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ON POSITIVE SOLUTIONS FOR A NONLINEAR BOUNDARY VALUE PROBLEM WITH IMPULSE

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Abstract. In this paper we study nonlinear second order differential equations subject to separated linear boundary conditions and to linear impulse conditions. Sign properties of an associated Green's function are investigated and existence results for positive solutions of the nonlinear boundary value problem with impulse are established. Upper and lower bounds for positive solutions are also given.

Keywords: impulse conditions, Green's function, completely continuous operator, fixed point theorem in cones

MSC 2000: 34A37, 34B15

1. Introduction

Positive solutions of abstract mathematical problems are investigated in the monographs by Krasnosel'skii [8], Guo and Lakshmikantam [7] making use of the theory of operators acting in Banach spaces with a cone and leaving this cone invariant. The significance of this investigation is due to the fact that in analysing nonlinear phenomena many mathematical models give rise to problems for which only nonnegative solutions make sense. In the mentioned monographs the idea of the method is illustrated by the example of nonlinear boundary value problem (BVP)

$$-y'' = f(x, y), \quad x \in [a, b], \tag{1.1}$$

$$y(a) = y(b) = 0. \tag{1.2}$$

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Later in [5], [6] instead of the simple boundary conditions (1.2) the general separated linear boundary conditions

$$\alpha y(a) - \beta y'(a) = 0, \quad \gamma y(b) + \delta y'(b) = 0, \tag{1.3}$$

and in [1] the periodic boundary conditions

$$y(a) = y(b), \quad y'(a) = y'(b) \tag{1.4}$$

were taken and the existence of positive solutions for the BVPs (1.1), (1.3) and (1.1), (1.4) was studied by a similar method.

In this paper, we study the existence of positive solutions for the boundary value problem with impulse (BVPI):

$$-[p(x)y']' + q(x)y = f(x, y), \quad x \in [a, c) \cup (c, b], \tag{1.5}$$

$$y(c-0) = d_1 y(c+0), \quad y^{[1]}(c-0) = d_2 y^{[1]}(c+0), \tag{1.6}$$

$$\alpha y(a) - \beta y^{[1]}(a) = 0, \quad \gamma y(b) + \delta y^{[1]}(b) = 0, \tag{1.7}$$

where $a < c < b$, $y = y(x)$ is a desired solution, $y^{[1]}(x) = p(x)y'(x)$ denotes the quasi-derivative of $y(x)$, $y(c-0)$ is the left-hand limit of $y(x)$ at c and $y(c+0)$ is the right-hand limit of $y(x)$ at c.

Note that a function $y(x)$ defined on $[a, c) \cup (c, b]$ is called a solution of (1.5)–(1.7) if its first derivative $y'(x)$ exists for each $x \in [a, c) \cup (c, b]$, $p(x)y'(x)$ is absolutely continuous on each closed subinterval of $[a, c) \cup (c, b]$, there exist finite values $y(c \pm 0)$ and $y^{[1]}(c \pm 0)$, the impulse conditions (1.6) and the boundary conditions (1.7) are satisfied, and the equation (1.5) is satisfied almost everywhere on $[a, c) \cup (c, b]$.

The paper is organized as follows.

In Section 2, we consider the linear homogeneous differential equation with impulse

$$-[p(x)y']' + q(x)y = 0, \quad x \in (-\infty, c) \cup (c, \infty),$$
$$y(c-0) = d_1 y(c+0), \quad y^{[1]}(c-0) = d_2 y^{[1]}(c+0),$$

as an auxiliary problem. Here, a uniqueness and existence theorem is presented, and a variation of constants formula for the corresponding nonhomogeneous equation is given.

In Section 3, the Green's function of the BVPI

$$-[p(x)y']' + q(x)y = h(x), \quad x \in [a, c) \cup (c, b], \tag{1.8}$$

$$y(c-0) = d_1 y(c+0), \quad y^{[1]}(c-0) = d_2 y^{[1]}(c+0), \tag{1.9}$$

$$\alpha y(a) - \beta y^{[1]}(a) = 0, \quad \gamma y(b) + \delta y^{[1]}(b) = 0, \tag{1.10}$$

is constructed.

In Section 4, sign properties of the Green's function are investigated.

In the last Section 5 existence results and upper and lower bounds for positive solutions of the BVPI (1.5)–(1.7) are established. For this, the Green's function of the linear BVPI (1.8)–(1.10) is used in order to reduce the nonlinear BVPI (1.5)–(1.7) to a fixed point problem.

Notice that positive solutions of boundary value problems for ordinary differential equations with impulse were studied earlier in [3], [4]. However, our problem and results in this paper are different from those in [3], [4]. For the basic concepts of impulsive differential equations we refer to [2], [10]. Differential equations with impulses are very special cases of generalized differential equations introduced by J. Kurzweil and further investigated by Š. Schwabik and M. Tvrdý [11]–[14].

Finally, for easy reference, we state here the fixed point theorem [8, p. 148; 7, p. 94] which is employed in this paper.

Let $\mathbb{B}$ be a real Banach space. A nonempty set $P \subset \mathbb{B}$ is called a *cone* if it satisfies the following three conditions:

(i) P is closed and convex;

(ii) $u \in P$, $\lambda \geqslant 0$ implies $\lambda u \in P$;

(iii) $u \in P$, $-u \in P$, implies $u = \theta$, where θ denotes the zero element of $\mathbb{B}$.

Theorem 1.1 (Krasnosel'skii Fixed Point Theorem). *Let $\mathbb{B}$ be a Banach space, and let $P \subset \mathbb{B}$ be a cone in $\mathbb{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathbb{B}$ with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $A\colon P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that, either*

(i) $\|Ay\| \leqslant \|y\|$, $y \in P \cap \partial\Omega_1$, and $\|Ay\| \geqslant \|y\|$, $y \in P \cap \partial\Omega_2$; or

(ii) $\|Ay\| \geqslant \|y\|$, $y \in P \cap \partial\Omega_1$ and $\|Ay\| \leqslant \|y\|$, $y \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2. Second order linear differential equations with impulse

Let c be a real number and d_1, d_2 be nonzero real numbers. Consider the linear homogeneous problem of the form

$$-[p(x)y']' + q(x)y = 0, \quad x \in (-\infty, c) \cup (c, \infty), \tag{2.1}$$

$$y(c-0) = d_1 y(c+0), \quad y^{[1]}(c-0) = d_2 y^{[1]}(c+0), \tag{2.2}$$

where $y = y(x)$ is a desired solution, and

$$y^{[1]}(x) = p(x)y'(x) \tag{2.3}$$

denotes the quasi-derivative of $y(x)$. We will assume that the coefficients $p(x)$ and $q(x)$ of the equation (2.1) are real valued measurable funtions on $(-\infty, +\infty)$ and for all finite real numbers c_1, c_2 with $c_1 < c_2$,

$$\int_{c_1}^{c_2} \frac{1}{|p(x)|}\, \mathrm{d}x < \infty, \quad \int_{c_1}^{c_2} |q(x)|\, \mathrm{d}x < \infty. \tag{2.4}$$

A function $y(x)$ defined on $(-\infty, c) \cup (c, \infty)$ is called a solution of (2.1), (2.2) if its first derivative $y'(x)$ exists, $p(x)y'(x)$ is absolutely continuous on each closed subinterval of $(-\infty, c) \cup (c, \infty)$ and moreover there exist finite values $y(c \pm 0)$, $y^{[1]}(c \pm 0)$ that satisfy the impulse conditions (2.2), and the equation (2.1) is satisfied almost everywhere on $(-\infty, c) \cup (c, \infty)$.

Theorem 2.1. *Let x_0 be a fixed point in $(-\infty, c) \cup (c, \infty)$ and c_0, c_1 be given constants. Then* (2.1), (2.2) *has a unique solution $y(x)$ such that*

$$y(x_0) = c_0, \quad y^{[1]}(x_0) = c_1. \tag{2.5}$$

Proof. Let $x_0 \in (-\infty, c)$. By the condition (2.4) and the well-known existence and uniqueness theorem (see, for example, [9, Kapitel 5]) it follows that the equation (2.1) has a unique solution $y(x)$ on $(-\infty, c)$ satisfying the initial conditions (2.5), and this solution has finite values $y(c-0)$ and $y^{[1]}(c-0)$. Further, according to the impulse conditions (2.2), we define

$$y(c+0) = \frac{1}{d_1} y(c-0), \quad y^{[1]}(c+0) = \frac{1}{d_2} y^{[1]}(c-0), \tag{2.6}$$

and hence we solve the equation (2.1) on (c, ∞) under these initial conditions. By the condition (2.4) and the well-known existence and uniqueness theorem, it follows that the initial value problem (2.1), (2.6) has a unique solution on (c, ∞).

So in the case $x_0 \in (-\infty, c)$ we get a unique solution $y(x)$ of the equation (2.1) satisfying the impulse conditions (2.2) and initial conditions (2.5). The case $x_0 \in (c, \infty)$ is considered in a similar way starting with the interval (c, ∞) and passing then to the interval $(-\infty, c)$. $\square$

If y and z are differentiable functions on $(-\infty, c) \cup (c, \infty)$, then their Wronskian is defined by

$$W_x(y, z) = y(x)z^{[1]}(x) - y^{[1]}(x)z(x) = p(x)[y(x)z'(x) - y'(x)z(x)]$$

for $x \in (-\infty, c) \cup (c, \infty)$.

Theorem 2.2. *The Wronskian of any two solutions y and z of* (2.1), (2.2) *is constant on each of the intervals $(-\infty, c)$ and (c, ∞),*

$$W_x(y,z) = \begin{cases} k, & x \in (-\infty, c), \\ l, & x \in (c, \infty), \end{cases} \tag{2.7}$$

where k and l are constants such that

$$k = d_1 d_2 l. \tag{2.8}$$

Proof. Let y and z be two solutions of (2.1), (2.2). Then for $x \in (-\infty, c) \cup (c, \infty)$

$$\begin{aligned} \{W_x(y,z)\}' &= \{p(x)[y(x)z'(x) - y'(x)z(x)]\}' \\ &= y(x)[p(x)z'(x)]' - [p(x)y'(x)]'z(x) \\ &= y(x)q(x)z(x) - q(x)y(x)z(x) \\ &= 0. \end{aligned}$$

Therefore (2.7) holds. Further, from (2.7) we have

$$k = W_{c-0}(y,z), \quad l = W_{c+0}(y,z),$$

and using the impulsive conditions (2.2) we get

$$\begin{aligned} k &= W_{c-0}(y,z) = y(c-0)z^{[1]}(c-0) - y^{[1]}(c-0)z(c-0) \\ &= d_1 y(c+0) d_2 z^{[1]}(c+0) - d_2 y^{[1]}(c+0) d_1 z(c+0) \\ &= d_1 d_2 W_{c+0}(y,z) \\ &= d_1 d_2 l. \end{aligned}$$

□

Corollary 2.3. *If y and z are both solutions of* (2.1), (2.2), *then either $W_x(y,z) = 0$ for all $x \in (-\infty, c) \cup (c, \infty)$, or $W_x(y,z) \neq 0$ for all $x \in (-\infty, c) \cup (c, \infty)$.*

The following two theorems are proved in exactly the same way as in the case of the equation (2.1) without the impulse conditions (2.2), by using Theorem 2.1.

Theorem 2.4. *Any two solutions of* (2.1), (2.2) *are linearly independent if and only if their Wronskian is nonzero.*

Theorem 2.5. *Problem* (2.1), (2.2) *has two linearly independent solutions and every solution of* (2.1), (2.2) *is a linear combination of these solutions.*

We say that y_1 and y_2 form a fundamental set (or fundamental system) of solutions for (2.1), (2.2) provided their Wronskian is nonzero.

Let us consider the nonhomogeneous equation

$$-[p(x)y']' + q(x)y = h(x), \quad x \in (-\infty, c) \cup (c, \infty), \tag{2.9}$$

with the impulse conditions (2.2), where $h(x)$ is a real valued function defined on $(-\infty, c) \cup (c, \infty)$ and satisfying the condition

$$\int_{x_1}^{x_2} |h(x)| \, \mathrm{d}x < \infty$$

for all finite real numbers x_1 and x_2 with $x_1 < x_2$.

Theorem 2.6. *Suppose that y_1 and y_2 form a fundamental set of solutions for the homogeneous problem* (2.1), (2.2). *Then the general solution of the nonhomogeneous problem* (2.9), (2.2) *is given by*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_c^x \frac{y_1(x)y_2(s) - y_1(s)y_2(x)}{W_s(y_1, y_2)} h(s) \, \mathrm{d}s, \tag{2.10}$$

where c_1 and c_2 are arbitrary constants.

P r o o f. It sufficies to show that the function

$$z(x) = \int_c^x \frac{y_1(x)y_2(s) - y_1(s)y_2(x)}{W_s(y_1, y_2)} h(s) \, \mathrm{d}s, \tag{2.11}$$

is a particular solution of (2.9), (2.2).

From (2.11) we have for $x \in (-\infty, c) \cup (c, \infty)$,

$$z'(x) = \int_c^x \frac{y_1'(x)y_2(s) - y_1(s)y_2'(x)}{W_s(y_1, y_2)} h(s) \, \mathrm{d}s \tag{2.12}$$

and

$$[p(x)z']' = -h(x) + q(x)z.$$

Besides, from (2.11) and (2.12) we have

$$z(c-0) = z(c+0) = 0, \quad z^{[1]}(c-0) = z^{[1]}(c+0) = 0.$$

Thus, $z(x)$ satisfies the equation (2.9) and the impulse conditions (2.2). □

3. Boundary value problems with impulse and Green's functions

Let a, b, c be fixed real numbers such that $a < c < b$. Consider the following boundary value problem with impulse (BVPI):

$$-[p(x)y']' + q(x)y = h(x), \quad x \in [a,c) \cup (c,b], \tag{3.1}$$

$$y(c-0) = d_1 y(c+0), \quad y^{[1]}(c-0) = d_2 y^{[1]}(c+0), \tag{3.2}$$

$$\alpha y(a) - \beta y^{[1]}(a) = 0, \quad \gamma y(b) + \delta y^{[1]}(b) = 0, \tag{3.3}$$

where $p(x)$, $q(x)$, and $h(x)$ are real valued measurable functions on $[a,b]$ such that

$$\int_a^b \frac{1}{|p(x)|}\,\mathrm{d}x < \infty, \quad \int_a^b |q(x)|\,\mathrm{d}x < \infty, \quad \int_a^b |h(x)|\,\mathrm{d}x < \infty; \tag{3.4}$$

d_1 and d_2 are nonzero real numbers; α, β, γ, and δ are real numbers such that $|\alpha| + |\beta| \neq 0$ and $|\gamma| + |\delta| \neq 0$.

Denote by $\varphi(x)$ and $\psi(x)$ the solutions of the homogenous problem

$$-[p(x)y']' + q(x)y = 0, \quad x \in [a,c) \cup (c,b], \tag{3.5}$$

$$y(c-0) = d_1 y(c+0), \quad y^{[1]}(c-0) = d_2 y^{[1]}(c+0), \tag{3.6}$$

satisfying the initial conditions

$$\varphi(a) = \beta, \quad \varphi^{[1]}(a) = \alpha, \tag{3.7}$$

$$\psi(b) = \delta, \quad \psi^{[1]}(b) = -\gamma. \tag{3.8}$$

So $\varphi(x)$ satisfies the first and $\psi(x)$ the second condition of (3.3). By Theorem 2.2 and the conditions (3.7), (3.8), we have for $x \in [a,c)$

$$W_x(\varphi,\psi) = \varphi(a)\psi^{[1]}(a) - \varphi^{[1]}(a)\psi(a) = \beta\psi^{[1]}(a) - \alpha\psi(a),$$

and for $x \in (c,b]$

$$W_x(\varphi,\psi) = \varphi(b)\psi^{[1]}(b) - \varphi^{[1]}(b)\psi(b) = -\gamma\varphi(b) - \delta\varphi^{[1]}(b).$$

Therefore, taking into account (2.8), we get

$$W_x(\varphi,\psi) = \begin{cases} d_1 d_2[-\gamma\varphi(b) - \delta\varphi^{[1]}(b)], & x \in [a,c), \\ -\gamma\varphi(b) - \delta\varphi^{[1]}(b), & x \in (c,b], \end{cases} \tag{3.9}$$

and also

$$W_x(\varphi, \psi) = \begin{cases} \beta\psi^{[1]}(a) - \alpha\psi(a), & x \in [a, c), \\ \dfrac{1}{d_1 d_2}[\beta\psi^{[1]}(a) - \alpha\psi(a)], & x \in (c, b]. \end{cases} \tag{3.10}$$

Notice that form (3.9) and (3.10) it follows that

$$\beta\psi^{[1]}(a) - \alpha\psi(a) = -d_1 d_2[\gamma\varphi(b) + \delta\varphi^{[1]}(b)]. \tag{3.11}$$

According to Theorem 2.4, we get from (3.9) that $\gamma\varphi(b) + \delta\varphi^{[1]}(b) \neq 0$ if and only if $\varphi(x)$ and $\psi(x)$ are linearly independent. The following theorem describes the condition $\gamma\varphi(b) + \delta\varphi^{[1]}(b) \neq 0$ from the other point of view.

Theorem 3.1. *$\gamma\varphi(b) + \delta\varphi^{[1]}(b) \neq 0$ if and only if the homogeneous problem* (3.5), (3.6) *has only the trivial solution satisfying the boundary conditions* (3.3).

P r o o f. If $\gamma\varphi(b) + \delta\varphi^{[1]}(b) = 0$, then by virtue of (3.7), $\varphi(x)$ will be a nontrivial solution of (3.5), (3.6) satisfying the boundary conditions (3.3). Let us now assume that $\gamma\varphi(b) + \delta\varphi^{[1]}(b) \neq 0$. Then $\varphi(x)$ and $\psi(x)$ will form a fundamental set of solutions of (3.5), (3.6) and therefore any solution of (3.5), (3.6), (3.3) will have the form

$$y(x) = c_1\varphi(x) + c_2\psi(x),$$

where c_1, c_2 are constants. Substituting this expression of $y(x)$ into the boundary conditions (3.3) and taking into account (3.7) and (3.8), we get

$$c_2[\alpha\psi(a) - \beta\psi^{[1]}(a)] = 0 \quad \text{and} \quad c_1[\gamma\varphi(b) + \delta\varphi^{[1]}(b)] = 0.$$

Since $\gamma\varphi(b) + \delta\varphi^{[1]}(b) \neq 0$ and also $\alpha\psi(a) - \beta\psi^{[1]}(a) \neq 0$ by (3.11), it follows that $c_1 = c_2 = 0$, that is, the solution $y(x)$ is trivial. □

Theorem 3.2. *If $\gamma\varphi(b) + \delta\varphi^{[1]}(b) \neq 0$, then the homogeneous BVPI* (3.1)–(3.3) *has a unique solution $y(x)$ for which the formula*

$$y(x) = \int_a^b G(x, s)h(s)\,\mathrm{d}s, \quad x \in [a, c) \cup (c, b] \tag{3.12}$$

holds, where the function $G(x, s)$ is called the Green's function of the BVPI (3.1)–(3.3) *and is defined for $x, s \in [a, c) \cup (c, b]$ by the formula*

$$G(x, s) = -\frac{1}{W_s(\varphi, \psi)} \begin{cases} \varphi(s)\psi(x), & a \leqslant s \leqslant x \leqslant b, \\ \varphi(x)\psi(s), & a \leqslant x \leqslant s \leqslant b. \end{cases} \tag{3.13}$$

P r o o f. Under the condition $\gamma\varphi(b)+\delta\varphi^{[1]}(b)\neq 0$, the solutions $\varphi(x)$ and $\psi(x)$ of the homogeneous problem (3.5), (3.6) are linearly independent and therefore by Theorem 2.6 the general solution of the nonhomogeneous problem (3.1), (3.2) has the form

$$y(x)=c_1\varphi(x)+c_2\psi(x)+\int_c^x \frac{\varphi(x)\psi(s)-\varphi(s)\psi(x)}{W_s(\varphi,\psi)}h(s)\,\mathrm{d}s, \tag{3.14}$$

where c_1 and c_2 are arbitrary constants. Now we try to choose the constants c_1 and c_2 so that the function $y(x)$ satisfies also the boundary conditions (3.3).

From (3.14) we have

$$y^{[1]}(x)=c_1\varphi^{[1]}(x)+c_2\psi^{[1]}(x)+\int_c^x \frac{\varphi^{[1]}(x)\psi(s)-\varphi(s)\psi^{[1]}(x)}{W_s(\varphi,\psi)}h(s)\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned} y(a)&=c_1\beta+c_2\psi(a)+\int_c^a \frac{\beta\psi(s)-\psi(a)\varphi(s)}{W_s(\varphi,\psi)}h(s)\,\mathrm{d}s,\\ y^{[1]}(a)&=c_1\alpha+c_2\psi^{[1]}(a)+\int_c^a \frac{\alpha\psi(s)-\psi^{[1]}(a)\varphi(s)}{W_s(\varphi,\psi)}h(s)\,\mathrm{d}s. \end{aligned}$$

Substituting these values of $y(a)$ and $y^{[1]}(a)$ into the first condition of (3.3) and taking into account that $\alpha\psi(a)-\beta\psi^{[1]}(a)\neq 0$ by (3.11), we find

$$c_2=-\int_a^c \frac{\varphi(s)}{W_s(\varphi,\psi)}h(s)\,\mathrm{d}s.$$

Similary from the second condition of (3.3) taking into account that $\alpha\psi(a)-\beta\psi^{[1]}(a)\neq 0$, we find

$$c_1=-\int_c^b \frac{\psi(s)}{W_s(\varphi,\psi)}h(s)\,\mathrm{d}s.$$

Putting these values of c_1 and c_2 in (3.14), we get the formula (3.12), (3.13). □

Remark 3.1. It can be verifed without diffuculty that for the solution $y(x)$ of the nonhomogeneous equation (3.1) with the impulse conditions (3.2) and nonhomogeneous boundary conditions

$$\alpha y(a)-\beta y^{[1]}(a)=\nu,\quad \gamma y(b)+\delta y^{[1]}(b)=\mu, \tag{3.15}$$

the formula

$$y(x)=w(x)+\int_a^b G(x,s)h(s)\,\mathrm{d}s,\quad x\in[a,c)\cup(c,b], \tag{3.16}$$

holds, where the function $G(x,s)$ is defined by (3.13), and

$$w(x) = -\frac{\mu}{W_b(\varphi,\psi)}\varphi(x) - \frac{\nu}{W_a(\varphi,\psi)}\psi(x). \tag{3.17}$$

4. Sign properties of the Green's function

Consider the BVPI (3.1)–(3.3). Here, in addition to the preceding hypotheses, we assume that

$$p(x) > 0 \text{ and } q(x) \geqslant 0 \quad \text{for all } x \in [a,c) \cup (c,b]; \tag{4.1}$$

$$d_1 > 0,\ d_2 > 0;\ \alpha,\beta,\gamma,\delta \geqslant 0,\ \alpha+\beta > 0,\ \gamma+\delta > 0. \tag{4.2}$$

Let $\varphi(x)$ and $\psi(x)$ be the solutions of the homogeneous problem (3.5), (3.6) satisfying the inital conditions (3.7) and (3.8), respectively. It is easy to see that for these solutions the equations

$$\varphi^{[1]}(x) = \alpha + \int_a^x q(s)\varphi(s)\,\mathrm{d}s, \quad x \in [a,c), \tag{4.3}$$

$$\varphi(x) = \beta + \alpha\int_a^x \frac{\mathrm{d}t}{p(t)} + \int_a^x \left[\int_s^x \frac{\mathrm{d}t}{p(t)}\right] q(s)\varphi(s)\,\mathrm{d}s, \quad x \in [a,c), \tag{4.4}$$

$$\varphi^{[1]}(x) = \varphi^{[1]}(c+0) + \int_c^x q(s)\varphi(s)\,\mathrm{d}s, \quad x \in (c,b], \tag{4.5}$$

$$\begin{aligned}\varphi(x) &= \varphi(c+0) + \varphi^{[1]}(c+0)\int_c^x \frac{\mathrm{d}t}{p(t)}\\ &\quad + \int_c^x \left[\int_s^x \frac{\mathrm{d}t}{p(t)}\right] q(s)\varphi(s)\,\mathrm{d}s, \quad x \in (c,b];\end{aligned} \tag{4.6}$$

$$\psi^{[1]}(x) = -\gamma - \int_x^b q(s)\psi(s)\,\mathrm{d}s, \quad x \in (c,b], \tag{4.7}$$

$$\psi(x) = \delta + \gamma\int_x^b \frac{\mathrm{d}t}{p(t)} + \int_x^b \left[\int_x^s \frac{\mathrm{d}t}{p(t)}\right] q(s)\psi(s)\,\mathrm{d}s, \quad x \in (c,b], \tag{4.8}$$

$$\psi^{[1]}(x) = \psi^{[1]}(c-0) - \int_x^c q(s)\psi(s)\,\mathrm{d}s, \quad x \in [a,c), \tag{4.9}$$

$$\begin{aligned}\psi(x) &= \psi(c-0) - \psi^{[1]}(c-0)\int_x^c \frac{\mathrm{d}t}{p(t)}\\ &\quad + \int_x^c \left[\int_x^s \frac{\mathrm{d}t}{p(t)}\right] q(s)\psi(s)\,\mathrm{d}s, \quad x \in [a,c),\end{aligned} \tag{4.10}$$

hold.

Lemma 4.1. *Let $K(x, s)$ be a nonnegative continuous function defined for $-\infty < x_1 \leqslant x,\ s \leqslant x_2 < \infty$, and $g(x)$ be a nonnegative integrable function on $[x_1, x_2]$. Then for arbitrary nonnegative continuous function $f(x)$ defined on $[x_1, x_2]$, the Volterra integral equation*

$$y(x) = f(x) + \int_{x_1}^{x} K(x, s) g(s) y(s)\, \mathrm{d}s, \quad x_1 \leqslant x \leqslant x_2, \tag{4.11}$$

has a unique solution $y(x)$. This solution is continuous and satisfies the inequality

$$y(x) \geqslant f(x), \quad x_1 \leqslant x \leqslant x_2. \tag{4.12}$$

Proof. We solve the equation (4.11) by the method of successive approximations setting

$$y_0(x) = f(x), \quad y_n(x) = \int_{x_1}^{x} K(x, s) g(s) y_{n-1}(s)\, \mathrm{d}s, \quad n = 1, 2, \ldots. \tag{4.13}$$

If the series $\sum\limits_{n=0}^{\infty} y_n(x)$ converges uniformly with respect to $x \in [x_1, x_2]$, then its sum will be, obviously, a continuous solution of the equation (4.11). To prove the uniform convergence of this series we put

$$\max_{x_1 \leqslant x \leqslant x_2} f(x) = c_1, \qquad \max_{x_1 \leqslant x, s \leqslant x_2} K(x, s) = c_2.$$

Then it is easy to get from (4.13) that

$$0 \leqslant y_n(x) \leqslant c_1 \frac{c_2^n}{n!} \left[\int_{x_1}^{x} g(s)\, \mathrm{d}s \right]^n, \quad n = 1, 2, \ldots.$$

It follows that the equation (4.11) has a continuous solution $y(x) = \sum\limits_{n=0}^{\infty} y_n(x)$ and for this solution the inequality (4.12) holds because $y_0(x) = f(x)$ and $y_n(x) \geqslant 0$, $n = 1, 2, \ldots$. The proof of uniqueness of the solution of (4.11) is trivial. □

Remark 4.1. Evidently, the statement of Lemma 4.1 is also valid for the Volterra equation of the form

$$y(x) = f(x) + \int_{x}^{x_2} K(x, s) g(s) y(s)\, \mathrm{d}s, \quad x_1 \leqslant x \leqslant x_2.$$

Lemma 4.2. *Under the conditions* (4.1), (4.2) *the solutions* $\varphi(x)$ *and* $\psi(x)$ *have the following properties:*

$$\begin{array}{lll}
(4.14) & \varphi(x) \geqslant 0,\ x \in [a,c) \cup (c,b]; & \psi(x) \geqslant 0,\ x \in [a,c) \cup (c,b]; \\
(4.15) & \varphi(x) > 0,\ x \in (a,c) \cup (c,b]; & \psi(x) > 0,\ x \in [a,c) \cup (c,b); \\
(4.16) & \varphi(c-0) > 0,\ \varphi(c+0) > 0; & \psi(c-0) > 0,\ \psi(c+0) > 0; \\
(4.17) & \varphi^{[1]}(x) \geqslant 0,\ x \in [a,c) \cup (c,b]; & \psi^{[1]}(x) \leqslant 0,\ x \in [a,c) \cup (c,b]; \\
(4.18) & \varphi^{[1]}(c-0) \geqslant 0,\ \varphi^{[1]}(c+0) \geqslant 0; & \psi^{[1]}(c-0) \leqslant 0,\ \psi^{[1]}(c+0) \leqslant 0.
\end{array}$$

Proof. From (4.4) and Lemma 4.1, we get

$$\varphi(x) \geqslant \beta + \alpha \int_a^x \frac{\mathrm{d}t}{p(t)}, \quad x \in [a,c).$$

Hence

$$\varphi(x) \geqslant 0 \ \text{ for } x \in [a,c); \quad \varphi(x) > 0 \ \text{ for } x \in (a,c); \quad \text{and} \quad \varphi(c-0) > 0.$$

Therefore, from (4.3), we get

$$\varphi^{[1]}(x) \geqslant 0,\ x \in [a,c); \quad \text{and} \quad \varphi^{[1]}(c-0) \geqslant 0.$$

Moreover, from the impulse conditions (3.2), we find

$$\varphi(c+0) = \frac{1}{d_1}\varphi(c-0) > 0, \quad \varphi^{[1]}(c+0) = \frac{1}{d_2}\varphi^{[1]}(c-0) \geqslant 0. \tag{4.19}$$

It follows now from (4.5) and (4.6) that

$$\varphi(x) > 0 \ \text{ for } x \in (c,b]; \quad \text{and} \quad \varphi^{[1]}(x) \geqslant 0 \ \text{ for } x \in (c,b].$$

So the claims about $\varphi(x)$ and $\varphi^{[1]}(x)$ are proved.

Finally, the claims about $\psi(x)$ and $\psi^{[1]}(x)$ can be proved similarly using (4.7)–(4.10). $\square$

Lemma 4.3. *Let conditions* (4.1) *and* (4.2) *hold. Besides, in the case* $q(x) \equiv 0$ *for* $x \in [a, c) \cup (c, b]$ *let* $\alpha + \gamma > 0$. *Then for the Wronskian of the solutions* $\varphi(x)$ *and* $\psi(x)$ *the inequality*

$$W_x(\varphi, \psi) < 0, \quad x \in [a, c) \cup (c, b] \cup \{c \pm 0\},$$

holds.

Proof. We have, by (3.9),

$$W_x(\varphi, \psi) = \begin{cases} -d_1 d_2[\gamma\varphi(b) + \delta\varphi^{[1]}(b)], & x \in [a, c) \cup \{c - 0\}, \\ -[\gamma\varphi(b) + \delta\varphi^{[1]}(b)], & x \in (c, b] \cup \{c + 0\}. \end{cases}$$

Next, using (4.5) and (4.6), we find

$$\gamma\varphi(b) + \delta\varphi^{[1]}(b) = \gamma\varphi(c+0) + \delta\varphi^{[1]}(c+0) + \gamma\varphi^{[1]}(c+0)\int_c^b \frac{\mathrm{d}t}{p(t)} + \gamma\int_c^b \left[\int_s^b \frac{\mathrm{d}t}{p(t)}\right] q(s)\,\mathrm{d}s + \delta\int_c^b q(s)\varphi(s)\,\mathrm{d}s. \tag{4.20}$$

It follows from (4.20) that $\gamma\varphi(b) + \delta\varphi^{[1]}(b) > 0$ if $q(x)$ is not identically zero on $[a, c) \cup (c, b]$.

If $q(x)$ is identically zero on $[a, c) \cup (c, b]$, then we have from (4.20) and (4.3), (4.4),

$$\begin{aligned} \gamma\varphi(b) + \delta\varphi^{[1]}(b) &= \gamma\varphi(c+0) + \delta\varphi^{[1]}(c+0) + \gamma\varphi^{[1]}(c+0)\int_c^b \frac{\mathrm{d}t}{p(t)}, \\ \varphi(c+0) &= \frac{1}{d_1}\varphi(c-0) = \frac{1}{d_1}\left\{\beta + \alpha\int_a^c \frac{\mathrm{d}t}{p(t)}\right\}, \\ \varphi^{[1]}(c+0) &= \frac{1}{d_2}\varphi^{[1]}(c-0) = \frac{\alpha}{d_2}, \end{aligned}$$

and hence

$$\gamma\varphi(b) + \delta\varphi^{[1]}(b) = \frac{\gamma}{d_1}\left\{\beta + \alpha\int_a^c \frac{\mathrm{d}t}{p(t)}\right\} + \frac{\alpha\delta}{d_2} + \frac{\alpha\gamma}{d_2}\int_c^b \frac{\mathrm{d}t}{p(t)} > 0 \ \text{ if } \alpha + \gamma > 0.$$

$\square$

From the formula (3.13), by Lemmas 4.2, 4.3, and (4.3)–(4.10) the following theorem follows.

Theorem 4.4. *Let conditions* (4.1) *and* (4.2) *hold. Besides, in the case* $q(x) \equiv 0$ *for* $x \in [a, c) \cup (c, b]$ *let* $\alpha + \gamma > 0$. *Then*

(i) $G(x, s) \geqslant 0$ *for* $x \in [a, c) \cup (c, b]$.

(ii) $G(x, s) > 0$ *for* $x \in (a, c) \cup (c, b) \cup \{c \pm 0\}$.

(iii) *If* $\beta > 0$ *and* $\delta > 0$, *then* $G(x, s) > 0$ *for* $x \in [a, c) \cup (c, b] \cup \{c \pm 0\}$.

5. Existence of positive solutions

In this section, we consider the BVPI

$$-[p(x)y']' + q(x)y = f(x, y), \quad x \in [a, c) \cup (c, b], \tag{5.1}$$

$$y(c - 0) = d_1 y(c + 0), \quad y^{[1]}(c - 0) = d_1 y^{[1]}(c + 0), \tag{5.2}$$

$$\alpha y(a) - \beta y^{[1]}(a) = 0, \quad \gamma y(b) + \delta y^{[1]}(b) = 0. \tag{5.3}$$

We will assume that the following conditions are satisfied:

(H1) $p(x) > 0$ and $q(x) \geqslant 0$ are measurable functions on $[a, b]$ and

$$\int_a^b \frac{\mathrm{d}x}{p(x)} < \infty, \qquad \int_a^b q(x)\,\mathrm{d}x < \infty.$$

(H2) $d_1 > 0$, $d_2 > 0$; $\alpha, \beta, \gamma, \delta \geqslant 0$, $\alpha + \beta > 0$, $\gamma + \delta > 0$; if $q(x) \equiv 0$ on $[a, c) \cup (c, b]$, then $\alpha + \gamma > 0$.

(H3) $f(x, \xi)$ is a real valued function continuous with respect to the collection of its arguments $x \in [a, c) \cup (c, b]$ and $\xi \in \mathbb{R}$, and $f(x, \xi) \geqslant 0$ for $\xi \in \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. Moreover, for each $\xi_0 \in \mathbb{R}$ there exist finite limits

$$\lim_{\substack{(x,\xi)\to(c,\xi_0)\\ x<c}} f(x, \xi) = f(c - 0, \xi_0) \quad \text{and} \quad \lim_{\substack{(x,\xi)\to(c,\xi_0)\\ x>c}} f(x, \xi) = f(c + 0, \xi_0).$$

By Lemma 4.3 and Theorem 3.2, the nonlinear BVPI (5.1)–(5.3) is equivalent to the integral equation

$$y(x) = \int_a^b G(x, s) f(s, y(s))\,\mathrm{d}s, \quad x \in [a, c) \cup (c, b], \tag{5.4}$$

where $G(x, s)$ is defined by (3.13) as Green's function of the linear BVPI (3.1)–(3.3).

We investigate the equation (5.4) in the Banach space $\mathbb{B}$ of all real-valued continuous functions $y(x)$ on $[a, c) \cup (c, b]$ for which the finite values $y(c - 0)$ and $y(c + 0)$ exist, with the norm

$$\|y\| = \sup |y(x)|, \quad x \in [a, c) \cup (c, b].$$

Solving equation (5.4) in $\mathbb{B}$ is equivalent to finding fixed points of the operator $A\colon \mathbb{B} \to \mathbb{B}$ defined by

$$Ay(x) = \int_a^b G(x,s) f(s, y(s))\,\mathrm{d}s, \quad x \in [a,c) \cup (c,b]. \tag{5.5}$$

Note that for each $y \in \mathbb{B}$ the function $Ay(x)$ belongs to $\mathbb{B}$ and satisfies the impulse conditions (5.2) and the boundary conditions (5.3) by the definition of the Green's function $G(x,s)$.

Let us set

$$P = \{y \in \mathbb{B}\colon\ y(x) \geqslant 0 \quad \text{for } x \in [a,c) \cup (c,b]\}. \tag{5.6}$$

Evidently P is a cone in $\mathbb{B}$. Moreover, for all $y \in P$, by Theorem 4.4 (i) and (H3), $Ay(x) \geqslant 0$ for all $x \in [a,c) \cup (c,b]$. Therefore the operator A leaves the cone P invariant, i.e., $A(P) \subset P$.

Remark 5.1. Assume that conditions (H1), (H2), and (H3) are satisfied. Assume also that the function $f(x,\xi)$ satisfies with respect to ξ the Lipschitz condition $|f(x,\xi_1) - f(x,\xi_2)| \leqslant L\,|\xi_1 - \xi_2|$, $\xi_1, \xi_2 \in \mathbb{R}$, where L is a positive constant not depending on x, ξ_1, ξ_2. If

$$L \cdot \sup_{x \in [a,c)\cup(c,b]} \int_a^b G(x,s)\,\mathrm{d}s < 1,$$

then applying the Banach fixed point theorem (contraction mapping theorem) to the operator $A\colon\ P \to P$ it is easy to see that the BVPI (5.1)–(5.3) has a unique solution $y(x)$ such that $y(x) \geqslant 0$ for $x \in [a,c) \cup (c,b]$.

Using the well-known Arzelá-Ascoli Theorem it is not difficult to show that a subset S of the space $\mathbb{B}$ is relatively compact (precompact) if and only if the functions of S are equi-bounded and equi-continuous on each of the intervals $[a,c)$ and $(c,b]$. On the basis of this fact, making use of the piece-wise continuity properties of $G(x,s)$ and $f(x,\xi)$ we can prove, in a standard way, that the operator A is completely continuous in the space $\mathbb{B}$.

Remark 5.2. Assume that conditions (H1), (H2), and (H3) are satisfied. Assume also that there is a number $R > 0$ such that

$$\left(\sup_{x \in [a,c)\cup(c,b]} \int_a^b G(x,s)\,\mathrm{d}s\right) \cdot \sup_{x \in [a,c)\cup(c,b], \xi \in [0,R]} f(x,\xi) \leqslant R. \tag{5.7}$$

Then applying the Schauder fixed point theorem to the operator $A\colon\ P_R \to P_R$, where

$$P_R = \{y \in \mathbb{B}\colon\ \|y\| \leqslant R,\ y(x) \geqslant 0 \text{ for all } x \in [a,c) \cup (c,b]\},$$

we can prove that the BVPI (5.1)–(5.3) has at least one solution $y(x)$ such that $0 \leqslant y(x) \leqslant R$, $x \in [a,c) \cup (c,b]$.

Notice that the condition (5.7) of Remark 5.2 is satisfied for sufficiently large R if $|f(x,\xi)| \leqslant c_1 + c_2 |\xi|^{\lambda}$, where c_1, c_2, λ are positive constants and $\lambda < 1$.

Next we will assume that instead of (H2) the following stronger condition holds:

(H2′) $d_1 > 0$, $d_2 > 0$; $\alpha, \gamma \geqslant 0$, $\beta > 0$, $\delta > 0$; if $q(x) \equiv 0$ on $[a,c) \cup (c,b]$, then $\alpha + \gamma > 0$.

In this case, by Theorem 4.4 (iii), we will have

$$G(x,s) > 0 \quad \text{for all } x \in [a,c) \cup (c,b] \cup \{c \pm 0\}. \tag{5.8}$$

Let us set

$$m = \inf G(x,s), \qquad M = \sup G(x,s), \qquad x \in [a,c) \cup (c,b] \tag{5.9}$$

and form the cone

$$\widetilde{P} = \Big\{ y \in P\colon \inf_{x \in [a,c) \cup (c,b]} y(x) \geqslant \frac{m}{M} \|y\| \Big\},$$

where the cone P is defined by (5.6).

Lemma 5.1. *Assume that conditions* (H1), (H2′), *and* (H3) *are satisfied and let A be the operator defined by* (5.5). *Then $Ay \in \widetilde{P}$ for all $y \in P$. In particular, the operator A leaves the cone $\widetilde{P}$ invariant, i.e., $A\widetilde{P} \subset \widetilde{P}$.*

P r o o f. For all $y \in P$ we obviously have $Ay(x) \geqslant 0$ for all $x \in [a,c) \cup (c,b]$. Further,

$$\begin{aligned}
\inf_{x \in [a,c) \cup (c,b]} Ay(x) &\geqslant m \int_a^b f(s, y(s))\, \mathrm{d}s \\
&\geqslant \frac{m}{M} \int_a^b \Big(\sup_{x \in [a,c) \cup (c,b]} G(x,s) \Big) f(s, y(s))\, \mathrm{d}s \\
&\geqslant \frac{m}{M} \cdot \sup_{x \in [a,c) \cup (c,b]} \int_a^b G(x,s) f(s, y(s))\, \mathrm{d}s \\
&= \frac{m}{M} \|Ay\|\,.
\end{aligned}$$

Therefore, $Ay \in \widetilde{P}$. $\square$

In the next theorem we also assume the following condition on $f(x,\xi)$.

(H4) There exist numbers $0<r<R<\infty$ such that for all $x \in [a,c)\cup(c,b]$:

$$f(x,\xi) \leqslant \frac{1}{(b-a)M} r \quad \text{if } 0 \leqslant \xi \leqslant r;$$
$$f(x,\xi) \geqslant \frac{M}{(b-a)m^2} R \quad \text{if } R \leqslant \xi < \infty,$$

where m and M are defined by (5.9).

Theorem 5.2. *Assume that conditions* (H1), (H2′), (H3), *and* (H4) *are satisfied. Then the BVPI* (5.1)–(5.3) *has at least one solution* $y(x)$ *such that*

$$\frac{m}{M} r \leqslant y(x) \leqslant \frac{M}{m} R, \quad x \in [a,c)\cup(c,b]. \tag{5.10}$$

Proof. For $y \in \widetilde{P}$ with $\|y\| = r$ (hence $0 \leqslant y(x) \leqslant r$ for $x \in [a,c)\cup(c,b]$), we have for all $x \in [a,c)\cup(c,b]$,

$$Ay(x) \leqslant M \int_a^b f(s,y(s))\,\mathrm{d}s \leqslant M \frac{1}{(b-a)M} r \int_a^b \mathrm{d}s = r = \|y\|. \tag{5.11}$$

Now if we let $\Omega_1 = \{y \in \mathbb{B}\colon \ \|y\| < r\}$, then (5.11) shows that $\|Ay\| \leqslant \|y\|$ for $y \in \widetilde{P} \cap \partial\Omega_1$. Further, let

$$R_1 = \frac{M}{m} R \quad \text{and} \quad \Omega_2 = \{y \in \mathbb{B}\colon \ \|y\| < R_1\}.$$

Then $y \in \widetilde{P}$ and $\|y\| = R_1$ implies

$$\inf_{x\in[a,c)\cup(c,b]} y(x) \geqslant \frac{m}{M}\|y\| = \frac{m}{M} R_1 = R,$$

hence $y(s) \geqslant R$ for all $s \in [a,c)\cup(c,b]$. Therefore, for all $x \in [a,c)\cup(c,b]$ we obtain that

$$Ay(x) \geqslant m \int_a^b f(s,y(s))\,\mathrm{d}s \geqslant m \frac{M}{(b-a)m^2} R \int_a^b \mathrm{d}s = \frac{M}{m} R = R_1 = \|y\|.$$

Hence $\|Ay\| \geqslant \|y\|$, for all $y \in \widetilde{P} \cap \partial\Omega_2$.

Consequently, by the first part of Theorem 1.1, it follows that A has a fixed point y in $\widetilde{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. We have $r \leqslant \|y\| \leqslant R_1$. Hence, since for $y \in \widetilde{P}$ we have $y(x) \geqslant \frac{m}{M}\|y\|$, $x \in [a,c)\cup(c,b]$, it follows that (5.10) holds. $\square$

Remark 5.3. If

$$\lim_{\xi\to 0^+}\frac{f(x,\xi)}{\xi}=0 \quad\text{and}\quad \lim_{\xi\to\infty}\frac{f(x,\xi)}{\xi}=\infty$$

uniformly on $x\in[a,c)\cup(c,b]$, then the condition (H4) will be satisfied for $r>0$ sufficiently small and $R>0$ sufficiently large.

Below, in Theorem 5.3, we assume the following condition on $f(x,\xi)$:

(H5) There exist numbers $0<r<R<\infty$ such that for all $x\in[a,c)\cup(c,b]$:

$$f(x,\xi)\geqslant\frac{M}{(b-a)m^2}\xi \quad\text{if } 0\leqslant\xi\leqslant r;$$
$$f(x,\xi)\leqslant\frac{1}{(b-a)M}\xi \quad\text{if } R\leqslant\xi<\infty.$$

Theorem 5.3. *Assume that conditions* (H1), (H2′), (H3), *and* (H5) *are satisfied. Then the BVPI* (5.1)–(5.3) *has at least one solution* $y(x)$ *with the property* (5.10).

The proof is analogous to that of Theorem 5.2 and uses the second part of Theorem 1.1.

Remark 5.4. If

$$\lim_{\xi\to 0^+}\frac{f(x,\xi)}{\xi}=\infty \quad\text{and}\quad \lim_{\xi\to\infty}\frac{f(x,\xi)}{\xi}=0$$

uniformly on $x\in[a,c)\cup(c,b]$, then the condition (H5) will be satisfied for $r>0$ sufficiently small and $R>0$ sufficiently large.

References

[1] *F. M. Atici and G. Sh. Guseinov*: On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions. J. Comput. Appl. Math. *132* (2001), 341–356. Zbl 0993.34022

[2] *D. D. Bainov and P. S. Simeonov*: Impulsive Differential Equations: Asymtotic Properties of the Solutions. World Scientific, Singapore, 1995. Zbl 0828.34002

[3] *P. W. Eloe and J. Henderson*: Positive solutions of boundary value problems for ordinary differential equations with impulse. Dynam. Contin. Discrete Impuls. Systems *4* (1998), 285–294. Zbl 0903.34013

[4] *P. W. Eloe and M. Sokol*: Positive solutions and conjugate points for a boundary value problem with impulse. Dynam. Systems Appl. *7* (1998), 441–449. Zbl 0945.34012

[5] *L. H. Erbe, S. Hu and H. Wang*: Multiple positive solutions of some boundary value problems. J. Math. Anal. Appl. *184* (1994), 640–648. Zbl 0805.34021

[6] *L. H. Erbe and H. Wang*: On the existence of positive solutions of ordinary differential equations. Proc. Amer. Math. Soc. *120* (1994), 743–748. Zbl 0802.34018

[7] *D. Guo and V. Lakshmikantham*: Nonlinear Problems in Abstract Cones. Academic Press, San Diego, 1998.

[8] *M. A. Krasnosel'skii*: Positive Solutions of Operator Equations. Noordhoff, Groningen, 1964. Zbl 0121.10604

[9] *M. A. Neumark*: Lineare Differential Operatoren. Akademie-Verlag, Berlin, 1967. Zbl 0114.28703

[10] *A. M. Samoilenko and N. A. Perestyuk*: Impulsive Differential Equations. World Scientific, Singapore, 1995. Zbl 0837.34003

[11] *Š. Schwabik, M. Tvrdý and O. Vejvoda*: Differential and Integral Equations: Boundary Value Problems and Adjoint. Academia and Reidel, Praha and Dordrecht, 1979. Zbl 0417.45001

[12] *Š. Schwabik*: Generalized Ordinary Differential Equations. World Scientific, Singapore, 1992. Zbl 0781.34003

[13] *M. Tvrdý*: Differential and integral equations in the space of regulated functions. Memoirs on Differential Equations and Mathematical Physics *25* (2002), 1–104. Zbl pre01836229

[14] *M. Tvrdý*: Linear distributional differential equations of the second order. Math. Bohem. *119* (1994), 415–436. Zbl 0819.34007

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