\in\Lambda} (N_\alpha)_-$.*

Proof. For any $\alpha \in \Lambda$, since $N_\alpha \leqslant \bigvee\limits_{\alpha\in\Lambda} N_\alpha$, it follows that if $F \ngeqslant N_\alpha$ then $F \ngeqslant \bigvee\limits_{\alpha\in\Lambda} N_\alpha$; hence $(N_\alpha)_- \leqslant \Big(\bigvee\limits_{\alpha\in\Lambda} N_\alpha\Big)_-$. So $\bigvee\limits_{\alpha\in\Lambda} (N_\alpha)_- \leqslant \Big(\bigvee\limits_{\alpha\in\Lambda} N_\alpha\Big)_-$.

Conversely, suppose that $F \ngeqslant \bigvee\limits_{\alpha\in\Lambda} N_\alpha$. If $F \geqslant N_\alpha$ for each $\alpha \in \Lambda$, then $F \geqslant \bigvee\limits_{\alpha\in\Lambda} N_\alpha$; hence, there exists $\alpha_0 \in \Lambda$ such that $F \ngeqslant N_{\alpha_0}$. Thus $F \leqslant \bigvee\limits_{\alpha\in\Lambda} (N_\alpha)_-$. Thus, $\Big(\bigvee\limits_{\alpha\in\Lambda} N_\alpha\Big)_- = \bigvee\{F\colon\ F \ngeqslant \bigvee\limits_{\alpha\in\Lambda} N_\alpha\} \leqslant \bigvee\limits_{\alpha\in\Lambda} (N_\alpha)_-$ and we are done. $\square$

Set $\mathcal{N} \otimes I = \{N \otimes I\colon\ N \in \mathcal{N}\}$; $\mathcal{N} \otimes I$ is a nest on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Lemma 2. *Suppose that $N \in \mathcal{N}$ and $M \in \mathcal{M}$, then $(N \otimes M)_- = (N_- \otimes I) \vee (I \otimes M_-)$ and $(N \otimes M)_-^\perp = N_-^\perp \otimes M_-^\perp$ in $\mathcal{N} \otimes \mathcal{M}$.*

Proof. First, we prove the following assertion:

$$(N \otimes M)_- = \bigvee\{F\colon\ F \ngeqslant N \otimes M\} = \bigvee\{E_1 \otimes E_2\colon\ E_1 \otimes E_2 \ngeqslant N \otimes M\}.$$

Indeed, suppose that $F \ngeqslant N \otimes M$. For any $E_1 \otimes E_2 \leqslant F$ we have $E_1 \otimes E_2 \ngeqslant N \otimes M$. Thus,

$$\{E_1 \otimes E_2\colon\ E_1 \otimes E_2 \leqslant F\} \subseteq \{E_1 \otimes E_2\colon\ E_1 \otimes E_2 \ngeqslant N \otimes M\}.$$

Hence it follows from [3] Proposition 2.4 that

$$F = \bigvee\{E_1 \otimes E_2\colon\ E_1 \otimes E_2 \leqslant F\} \leqslant \bigvee\{E_1 \otimes E_2\colon\ E_1 \otimes E_2 \ngeqslant N \otimes M\}$$

and

$$(N \otimes M)_- = \bigvee\{F\colon\ F \ngeqslant N \otimes M\} \leqslant \bigvee\{E_1 \otimes E_2\colon\ E_1 \otimes E_2 \ngeqslant N \otimes M\}.$$

The converse inequality is obvious.

Secondly, we show that $E_1 \otimes E_2 \geqslant N \otimes M$ if and only if $E_1 \geqslant N$ and $E_2 \geqslant M$. Suppose that $E_1 \otimes E_2 \geqslant N \otimes M$. If $E_1 < N$, choose nonzero vectors $x_1 \in N \ominus E_1$ and $x_2 \in M$. Thus $x_1 \otimes x_2 \in N \otimes M \subseteq E_1 \otimes E_2$. But $(E_1 \otimes E_2)(x_1 \otimes x_2) = 0$ shows that $x_1 \otimes x_2 \notin E_1 \otimes E_2$. This contradiction shows that $E_1 \geqslant N$. Similarly, $E_2 \geqslant M$.

The converse implication is obvious. Hence $E_1 \otimes E_2 \not\geq N \otimes M$ if and only if $E_1 \not\geq N$ or $E_2 \not\geq M$.

Therefore

$$\begin{aligned}(N \otimes M)_- &= \bigvee\{E_1 \otimes E_2\colon\ E_1 \otimes E_2 \not\geq N \otimes M\} \\ &= \bigvee\{E_1 \otimes E_2\colon\ E_1 < N \ \text{ or } \ E_2 < M\} \\ &= (N_- \otimes I) \vee (I \otimes M_-).\end{aligned}$$

We can easily prove that $(N_- \otimes I)^\perp = N_-^\perp \otimes I$, thus

$$(N \otimes M)_-^\perp = (N_- \otimes I)^\perp \wedge (I \otimes M_-)^\perp = (N_-^\perp \otimes I) \wedge (I \otimes M_-^\perp) = N_-^\perp \otimes M_-^\perp.$$

□

The following result of Longstaff [8] is essential to this paper.

Lemma 3. *Let $\mathcal{L}$ be a subspace lattice. Then $xy^* \in \operatorname{Alg}\mathcal{L}$ if and only if there is an element $L \in \mathcal{L}$ such that $x \in L$ and $y \in L_-^\perp$.*

Lemma 4. *The rank one operator xy^* belongs to $\operatorname{Alg}(\mathcal{N} \otimes \mathcal{M})$ if and only if there exist $N \in \mathcal{N}$ and $M \in \mathcal{M}$ such that $x \in N \otimes M$ and $y \in N_-^\perp \otimes M_-^\perp$.*

Proof. Since $\mathcal{N} \otimes \mathcal{M} = (\mathcal{N} \otimes I) \vee (I \otimes \mathcal{M})$, so

$$\operatorname{Alg}(\mathcal{N} \otimes \mathcal{M}) = \operatorname{Alg}(\mathcal{N} \otimes I) \cap \operatorname{Alg}(I \otimes \mathcal{M}).$$

Now suppose that $xy^* \in \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M})$. Thus $xy^* \in \operatorname{Alg}(\mathcal{N} \otimes I)$; by the definition of $\mathcal{N} \otimes I$ and Lemma 2 and Lemma 3, there is an element $N \in \mathcal{N}$ such that $x \in N \otimes I$ and $y \in (N \otimes I)_-^\perp = N_-^\perp \otimes I$. Similarly, there exists $M \in \mathcal{M}$ such that $x \in I \otimes M$ and $y \in I \otimes M_-^\perp$. Hence, $x \in N \otimes M$ and $y \in N_-^\perp \otimes M_-^\perp$.

For the converse, if $x \in N \otimes M$ and $y \in N_-^\perp \otimes M_-^\perp$ then, in particular, $x \in N \otimes I$ and $y \in N_-^\perp \otimes I$. Lemma 2 and Lemma 3 show that $xy^* \in \operatorname{Alg}(\mathcal{N} \otimes I)$. Similarly, $xy^* \in \operatorname{Alg}(I \otimes \mathcal{M})$. Hence

$$xy^* \in \operatorname{Alg}(\mathcal{N} \otimes I) \cap \operatorname{Alg}(I \otimes \mathcal{M}) = \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M}).$$

□

As an application of Lemma 4 we give a simple proof of the tensor product formula in [3].

Theorem 5 ([3], Theorem 2.6). $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M}) = \operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M}$.

Proof. Each of the operators which generate $\operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M}$ leaves invariant each of the projections which generate $\mathcal{N}\otimes\mathcal{M}$; therefore

$$\operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M} \subseteq \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M}).$$

It remains to show that $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M}) \subseteq \operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M}$. It follows from [5, Theorem 10] that $\mathcal{N}\otimes\mathcal{M}$ is a completely distributive CSL. Thus, by virtue of [7, Theorem 3], $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$ is weakly generated by the rank one operators in itself. So it suffices to show that each rank one operator in $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$ belongs to $\operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M}$. Now for any $N\in\mathcal{N}$, $M\in\mathcal{M}$ and $x_i, y_i\in\mathcal{H}_i$ $(i=1,2)$, we have that

$$\begin{aligned}(N\otimes M)&[(x_1\otimes x_2)(y_1\otimes y_2)^*](N_-^\perp\otimes M_-^\perp)\\ &= (N\otimes M)[(x_1y_1^*)\otimes(x_2y_2^*)](N_-^\perp\otimes M_-^\perp)\\ &= N(x_1y_1^*)N_-^\perp\otimes M(x_2y_2^*)M_-^\perp \in \operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M}.\end{aligned}$$

(It is routine to verify that $(x_1\otimes x_2)(y_1\otimes y_2)^* = (x_1y_1)^*\otimes(x_2y_2^*)$.)

For any rank one operator $zw^*\in\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$, it follows from Lemma 4 that there exist $N\in\mathcal{N}$ and $M\in\mathcal{M}$ such that $z\in N\otimes M$ and $w\in N_-^\perp\otimes M_-^\perp$. Since $z,w\in\mathcal{H}_1\otimes\mathcal{H}_2$, there exist sequences $\{z_n\}$ and $\{w_n\}$ such that

$$z_n \xrightarrow{\|\cdot\|} z \quad\text{and}\quad w_n \xrightarrow{\|\cdot\|} w,$$

where $\{z_n\}$, $\{w_n\}$ are finite linear combinations of simple tensors. Thus,

$$(N\otimes M)(z_nw_n^*)(N_-^\perp\otimes M_-^\perp) \xrightarrow{\|\cdot\|} (N\otimes M)(zw^*)(N_-^\perp\otimes M_-^\perp) = zw^*.$$

The above paragraph shows that

$$(N\otimes M)(z_nw_n^*)(N_-^\perp\otimes M_-^\perp)\in\operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M},$$

so $zw^*\in\operatorname{Alg}\mathcal{N}\otimes_w \operatorname{Alg}\mathcal{M}$. This completes the proof. □

Lemma 6. *If $(N\ominus N_-)\otimes(M\ominus M_-)\neq 0$, then it is an atom of $\mathcal{N}\otimes\mathcal{M}$.*

Proof. Recall that an atom P of $\mathcal{N}\otimes\mathcal{M}$ is an interval projection from $\mathcal{N}\otimes\mathcal{M}$ such that for any $E\in\mathcal{N}\otimes\mathcal{M}$, either $P\leqslant E$ or $PE=0$ (see [4]). Set $P=(N\ominus N_-)\otimes(M\ominus M_-)$. $P = N\otimes M - [(N_-\otimes M)\vee(N\otimes M_-)]$ is an interval projection. For any $E=E_1\otimes E_2\in\mathcal{N}\otimes\mathcal{M}$, since $\mathcal{N}$ is totally ordered, either $E_1\leqslant N_-$ or $E_1\geqslant N$. If $E_1\leqslant N_-$ then $P(E_1\otimes E_2)=0$; if $E_1\geqslant N$, since $\mathcal{M}$ is also totally ordered, either $E_2 \leqslant M_-$ or $E_2 \geqslant M$. If $E_2 \leqslant M_-$ then $P(E_1 \otimes E_2) = 0$; and if $E_2 \geqslant M$ then $P \leqslant E_1 \otimes E_2$. Hence for any $E = E_1 \otimes E_2$, either $P \leqslant E_1 \otimes E_2$ or $P(E_1 \otimes E_2) = 0$.

Now for any $E \in \mathcal{N} \otimes \mathcal{M}$, by virtue of [3, Proposition 2.4] we have

$$E = \bigvee \{E_1 \otimes E_2 \colon\ E_1 \otimes E_2 \leqslant E\}.$$

If $P(E_1 \otimes E_2) = 0$ for any $E_1 \otimes E_2 \leqslant E$, then $PE = 0$; if there exist E_1, E_2 with $E_1 \otimes E_2 \leqslant E$ such that $P(E_1 \otimes E_2) \neq 0$ then it follows from the result of the above paragraph that $P \leqslant E_1 \otimes E_2$ and $P \leqslant E$. □

Proposition 7. *If a rank-one operator xy^* belongs to* $\mathrm{Alg}(\mathcal{N} \otimes \mathcal{M})$, *then the following statements are equivalent:*

1) $xy^* \in \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$;
2) *there exists $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x \in L$ and $y \in L^\perp$.*

Proof. 1) $\Rightarrow$ 2) Since $xy^* \in \mathrm{Alg}(\mathcal{N} \otimes \mathcal{M})$, it follows from Lemma 4 that there exist $N \in \mathcal{N}$ and $M \in \mathcal{M}$ such that $x \in N \otimes M$ and $y \in N_-^\perp \otimes M_-^\perp$. Set $G_1 = (N \ominus N_-) \otimes (M \ominus M_-)$, $G_2 = (N \otimes M) \ominus G_1 = (N_- \otimes M) \vee (N \otimes M_-)$ and $G_3 = (N_-^\perp \otimes M_-^\perp) \ominus G_1 = (N^\perp \otimes M_-^\perp) \vee (N_-^\perp \otimes M^\perp)$. If $G_1 = 0$ then $N \ominus N_- = 0$ or $M \ominus M_- = 0$. In this case $L = N \otimes M$ satisfies the condition in 2). Now we suppose that $G_1 \neq 0$. Since $N \otimes M = G_1 + G_2$ and $N_-^\perp \otimes M_-^\perp = G_1 + G_3$, we have

$$\begin{aligned} xy^* &= (G_1 + G_2)(xy^*)(G_1 + G_3) \\ &= (N \otimes M)(xy^*)G_3 + G_2(xy^*)G_1 + G_1(xy^*)G_1. \end{aligned}$$

Since $xy^* \in \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and G_1 is an atom of $\mathcal{N} \otimes \mathcal{M}$, it follows from [1, Theorem 4.8] that $G_1(xy^*)G_1 = 0$. Hence $x \in G_1^\perp$ or $y \in G_1^\perp$. If $x \in G_1^\perp$ then $x \in G_2$ and $y \in G_1 + G_3 = N_-^\perp \otimes M_-^\perp \subseteq G_2^\perp$; if $y \in G_1^\perp$, then $y \in G_3 \subseteq (N \otimes M)^\perp$ and $x \in N \otimes M$.

2) $\Rightarrow$ 1) If there exists $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x \in L$ and $y \in L^\perp$, then for any $T \in \mathrm{Alg}(\mathcal{N} \otimes \mathcal{M})$ we have $L^\perp T L = 0$ and

$$[(xy^*)T]^n = [L(xy^*)L^\perp T]^n = 0 \quad \forall\, n \geqslant 2.$$

So $(xy^*)T$ is quasinilpotent. It follows from the definition of $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and from $xy^* \in \mathrm{Alg}(\mathcal{N} \otimes \mathcal{M})$ that $xy^* \in \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. □

Theorem 8. *Each finite rank operator in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ can be written as a finite sum of rank one operators in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$.*

Proof. Suppose that F is a finite rank operator in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. Since $F\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}\subseteq \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$, it follows from [6, Corollary 7] that F can be written as a finite sum of rank one operators in $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$. Write

$$F=\sum_{i=1}^{n}x_iy_i^*,\quad \text{where}\quad x_iy_i^*\in\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})\quad\text{for }\ i=1,\ldots,n.$$

For any fixed i ($1\leqslant i\leqslant n$), since $x_iy_i^*\in\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$, it follows from Lemma 4 that there exist $N_i\in\mathcal{N}$ and $M_i\in\mathcal{M}$ such that

$$x_i\in N_i\otimes M_i\quad\text{and}\quad y_i\in N_{i-}^{\perp}\otimes M_{i-}^{\perp}.$$

If $N_i=N_{i-}$ or $M_i=M_{i-}$, Proposition 7 shows that $x_iy_i^*\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. Without loss of generality, we can suppose that $N_i\neq N_{i-}$ and $M_i\neq M_{i-}$. Set

$$\begin{aligned}G_i^{(1)}&=(N_i\ominus N_{i-})\otimes(M_i\ominus M_{i-}),\\ G_i^{(2)}&=(N_i\otimes M_i)\ominus G_i^{(1)},\\ G_i^{(3)}&=(N_{i-}^{\perp}\otimes M_{i-}^{\perp})\ominus G_i^{(1)}.\end{aligned}$$

Thus

$$\begin{aligned}x_iy_i^*&=(G_i^{(1)}+G_i^{(2)})(x_iy_2^*)(G_i^{(1)}+G_i^{(3)})\\ &=(N_i\otimes M_i)(x_iy_i^*)G_i^{(3)}+G_i^{(2)}(x_iy_i^*)G_i^{(1)}+G_i^{(1)}(x_iy_i^*)G_i^{(1)}.\end{aligned}$$

Since $N_i\otimes M_i\perp G_i^{(3)}$, $G_i^{(2)}\perp G_i^{(1)}$ and $x_iy_i^*\in\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$, so $(N_i\otimes M_i)(x_iy_i^*)G_i^{(3)}$ and $G_i^{(2)}(x_iy_i^*)G_i^{(1)}$ belong to $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ by Proposition 7. Now we consider the operator $G_i^{(1)}(x_iy_i^*)G_i^{(1)}$.

Set $\Lambda_i=\{j\colon\ G_j^{(1)}=G_i^{(1)}\}$. Since $G_i^{(1)}$ is an atom of $\mathcal{N}\otimes\mathcal{M}$ and $G_i^{(1)}\in\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$, we have

$$G_i^{(1)}FG_i^{(1)}=\sum_{j\in\Lambda_i}G_j^{(1)}(x_jy_j^*)G_j^{(1)}\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}.$$

By virtue of [1, Theorem 4.8], $G_i^{(1)}FG_i^{(1)}=0$. Owing to the arbitrariness of i, we obtain that

$$\sum_{j=1}^{n}G_j^{(1)}(x_jy_j^*)G_j^{(1)}=0.$$

Hence

$$F=\sum_{i=1}^{n}x_iy_i^*=\sum_{i=1}^{n}(N_i\otimes M_i)(x_iy_i^*)G_i^{(3)}+G_i^{(2)}(x_iy_i^*)G_i^{(1)}.$$

Thus, F can be written as a finite sum of rank one operators in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. $\square$

Lemma 9. *Suppose that $\mathcal{U}_\tau$ is a weakly closed* $\mathrm{Alg}(\mathcal{N} \otimes \mathcal{M})$*-module determined by an order homomorphism τ from $\mathcal{N} \otimes \mathcal{M}$ into itself. Then a rank one operator xy^* belongs to $\mathcal{U}_\tau$ if and only if there exists an element $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x \in L$ and $y \in L_\sim^\perp$, where $L_\sim = \bigvee\{G \colon L \nleqslant \tau(G)\}$.*

P r o o f. Suppose that there exists an element $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x \in L$ and $y \in L_\sim^\perp$. For any $G \in \mathcal{N} \otimes \mathcal{M}$, if $L \leqslant \tau(G)$ then

$$(xy^*)G = L(xy^*)L_\sim^\perp G \leqslant L \leqslant \tau(G);$$

if $L \nleqslant \tau(G)$, then $G \leqslant L_\sim$ and

$$(xy^*)G = L(xy^*)L_\sim^\perp G = (0) \subseteq \tau(G).$$

Thus the rank one operator xy^* belongs to $\mathcal{U}_\tau$.

Conversely, suppose that $xy^* \in \mathcal{U}_\tau$. Set $L = \bigwedge\{G \in \mathcal{N} \otimes \mathcal{M} \colon Gx = x\}$, certainly $x \in L$. For any $G \in \mathcal{N} \otimes \mathcal{M}$ and $L \nleqslant \tau(G)$, it follows from the definition of L that $\tau(G)x \neq x$. If $Gy \neq 0$, since $(xy^*)G = \tau(G)(xy^*)G$, we have that

$$[(xy^*)G](Gy) = [\tau(G)(xy^*)G](Gy)$$

and

$$\|Gy\|^2 x = \|Gy\|^2 \tau(G)x.$$

This contradicts $\tau(G)x \neq x$, so $Gy = 0$. From the definition of $L_\sim$ we have $L_\sim y = 0$ and $y \in L_\sim^\perp$. $\square$

Lemma 10. *Let $\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \colon TL \subseteq L_- \ \forall L \in \mathcal{N} \otimes \mathcal{M}\}$. Then a rank one operator xy^* belongs to $\mathcal{U}$ if and only if there exists $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x \in L$ and $y \in L^\perp$.*

P r o o f. Necessity. It follows from Lemma 9 that if $xy^* \in \mathcal{U}$ then there is $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x \in L$ and $y \in L_\sim^\perp$, where $L_\sim = \bigvee\{E \colon L \nleqslant E_-\}$. Now we compute $L_\sim$. Since $L = \bigvee\{L_1 \otimes L_2 \colon L_1 \otimes L_2 \leqslant L\}$, it is easy to show that

$$\{E \colon L \nleqslant E_-\} = \bigcup_{L_1 \otimes L_2 \leqslant L} \{E \colon L_1 \otimes L_2 \nleqslant E_-\}.$$

Since $E = \bigvee\{E_1 \otimes E_2 \colon E_1 \otimes E_2 \leqslant E\}$, it follows from Lemma 1 that

$$E_- = \bigvee\{(E_1 \otimes E_2)_- \colon E_1 \otimes E_2 \leqslant E\}.$$

We first verify the following assertion:

$$\bigvee\{E\colon\ L_1\otimes L_2 \nleqslant E_-\} = \bigvee\{N\otimes M\colon\ L_1\otimes L_2 \nleqslant (N\otimes M)_-\}.$$

For $E\in\mathcal{N}\otimes\mathcal{M}$ and $L_1\otimes L_2\nleqslant E_- = \vee\{(E_1\otimes E_2)_-\colon\ E_1\otimes E_2\leqslant E\}$, we have

$$L_1\otimes L_2\nleqslant (E_1\otimes E_2)_- \quad \text{for any}\ \ E_1\otimes E_2\leqslant E.$$

Thus

$$E_1\otimes E_2\in\{N\otimes M\colon\ L_1\otimes L_2\nleqslant (N\otimes M)_-\}$$

and

$$E=\bigvee\{E_1\otimes E_2\colon\ E_1\otimes E_2\leqslant E\}\leqslant\bigvee\{N\otimes M\colon\ L_1\otimes L_2\nleqslant (N\otimes M)_-\}.$$

Hence

$$\bigvee\{E\colon\ L_1\otimes L_2 \nleqslant E_-\} \leqslant \bigvee\{N\otimes M\colon\ L_1\otimes L_2 \nleqslant (N\otimes M)_-\}.$$

The converse inequality is obvious. Thus, we have

$$\begin{aligned}
L_\sim = \bigvee\{E\colon\ L\nleqslant E_-\} &= \bigvee\bigcup_{L_1\otimes L_2\leqslant L}\{E\colon\ L_1\otimes L_2\nleqslant E_-\}\\
&= \bigvee_{L_1\otimes L_2\leqslant L}\bigvee\{E\colon\ L_1\otimes L_2\nleqslant E_-\}\\
&= \bigvee_{L_1\otimes L_2\leqslant L}\bigvee\{N\otimes M\colon\ L_1\otimes L_2\nleqslant (N\otimes M)_-\}\\
&= \bigvee_{L_1\otimes L_2\leqslant L}\bigvee\{N\otimes M\colon\ N_- < L_1 \quad\text{or}\quad M_- < L_2\}\\
&= \bigvee\{(L_1\otimes I)\vee(I\otimes L_2)\colon\ L_1\otimes L_2\leqslant L\}\\
&\geqslant \bigvee\{L_1\otimes L_2\colon\ L_1\otimes L_2\leqslant L\} = L.
\end{aligned}$$

The fourth equality follows from $(N\otimes M)_- = (N_-\otimes I)\vee(I\otimes M_-)$. Hence $L_\sim^\perp\leqslant L^\perp$.

Sufficiency. Suppose that there exists $L\in\mathcal{N}\otimes\mathcal{M}$ such that $x\in L$ and $y\in L^\perp$. For any $M\in\mathcal{N}\otimes\mathcal{M}$, if $M\leqslant L$, then $(xy^*)M = L(xy^*)L^\perp M = (0)\subseteq M_-$; if $M\nleqslant L$, then $(xy^*)M\subseteq L\leqslant M_-$. Thus, by the definition of $\mathcal{U}$, $xy^*\in\mathcal{U}$. $\square$

Theorem 11.

$$\begin{aligned}\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} &= \{T \in \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})\colon\ T(N\otimes M) \subseteq (N\otimes M)_- \quad \forall\, N \in \mathcal{N},\ \ M \in \mathcal{M}\}\\ &= \{T \in \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})\colon\ TL \subseteq L_- \quad \forall\, L \in \mathcal{N}\otimes\mathcal{M}\}.\end{aligned}$$

Proof. By [3, Proposition 2.4], $L = \bigvee\{N\otimes M\colon\ N\otimes M \leqslant L\}$ for all $L \in \mathcal{N}\otimes\mathcal{M}$. It follows from Lemma 1 that $L_- = \bigvee\{(N\otimes M)_-\colon\ N\otimes M \leqslant L\}$. Thus it is routine to prove that $\{T \in \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})\colon\ T(N\otimes M) \subseteq (N\otimes M)_-\ \forall\, N \in \mathcal{N},\ \ M \in \mathcal{M}\} = \{T \in \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})\colon\ TL \subseteq L_-\ \forall\, L \in \mathcal{N}\otimes\mathcal{M}\}$.

Suppose that $T \in \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and let $\mathcal{F}_{\mathcal{N}\otimes\mathcal{M}}$ be the linear span of rank one operators in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. It follows from [7, Theorem 3] that there exists a net $\{F_\alpha\} \subseteq \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$ such that

$$F_\alpha \xrightarrow{w} I,$$

where F_α is a finite linear combination of rank one operators in $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$. Thus

$$F_\alpha T \xrightarrow{w} T$$

and $F_\alpha T$ belongs to $\mathcal{F}_{\mathcal{N}\otimes\mathcal{M}}$. Hence

$$\mathcal{F}^w_{\mathcal{N}\otimes\mathcal{M}} \supseteq \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$$

and

$$\mathcal{F}^w_{\mathcal{N}\otimes\mathcal{M}} = \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}.$$

If $xy^* \in \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}} \subseteq \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$, then there exists $E \in \mathcal{N}\otimes\mathcal{M}$ such that $x \in E$ and $y \in E^\perp$ by Proposition 7. For any $L \in \mathcal{N}\otimes\mathcal{M}$, if $L \leqslant E$ then $(xy^*)L = E(xy^*)E^\perp L = (0)$; if $L \nleqslant E$ then $(xy^*)L = E(xy^*)E^\perp L \subseteq E \subseteq L_-$. Thus

$$xy^* \in \{T \in \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})\colon\ TL \subseteq L_- \quad \forall\, L \in \mathcal{N}\otimes\mathcal{M}\}$$

and

$$\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} = \mathcal{F}^w_{\mathcal{N}\otimes\mathcal{M}} \subseteq \{T \in \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})\colon\ TL \subseteq L_- \quad \forall\, L \in \mathcal{N}\otimes\mathcal{M}\}.$$

Conversely, set $\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)\colon\ TL \subseteq L_-\}$. Then $\mathcal{U} \cap \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$ is a weakly closed module of $\operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$. Just like in the above paragraph, we can show that $\mathcal{U} \cap \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M})$ is weakly generated by rank one operators in itself. For any rank one operator $xy^* \in \mathcal{U} \cap \operatorname{Alg}(\mathcal{N}\otimes\mathcal{M}) \subseteq \mathcal{U}$, it follows from Lemma 10 that there exists $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x \in L$ and $y \in L^{\perp}$. Since $xy^* \in \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M})$, so $xy^* \in \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ by Proposition 7. Hence

$$\mathcal{U} \cap \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M}) \subseteq \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$$

and

$$\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} = \{T \in \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M}) \colon \ TL \subseteq L_- \quad \forall L \in \mathcal{N} \otimes \mathcal{M}\}.$$

□

Corollary 12. *$\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and $\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$ have the same rank one operators.*

Proof. If $xy^* \in \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$, it follows from Lemma 10 and Theorem 11 that there exist $L \in \mathcal{N} \otimes \mathcal{M}$ and $x \in L$ and $y \in L^{\perp}$. By Proposition 7, $xy^* \in \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. □

Corollary 13. *$\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} = \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M})$ if and only if at least one of $\mathcal{N}$, $\mathcal{M}$ is continuous.*

Proof. Without loss of generality, we suppose that $\mathcal{N}$ is continuous. It follows from Lemma 2 that for any $T \in \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M})$ and $N \in \mathcal{N}, M \in \mathcal{M}$, we have

$$T(N \otimes M) \subseteq N \otimes M \subseteq (N \otimes I) \vee (I \otimes M_-) = (N \otimes M)_-.$$

So $T \in \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$ by Theorem 11 and $\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} = \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M})$.

Conversely, suppose that $\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} = \operatorname{Alg}(\mathcal{N} \otimes \mathcal{M})$ and $\mathcal{N}$, $\mathcal{M}$ are not continuous. Thus there exist $N \in \mathcal{N}$ and $M \in \mathcal{M}$ such that $N \neq N_-$ and $M \neq M_-$. Thus we can choose non-zero vectors $x_1 \in N \ominus N_-$ and $x_2 \in M \ominus M_-$. By virtue of Lemma 4, the rank one operator $(x_1 \otimes x_2)(x_1 \otimes x_2)^*$ belongs to $\operatorname{Alg}(\mathcal{N} \otimes \mathcal{M}) = \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$. But it follows from Lemma 10 and Theorem 11 that there exists $L \in \mathcal{N} \otimes \mathcal{M}$ such that $x_1 \otimes x_2 \in L$ and $x_1 \otimes x_2 \in L^{\perp}$. This contradiction shows that at least one of $\mathcal{N}$, $\mathcal{M}$ is continuous. □

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