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Invariant subspaces of X^{**} under the action of biconjugates

Terms of use:

INVARIANT SUBSPACES OF X^{**} UNDER THE ACTION OF BICONJUGATES

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Abstract. We study conditions on an infinite dimensional separable Banach space X implying that X is the only non-trivial invariant subspace of X^{**} under the action of the algebra $\mathbb{A}(X)$ of biconjugates of bounded operators on X: $\mathbb{A}(X) = \{T^{**}\colon T \in \mathcal{B}(X)\}$. Such a space is called *simple*. We characterize simple spaces among spaces which contain an isomorphic copy of c_0, and show in particular that any space which does not contain ℓ_1 and has property (u) of Pelczynski is simple.

Keywords: algebras of operators with only one non-trivial invariant subspace, invariant subspaces under the action of the algebra of biconjugates operators, transitivity, property (u) of Pelczynski

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1. Introduction

Let X be a non-reflexive Banach space. It may happen that all the elements $x^{**} \in X^{**} \backslash X$ somehow "look alike" when considered as affine functions on (B_{X^*}, w^*), but it is not always so. The present work is an attempt to elucidate this question. We will be concerned with the study of the algebra of biconjugates of bounded operators on a Banach space X. We consider X as a subspace of its bidual X^{**} and write $X \subseteq X^{**}$ without mentioning the canonical inclusion. Let $\mathcal{B}(X)$ denote the algebra of bounded operators on X, and $\mathbb{A}(X)$ the subalgebra of $\mathcal{B}(X^{**})$ defined by

$$\mathbb{A}(X) = \{T^{**}\colon T \in \mathcal{B}(X)\}.$$

The "homogeneity" of the bidual space can be understood in terms of invariant subspaces under the action of the algebra $\mathbb{A}(X)$ on X^{**}. Recall that a closed

subspace E of X^{**} is said to be *invariant* under $\mathbb{A}(X)$ when $T^{**}(E) \subseteq E$ for every operator $T \in \mathcal{B}(X)$. The space E is said to be *non-trivial* when it is distinct from $\{0\}$ and from X^{**}. Of course X is always a non-trivial invariant subspace of X^{**} under the action of $\mathbb{A}(X)$.

Definition 1.1. The space X is said to be *simple* when the only non-trivial invariant subspace of X^{**} under the action of $\mathbb{A}(X)$ is X.

This makes sense only for non-reflexive Banach spaces. If x^{**} is any vector in X^{**}, the norm-closure $Y(x^{**})$ of the vectors $T^{**}x^{**}$, $T \in \mathcal{B}(X)$, is the smallest closed $\mathbb{A}(X)$-invariant subspace of X containing x^{**}. The space X is simple if and only if all the subspaces $Y(x^{**})$ are equal to X^{**} as soon as x^{**} does not belong to X.

This notion of simplicity has its origin in the work of E. Kissin, V. Lomonosov and V. Shulman ([12]), who investigate the structure of norm-closed operator algebras $\mathcal{B}$ on a Banach space X with only one non-trivial invariant subspace $L \subseteq X$. This study is carried out in the reflexive case, and the notion of simplicity arises naturally when considering non-reflexive cases: see Example 4.6 and Remark 4.7 of [12].

In Section 2 of this note we give some necessary conditions for a Banach space to be simple. These conditions are obtained by considering natural classes of functions on the unit ball of the dual space X^* equipped with the weak-star topology, such as Baire-1 class functions (Proposition 2.8), functions of a given oscillation rank (Proposition 2.4), or functions which can be written as differences of bounded semi-continuous functions (Proposition 2.7).

The main result of Section 3 yields a necessary and sufficient condition for a separable space containing c_0 to be simple. We denote by $\mathrm{DBSC}(X)$ the space of elements x^{**} in the bidual space X^{**} which can be written as a difference of bounded semi-continuous functions on B_{X^*}, i.e. $x^{**} = u_1 - u_2$ if the space is real, and $x^{**} = (u_1 - u_2) + \mathrm{i}(u_3 - u_4)$ if the space is complex, where the u_j's are real-valued bounded semi-continuous functions on (B_{X^*}, w^*). With this notation, a separable Banach space containing an isomorphic copy of c_0 is simple if and only if $\mathrm{DBSC}(X)$ is norm-dense in X^{**} (Theorem 3.4, Corollary 3.5). The fourth section is devoted to applications and examples. We show in particular that any separable space which does not contain ℓ_1 and has property (u) of Pelczynski is simple. Section 5 contains some remarks and questions.

2. What kind of spaces can be simple?

Let X be an infinite dimensional real or complex Banach space, which will in general be assumed to be separable. We denote by B_{X^*} the closed unit ball of the dual space X^*, and by w^* the weak-star topology on X^*. Whenever X is separable, (B_{X^*}, w^*) is a compact metrizable space. We will also always suppose that X is not reflexive.

Notation 2.1. Let $\mathcal{B}_1(X)$ be the set of elements x^{**} of X^{**} which are of first Baire class in the following sense: $x^{**} \in \mathcal{B}_1(X)$ if and only if there exists a sequence (x_n) of elements of X such that x_n tends w^* to x^{**} as n tends to infinity. Let also $\widetilde{\mathcal{B}}_1(X)$ be the set of elements x^{**} of X^{**} such that for every closed subset F of (B_{X^*}, w^*) the restriction of x^{**} to F has a point of w^*-continuity. Of course $\mathcal{B}_1(X) \subseteq \widetilde{\mathcal{B}}_1(X)$. It is easy to see that $\mathcal{B}_1(X)$ and $\widetilde{\mathcal{B}}_1(X)$ are $\mathbb{A}(X)$-invariant subsets of X^{**} that contain X.

A first necessary condition for X to be simple is:

Proposition 2.2. *If X is a simple separable Banach space X, then X contains no isomorphic copy of ℓ_1.*

P r o o f. The cardinality of $\mathbb{A}(X)$ is $2^{\aleph_0}$, and since X is simple, this implies that the cardinality of X^{**} is $2^{\aleph_0}$, i.e. the cardinality of X. Hence by the Odell-Rosenthal theorem ([17]), ℓ_1 does not embed in X. $\square$

Thus when X is simple and separable every element x^{**} of X^{**} is of first Baire class. This suggests that we consider the oscillation rank of such elements:

Notation 2.3. We recall here the definition of the oscillation rank ([11]) of a function $f \colon K \to \mathbb{R}$ where K is any compact metrizable set:

$$\mathrm{osc}(f, x) = \inf \Big\{ \sup_{x_1, x_2 \in V} |f(x_1) - f(x_2)| \colon V \text{ open set in } K \text{ containing } x \Big\}.$$

If F is any closed set in K, $\mathrm{osc}(f, x, F) = \mathrm{osc}(f|_F, x)$. If now ε is any positive real number and α is any ordinal, define inductively the αth derivative of K by: $K_{f,\varepsilon}^{(0)} = K$,

$$K_{f,\varepsilon}^{(\alpha+1)} = \Big\{ x \in K_{f,\varepsilon}^{(\alpha)} \colon \mathrm{osc}(f, x, K_{f,\varepsilon}^{(\alpha)}) \geqslant \varepsilon \Big\},$$

and

$$K_{f,\varepsilon}^{(\alpha)} = \bigcap_{\beta < \alpha} K_{f,\varepsilon}^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal.}$$

Then let $\beta_K(f, \varepsilon)$ be equal to the infimum of the α's such that $K_{f,\varepsilon}^{(\alpha)} = \emptyset$ if such an α exists, and to ω_1 otherwise. The oscillation rank $\beta_K(f)$ of f with respect to K is defined by

$$\beta_K(f) = \sup_{\varepsilon > 0} \beta_K(f, \varepsilon).$$

With this definition, f is of Baire class 1 ($f \in \mathcal{B}_1(X)$) if and only if $\beta_K(f) < \omega_1$. Moreover, if (f_n) is a sequence of functions on K that converge uniformly to a function f on K, then $\beta_K(f) \leqslant \sup_n \beta_K(f_n)$.

In our setting, $K = (B_{X^*}, w^*)$ and whenever X does not contain ℓ_1, $\beta(x^{**})$ is countable for every element x^{**} of the bidual. Of course $\beta(x^{**}) = 1$ exactly means that x^{**} belongs to X. There is a natural class of $\mathbb{A}(X)$-invariant subspaces associated to the oscillation rank: let ξ be any countable ordinal, and denote by $\mathcal{B}_1^\xi(X)$ the set of elements x^{**} of X^{**} such that $\beta(x^{**}) \leqslant \xi$. By the remark above, this is a closed subspace of X^{**}. Simplicity implies that all non-trivial functionals in the bidual have the same complexity:

Proposition 2.4. *For every countable ordinal ξ, $\mathcal{B}_1^\xi(X)$ is an $\mathbb{A}(X)$-invariant subspace of X^{**}. Thus if X is simple, the oscillation rank β is constant on $X^{**} \setminus X$.*

P r o o f.   The only thing to prove is that if x^{**} is any element in $X^{**} \setminus X$ and T is a bounded operator on X, then $\beta(T^{**}x^{**}) \leqslant \beta(x^{**})$. There is no loss of generality in assuming that the norm of T is smaller than 1. Let $\varepsilon > 0$ and F be any w^*-closed subset of B_{X^*}. Then $T^*(F)$ is also a w^*-closed subset of B_{X^*}, and if $\mathrm{osc}(T^{**}x^{**}, x^*, F) \geqslant \varepsilon$ then $\mathrm{osc}(x^{**}, T^*x^*, T^*(F)) \geqslant \varepsilon$. Hence $T^*(F_{T^{**}x^{**},\varepsilon}^{(1)}) \subseteq T^*(F)_{x^{**},\varepsilon}^{(1)}$ for every w^*-closed subset of B_{X^*}. Since $T^*(B_{X^*}) \subseteq B_{X^*}$, a transfinite induction shows that for every ordinal α,

$$T^*\big((B_{X^*})_{T^{**}x^{**},\varepsilon}^{(\alpha)}\big) \subseteq (B_{X^*})_{x^{**},\varepsilon}^{(\alpha)}.$$

Hence if $(B_{X^*})_{x^{**},\varepsilon}^{(\alpha)}$ is empty, $(B_{X^*})_{T^{**}x^{**},\varepsilon}^{(\alpha)}$ is also empty, and thus $\beta(T^{**}x^{**}, \varepsilon) \leqslant \beta(x^{**}, \varepsilon)$. This yields that $\beta(T^{**}x^{**}) \leqslant \beta(x^{**})$. If X is simple, for every $\varepsilon > 0$, x^{**} and y^{**} in $X^{**} \setminus X$, there exists an operator $T \in \mathcal{B}(X)$ and a vector x of X such that $\|T^{**}x^{**} - y^{**}\| < \varepsilon$, because $Y(x^{**})$ is norm-dense in X^{**}. Hence $\beta(y^{**}, 2\varepsilon) \leqslant \beta(T^{**}x^{**}, \varepsilon) \leqslant \beta(x^{**}, \varepsilon)$, and this proves our claim. Hence if $\beta(x^{**}) \leqslant \xi$, then $\beta(T^{**}x^{**}) \leqslant \xi$ for every operator $T \in \mathcal{B}(X)$. $\square$

This necessary condition for simplicity is in fact too rough, and very different elements in the bidual can have the same oscillation rank. We will see in Section 4 an example of a non-simple space such that β is equal to ω on $X^{**} \setminus X$.

The class DBSC(X) is far more interesting. It consists also of Baire-1 class functionals, but it bears no immediate relationship to the class considered above. Let us first recall some notation and facts about this class:

Notation 2.5. We say that an element x^{**} in the bidual space X^{**} belongs to the class DBSC(X) if the following is true: if the space is real, x^{**} can be written as the difference of two bounded semi-continuous functions on (B_{X^*}, w^*), and if the space is complex, x^{**} can be written as $x^{**} = (u_1 - u_2) + \mathrm{i}(u_3 - u_4)$ where u_1, u_2, u_3, u_4 are four real-valued bounded semi-continuous functions on (B_{X^*}, w^*). Of course these functions are not affine on B_{X^*} unless x^{**} itself is continuous on (B_{X^*}, w^*), i.e. x^{**} belongs to X. We have DBSC$(X) \subseteq \mathcal{B}_1(X)$.

When the space X is separable, DBSC(X) coincides with the class LWUC(X), which stands for "limits of weakly unconditionally convergent series": x^{**} belongs to LWUC(X) if x^{**} is the w^* sum of a weakly unconditionally convergent (w.u.c.) series $\sum x_n$ of vectors of X. This means that x^{**} is the w^*-limit in X^{**} of the partial sums $\sum_{k=1}^{n} x_k$ as n tends to infinity. We denote this by $x^{**} = \overset{*}{\sum} x_n$. Recall that a series $\sum x_n$ with x_n in X is said to be w.u.c. if for every x^* in X, $\sum |x^*(x_n)|$ is convergent.

In the sequel we will only retain the notation DBSC(X) when X is separable. But when considering dual spaces that are not necessarily separable, we will make use of the class LWUC(X^*) instead. The inclusions $X \subseteq$ LWUC$(X) \subseteq$ DBSC(X) remain true in either case. We include the simple proof since it yields a particular representation of an element x^{**} of LWUC(X) as a difference of bounded lower semi-continuous functions on B_{X^*} which will be of use in the sequel: suppose first that the space X is real, and let $\sum x_n$ be a w.u.c. series such that $x^{**} = \overset{*}{\sum} x_n$. Setting $f(x^*) = \sum |x^*(x_n)|$ and $g(x^*) = \sum (|x^*(x_n)| - x^*(x_n))$, it is easy to check that f and g are two bounded lower semi-continuous functions on B_{X^*} such that $x^{**} = f - g$. If the space X is complex, it suffices to consider separately the real and imaginary parts: $x^{**} = (u_1 - u_2) + \mathrm{i}(u_3 - u_4)$ where $u_1(x^*) = \sum |\Re e\, x^*(x_n)|$, $u_2(x^*) = \sum (|\Re e\, x^*(x_n)| - \Re e\, x^*(x_n))$, $u_3(x^*) = \sum |\Im m\, x^*(x_n)|$, and $u_4(x^*) = \sum (|\Im m\, x^*(x_n)| - \Im m\, x^*(x_n))$.

Let us now recall a result, essentially due to Bessaga and Pelczynski (see for instance [15], p. 98) which establishes a link between the existence of a non-trivial element in DBSC(X) and the existence of an isomorphic copy of c_0 in X:

Theorem 2.6. *Let X be a separable Banach space. The following assertions are equivalent:*

(1) *x^{**} belongs to DBSC(X);*

(2) *there exists a subspace Y of X which is isomorphic to c_0 such that x^{**} belongs to $Y^{\perp\perp} \subseteq X^{**}$.*

In particular X contains an isomorphic copy of c_0 if and only if $\mathrm{DBSC}(X) \neq X$.

Bearing this is mind, it is easy to see the following:

Proposition 2.7. *Let X be a separable Banach space. The linear space $\mathrm{DBSC}(X)$ is $\mathbb{A}(X)$-invariant, and thus its norm-closure $\overline{\mathrm{DBSC}}(X)$ in X^{**} is also $\mathbb{A}(X)$-invariant. As a consequence, if X is simple, then either $\mathrm{DBSC}(X)$ is norm-dense in X^{**} or $\mathrm{DBSC}(X) = X$ (i.e. X contains no isomorphic copy of c_0).*

The first alternative will be the subject of Section 3, and some examples related to the case when $\mathrm{DBSC}(X)$ is trivial will be given in Section 4.

Consider now the classes $\widetilde{\mathcal{B}}_1(X^*)$ and $\mathrm{LWUC}(X^*)$: these are linear submanifolds of X^{***} which contain X^* and are invariant under any operator T^{***}, $T \in \mathcal{B}(X)$. Remembering that X^{***} can be written as the topological sum $X^{***} = X^* \oplus X^{\perp}$ where $X^{\perp}$ is the set of elements of X^{***} which vanish on X, we consider $E = \mathrm{LWUC}(X^*) \cap X^{\perp}$ and $F = \widetilde{\mathcal{B}}_1(X^*) \cap X^{\perp}$. The pre-orthogonals $E_{\perp}$ and $F_{\perp}$ are then two closed subspaces of X^{**} which contain X and are $\mathbb{A}(X)$-invariant. Thus they are equal to either X or X^{**} if X is simple, and since $F_{\perp} \subseteq E_{\perp}$, one of the following assertions must hold:

(a) $E_{\perp} = X$;

(b) $E_{\perp} = X^{**}$ and $F_{\perp} = X$;

(c) $F_{\perp} = X^{**}$

which are equivalent to

(a) $\mathrm{LWUC}(X^*) \cap X^{\perp}$ is w^*-dense in $X^{\perp}$;

(b) $\mathrm{LWUC}(X^*) = X^*$ and $\widetilde{\mathcal{B}}_1(X^*) \cap X^{\perp}$ is w^*-dense in $X^{\perp}$;

(c) $\widetilde{\mathcal{B}}_1(X^*) = X^*$,

respectively. The assertion (a) exactly means that X has property (X) of Godefroy and Talagrand [5], see [6]). For the reader's convenience, we recall here some facts about property (X):

The frame $C(Y)$ of a Banach space Y is the subspace of Y^{**} consisting of the elements y^{**} of Y^{**} such that for every w.u.c. series $\sum y_n^*$ of vectors of Y^*, $y^{**}\left(\sum^* y_n^*\right) = \sum^* y^{**}(y_n^*)$. The space Y is said to have property (X) if $C(Y) = Y$. As already told above, Y has property (X) if the subspace $\mathrm{LWUC}(Y^*) \cap Y^{\perp}$ is large enough, i.e. w^*-dense in $Y^{\perp}$. Any space Y which has property (X) is weakly sequentially complete, and it contains a complemented copy of ℓ_1 as soon as it is non-reflexive. This property was introduced in [5] in connection with questions regarding the existence of a unique predual of a Banach space: if Y has property (X), then Y is the unique isometric predual of its dual Y^*. The property (X) is hereditary, i.e. passes from a space to its subspaces. Quite a large collection of spaces have property (X),

among them separable weakly sequentially complete Banach lattices, duals of separable spaces with property (u) which do not contain ℓ_1, or preduals of Von Neumann algebras ([5], Theorem 7, or [6], Examples V.4.). For more about these questions, see [6].

If X is separable and simple, X^* does not contain c_0 because otherwise X would contain a complemented copy of ℓ_1 (see for instance [15], p. 103). Moreover (a) implies that X is reflexive, (b) implies that $\widetilde{\mathcal{B}}_1(X^*) \cap X^\perp$ is w^*-dense in $X^\perp$, and the assertion (c) means that X^* is weakly sequentially complete. We have proved:

Proposition 2.8. *If a separable non-reflexive Banach space X is simple, then X^* does not contain c_0. Moreover, either $\widetilde{\mathcal{B}}_1(X^*) \cap X^\perp$ is w^*-dense in $X^\perp$, or X^* is weakly sequentially complete.*

It is important to remark here that the condition "$\widetilde{\mathcal{B}}_1(X^*) \cap X^\perp$ is w^*-dense in $X^\perp$" is much weaker than the condition "$\ell_1 \not\hookrightarrow X^*$". This distinction illustrates the following fact: whenever $\ell_1 \not\hookrightarrow X^*$, X is unique predual of its dual X^* and there exists a unique projection π of norm 1 of X^{***} onto X^*, which is the projection with kernel $X^\perp$ ([6], Theorems II.1 and II.3). The condition "$\widetilde{\mathcal{B}}_1(X^*) \cap X^\perp$ is w^*-dense in $X^\perp$" also implies that X is the unique predual of its dual X^*, but in this case one can only assert that there exists a unique projection π of norm 1 of X^{***} onto X^* such that $\ker(\pi)$ is w^*-closed, which is the projection with kernel $X^\perp$. For instance the space L_∞ contains ℓ_1, but $\widetilde{\mathcal{B}}_1(L_1^*) \cap L_1^\perp$ is w^*-dense in $L_1^\perp$. The space L_1 has property (X), but there exist infinitely many projections of norm 1 of L_1^{***} onto L_∞ ([6], Proposition IV.1).

We will see examples of all this in Section 4.

3. Simplicity through the study of differences of bounded semi-continuous functions

We first present a stability property of simple spaces:

Lemma 3.1. *Let X be a simple Banach space, and E any complemented subspace of X. Then E is also simple.*

P r o o f.   Let e^{**} be any element in $E^{**} \setminus E$. It can be extended to an element x^{**} of $X^{**} \setminus X$, and the set $\{T^{**}x^{**}, \ T \in \mathcal{B}(X)\}$ is norm-dense in X^{**}. Let P_E denote a bounded projection onto E. The set $\{P_{E^{**}}T^{**}e^{**}, \ T \in \mathcal{B}(X)\}$ is norm-dense in E^{**}. Since $P_{E^{**}} = P_E^{**}$, the set $\{(P_E T P_E)^{**}e^{**}, \ T \in \mathcal{B}(X)\}$ is norm-dense in E^{**}, and hence E is simple. $\qquad\square$

We derive simplicity from the transitivity of the action of $\mathbb{A}(X)$ on the set $X^{**} \setminus X$. Recall that $\mathbb{A}(X)$ is said to be 1-transitive on $X^{**} \setminus X$ when for every elements x^{**}, y^{**} of $X^{**} \setminus X$, there exists an operator $T \in \mathcal{B}(X)$ such that $T^{**}x^{**} = y^{**}$. It is clear that whenever $\mathbb{A}(X)$ is 1-transitive on $X^{**} \setminus X$, X is simple. The cornerstone of all what follows is:

Proposition 3.2. *If $X = c_0$, then $\mathbb{A}(X)$ is 1-transitive on $X^{**} \setminus X = \ell_\infty \setminus c_0$. In particular, c_0 is simple.*

P r o o f. Let x^{**} and y^{**} be two vectors of $\ell_\infty \setminus c_0$. Since x^{**} does not belong to c_0, there exists an $\varepsilon > 0$ and a sequence (n_k) of integers such that for every k, $|x^{**}(n_k)| \geqslant \varepsilon$. Let $T\colon c_0 \longrightarrow c_0$ be defined by

$$Tx = \left(\frac{y^{**}(k)}{x^{**}(n_k)} x(n_k) \right)_{k \geqslant 1}$$

for every $x = (x_n)_{n \geqslant 1}$ in c_0. Then $T^{**}x^{**} = y^{**}$ and $\mathbb{A}(c_0)$ is 1-transitive on $\ell_\infty \setminus c_0$. $\square$

It is natural to study the transitivity of $\mathbb{A}(X)$ between elements of $\mathrm{DBSC}(X)$: each element of $\mathrm{DBSC}(X) \setminus X$ carries with him a good copy of c_0, and mapping one element of $\mathrm{DBSC}(X) \setminus X$ onto another exactly means mapping one element of $\ell_\infty \setminus c_0$ onto another. This is made precise by the following proposition:

Proposition 3.3. *Let X be a separable Banach space. Then $\mathbb{A}(X)$ is 1-transitive on the set $\mathrm{DBSC}(X) \setminus X$.*

P r o o f. Let x_1^{**} and x_2^{**} be two vectors of $\mathrm{DBSC}(X) \setminus X$. By Theorem 2.6, there exist two subspaces Y_1 and Y_2 of X isomorphic to c_0 such that for $i = 1, 2$ x_i^{**} belongs to $Y_i^{\perp\perp} \setminus Y_i$. Let $L_i\colon Y_i \to c_0$ be an isomorphism and $y_i^{**} = L_i^{**}x_i^{**}$. The vectors y_i^{**} belong to $\ell_\infty \setminus c_0$. So there exists an operator $S \in \mathcal{B}(c_0)$ such that $S^{**}y_1^{**} = y_2^{**}$. Consider now $T = L_2^{-1}SL_1$: T is defined on Y_1 with values in Y_2 and satisfies $T^{**}x_1^{**} = x_2^{**}$. Since Y_2 is isomorphic to c_0 and X is separable, the Sobczyk Theorem (see for instance [15], p. 106) allows us to extend T to an operator $\widetilde{T}$ on X such that the restriction of $\widetilde{T}$ to Y_1 is equal to T. Now a simple computation shows that the restriction of $\widetilde{T}^{**}$ to $Y_1^{\perp\perp}$ is equal to T^{**} and thus $\widetilde{T}^{**}x_1^{**} = x_2^{**}$, which proves our claim. $\square$

Theorem 3.4. *Let X be a separable non-reflexive Banach space such that* $\mathrm{DBSC}(X)$ *is norm-dense in* X^{**}. *Then X is simple.*

P r o o f.   Let x^{**} belong to $X^{**} \setminus X$. There exists a bounded operator $S \colon X \to c_0$ such that $S^{**}x^{**}$ belongs to $\ell_\infty \setminus c_0$. Indeed, x^{**} is not w^*-continuous on X^* so there exists a sequence $(x_n^*)_{n \geqslant 1}$ of vectors of X^* such that (x_n^*) tends w^* to 0 and $(x^{**}(x_n^*))$ does not tend to 0. It then suffices to define S on X by $Sx = (x_n^*(x))_{n \geqslant 1}$.

Since $\mathrm{DBSC}(X) \neq X$, X contains a subspace E which is isomorphic to c_0 by an isomorphism $L \colon E \to c_0$. Consider $T_1 = L^{-1}S$: since $S^{**}x^{**}$ does not belong to c_0, $T_1^{**}x^{**} = L^{-1**}S^{**}x^{**}$ belongs to $\mathrm{DBSC}(X) \setminus X$. If now z is any vector in $\mathrm{DBSC}(X) \setminus X$, Proposition 3.3 implies that there exists an operator T_2 on X such that $T_2^{**}T_1^{**}x^{**} = z^{**}$. Setting $T = T_1T_2$ yields that $T^{**}x^{**} = z^{**}$. This implies that $\mathbb{A}(X)$ is 1-transitive between $X^{**} \setminus X$ and $\mathrm{DBSC}(X) \setminus X$. Since $\mathrm{DBSC}(X)$ is norm-dense in X^{**}, X is simple. $\qquad\square$

Theorem 3.4 implies the following characterization of simple separable Banach space containing c_0:

Corollary 3.5. *Let X be a separable Banach space containing an isomorphic copy of c_0. Then X is simple if and only if* $\mathrm{DBSC}(X)$ *is norm-dense in* X^{**}.

P r o o f.   If X is simple and $c_0 \hookrightarrow X$, then $\mathrm{DBSC}(X) \neq X$ and hence $\overline{\mathrm{DBSC}}(X) = X^{**}$ by Proposition 2.7. The converse is Theorem 3.4. $\qquad\square$

4. Examples

4.1. Spaces containing c_0

Let us first derive some consequences of Theorem 3.4 and Corollary 3.5:

Recall that a Banach space X is said to have property (u) when $\mathrm{LWUC}(X) = \mathcal{B}_1(X)$, i.e. when every Baire 1 class element x^{**} can be written as $x^{**} = \sum^* x_n$, where $\sum x_n$ is a weakly unconditionally converging series of vectors of X ([18], see also [16], p. 31). If X is separable, X does not contain ℓ_1 and has property (u) if and only if $\mathrm{DBSC}(X) = X^{**}$.

Corollary 4.1. *If X is a separable Banach space not containing ℓ_1 and with property (u), then X is simple. In particular the following spaces are simple, as well as all their closed subspaces:*
 (1) *separable spaces which are M-ideals in their biduals;*
 (2) *separable Banach lattices with an order-continuous norm which do not contain ℓ_1, in particular spaces with an unconditional basis which do not contain ℓ_1.*

P r o o f .   (1) By [3] (see also [4]), any space which is an M-ideal in its bidual has property (u). Note that this is also a consequence of a result of Rosenthal ([21], [1]) which states that X does not contain ℓ_1 and has (u) if and only if Y^* is weakly sequentially complete for every subspace Y of X, and of the fact that spaces which are L-complemented in their bidual are weakly sequentially complete (see [6]).

(2) Banach lattices with an order-continuous norm have property (u) (see [16], p. 31).

Moreover, property (u) is hereditary (see [16], p. 32), hence all closed subspaces of the spaces above are also simple. □

Remark 4.2. If $\mathrm{DBSC}(X)$ is norm-dense in X^{**}, then X^* has property (X) ([6]), in particular X^* is weakly sequentially complete. Hence all the spaces of Corollary 4.1 satisfy the second alternative of Proposition 2.8. The oscillation rank of non-trivial elements of X^{**} is equal to ω in this case ([11]), but the converse is not true: if every non-trivial element of X^{**} has an oscillation rank equal to ω, then for every element x^{**} of X^{**}, the restriction of x^{**} to B_{X^*} can be written as a uniform limit of a sequence $(f_n)_{n \geqslant 1}$ of differences of bounded semi-continuous functions on B_{X^*} (the limit is taken with respect to the sup norm on B_{X^*}), but the functions f_n are not assumed to be affine here, and hence x^{**} does not necessarily belong to $\mathrm{DBSC}(X)$.

Example 4.3. Let H be an infinite dimensional separable Hilbert space, and $X = \mathcal{K}(H)$ be the ideal of compact operators on H. It is proved in [12], using special features of $\mathcal{K}(H)$, that $\mathcal{K}(H)$ is simple. We obtain this here directly as a consequence of Corollary 4.1, because $\mathcal{K}(H)$ is an M-ideal in its bidual $\mathcal{B}(H)$. See [7] for related examples of spaces which are u-ideals or h-ideals in their bidual.

Corollary 3.5 also provides a quick way to show that some spaces containing c_0 are not simple. For instance:

Example 4.4. Suppose that X and Y are two non-reflexive separable spaces not containing ℓ_1 such that $c_0 \hookrightarrow X$ and $c_0 \not\hookrightarrow Y$: the space $Z = X \oplus Y$ is non-reflexive, does not contain ℓ_1 and Theorem 2.6 easily implies that $\mathrm{DBSC}(Z) = \mathrm{DBSC}(X) \oplus \mathrm{DBSC}(Y)$. Since $c_0 \hookrightarrow X$ and $c_0 \not\hookrightarrow Y$, $\mathrm{DBSC}(X) \neq X$ and $\mathrm{DBSC}(Y) = Y$, hence $\overline{\mathrm{DBSC}}(Z)$ cannot be equal to either Z or Z^{**}. Thus Z is non-simple.

This example should be compared with the following proposition:

Proposition 4.5. *Let Y and Z be two non-reflexive Banach spaces such that every bounded operator from Y to Z is weakly compact. Then the space $X = Y \oplus Z$ is non-simple.*

P r o o f .   Every bounded linear operator $T \in \mathcal{B}(X)$ can be viewed as a 2×2 operator-valued matrix $T = (T_{YY}, T_{ZY}, T_{YZ}, T_{ZZ})$. The operator T_{YZ}, being weakly

compact, satisfies $T^{**}_{YZ}(Y^{**}) \subseteq Z$. Hence $T^{**}(Y^{**} \oplus Z) \subseteq Y^{**} \oplus Z$, and $Y^{**} \oplus Z$ is a non-trivial $\mathbb{A}(X)$-invariant subspace of X^{**}. $\qquad\square$

In particular if X and Y are separable non-reflexive spaces, X has property (V) of Pelczynski ([18]) and Y does not contain c_0, then $X \oplus Y$ is non-simple (this is a special case of Example 4.4). Recall that X has property (V) when it satisfies the following: for all Banach spaces Y and all operators $T \colon X \longrightarrow Y$ which are not weakly compact, there is a subspace X_0 of X isomorphic to c_0 such that the restriction of T to X_0 is an isomorphism. In particular every non-reflexive space with (V) contains c_0.

Remark 4.6. The separability assumption is essential in Theorem 3.4. The space $c_0(\Gamma)$ where Γ is an uncountable set is an M-ideal in its bidual but is non-simple. This follows from the fact that

$$\ell^c_\infty(\Gamma) = \{x \in \ell_\infty(\Gamma) \text{ such that } x_\gamma = 0 \text{ for all but countably many } \gamma \in \Gamma\}$$

is the w^*-sequential closure of $c_0(\Gamma)$ in $\ell_\infty(\Gamma)$, and it is also norm-closed. Hence it is a $\mathbb{A}(c_0(\Gamma))$-invariant subspace of $c_0(\Gamma)^{**} = \ell_\infty(\Gamma)$ which is obviously non-trivial.

Let us now derive a necessary and sufficient condition for spaces $C(K)$ to be simple. Recall that if K is a metrizable scattered compact, then K is homeomorphic to a successor ordinal $\widetilde{K} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ where $\alpha_1 \geqslant \ldots \geqslant \alpha_k$. We denote by $o(K)$ the ordinal α_1. With this notation, the Cantor index of K is equal to $\omega^{o(K)} + 1$. The Szlenk index of the space $C(K)$ is then equal to $\omega^{o(K)+1}$ ([13]).

Proposition 4.7. *Let K be an infinite compact metrizable space. The following assertions are equivalent:*
(1) *$C(K)$ is simple;*
(2) *K is scattered and $o(K) < \omega$;*
(3) *$C(K)$ is isomorphic to c_0.*

P r o o f.   For the equivalence between (2) and (3) see [13]. The fact that (3) implies (1) is obvious. Suppose now that $C(K)$ is simple. Then $\ell_1 \not\hookrightarrow C(K)$ and thus K is scattered. Assume now that $o(K) \geqslant \omega$. We will first work out the case when K is equal to $\omega^\omega + 1$ and show that $C(\omega^\omega + 1)$ is not simple.

It is known ([9], [1]) that there exists a bounded function f on K which belongs to $\mathcal{B}_1(K) \setminus \mathcal{B}_{\frac{1}{2}}(K)$, i.e. f is of Baire-class 1 but cannot be written as the limit of a sequence of differences of bounded semi-continuous functions on K. This function f induces an element x^{**} of $C(K)^{**}$ by the formula $x^{**}(\mu) = \int_K f \, d\mu$ for every bounded regular Borel measure with finite variation on K. Let us show that

x^{**} does not belong to $\overline{\mathrm{DBSC}}(C(K))$. Suppose that there exists a sequence $(x_n{}^{**})$ of $\mathrm{DBSC}(C(K))$ such that $\|x_n{}^{**} - x^{**}\|$ tends to zero as n tends to infinity. Since K is countable, $C(K)^{**}$ is isometric to $\ell_\infty(K)$ and there exist bounded functions f_n on K such that for every $\mu \in \mathcal{M}(K)$, $x_n{}^{**}(\mu) = \int_K f_n \, \mathrm{d}\mu$. For every point x in K, $f_n(x) = x_n{}^{**}(\delta_x)$. By Theorem 2.6, there exists a sequence $(g_k^{(n)})$ of elements of $C(K)$ such that $\sum_k g_k^{(n)}$ is w.u.c. and $x_n{}^{**} = \sum_k^* g_k^{(n)}$. Then for every $\mu \in \mathcal{M}(K)$, $x_n{}^{**}(\mu) = \varphi_n(\mu) - \psi_n(\mu)$ where $\varphi_n(\mu) = \sum_k |\int_K g_k^{(n)} \, \mathrm{d}\mu|$ and $\psi_n(\mu) = \sum_k \left(|\int_K g_k^{(n)} \, \mathrm{d}\mu| - \int_K g_k^{(n)} \, \mathrm{d}\mu \right)$. The functions φ_n and ψ_n are bounded semi-continuous on the unit ball of $\mathcal{M}(K)$ with the w^*-topology. In particular for every $x \in K$, $f_n(x) = \alpha_n(x) - \beta_n(x)$ where $\alpha_n(x) = \varphi_n(\delta_x) = \sum_k |g_k^{(n)}(x)|$ and $\beta_n(x) = \psi_n(\delta_x) = \sum_k (|g_k^{(n)}(x)| - g_k^{(n)}(x))$. Now α_n and β_n are bounded semi-continuous functions on K, and $\sup_{x \in K} |f_n(x) - f(x)| \leqslant \|x_n{}^{**} - x^{**}\|$, which tends to zero as n goes to infinity. This contradiction shows that x^{**} is not an element of $\overline{\mathrm{DBSC}}(C(K))$, and this implies that $\overline{\mathrm{DBSC}}(C(K))$ cannot be equal to $C(K)^{**}$. Since c_0 embeds in $C(K)$, it cannot be equal to $C(K)$ either: $C(\omega^\omega + 1)$ is not simple.

If now K is a general scattered compact with $o(K) \geqslant \omega$, then K is homeomorphic to a successor ordinal $\widetilde{K} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ where $\alpha_1 \geqslant \ldots \geqslant \alpha_k$ and $\alpha_1 \geqslant \omega$. Then $C(K)$ and $C(\widetilde{K})$ are isomorphic and it suffices to show that $C(\widetilde{K})$ is non-simple. But since $\omega^\omega + 1 \subseteq \widetilde{K}$, $C(\widetilde{K})$ contains a complemented copy of $C(\omega^\omega + 1)$ by Milutin's Theorem (see for instance [24], p. 160). By Lemma 3.1, $C(\widetilde{K})$ is non-simple. This completes the proof. $\qquad\square$

Remark 4.8. If a Banach space X does not contain ℓ_1 and has property (u), then X has property (V). In view of this fact one might wonder whether every Banach space with (V) which does not contain ℓ_1 is simple. The above example shows that this is not the case, since every space $C(K)$ has property (V) ([18]).

4.2. Spaces not containing c_0

Let us now say a few words concerning the second alternative of Proposition 2.7 when $\mathrm{DBSC}(X)$ is equal to X. We will be mainly concerned with quasi-reflexive spaces:

Proposition 4.9. *Any quasi-reflexive space which is of codimension 1 in its bidual is simple, and the algebra $\mathbb{A}(X)$ is even transitive on $X^{**} \setminus X$.*

P r o o f. If $X^{**} = X \oplus \mathrm{sp}(u^{**})$, $x_1^{**} = \lambda_1 u^{**} + x_1$ and $x_2^{**} = \lambda_2 u^{**} + x_2$ with $\lambda_1 \neq 0$, $x_1 \in X$ and $x_2 \in X$, let $x_0^* \in X^*$ be such that $x_0^*(x_1^{**}) = 1$. Setting

$T = \mu I + R$ where $\mu = \lambda_2/\lambda_1$ and $R(x) = x_0^*(x)(x_2 - \mu x_1)$, it is immediate that $T^{**}x_1^{**} = x_2^{**}$. $\square$

Example 4.10. Every quasi-reflexive space X which is of codimension 1 in its bidual satisfies the first assertion of Proposition 2.8, because X^* does not contain ℓ_1. For instance the James space J ([10], see also [15], p. 25) is simple. Since this space is isometric to its bidual, every non-trivial element of J^{**} has an oscillation rank equal to ω ([23]). Recall now that by Example 4.4, the space $c_0 \oplus J$ is non-simple. But every non-trivial functional of $\ell_\infty \oplus J^{**}$ has an oscillation rank equal to ω. Thus the necessary condition of Proposition 2.4 cannot be sufficient.

Example 4.11. For every ordinal number $\gamma < \omega_1$ there exists a separable quasi-reflexive space J_γ of codimension 1 in its bidual such that every non-trivial element of J_γ^{**} has an oscillation rank equal to γ ([9], see also [22]). Whenever γ_1 and γ_2 are two distinct ordinals, the space $J_{\gamma_1} \oplus J_{\gamma_2}$ is not simple by Proposition 2.4.

We now construct non-simple Banach spaces with all conjugates separable. These spaces are moreover quasi-reflexive and of codimension 2 in their biduals:

Example 4.12. We consider in this example a class of generalized James spaces J_p for $p \in (1, \infty)$ and show that whenever $p > q$, $J_p \oplus J_q$ is not simple. The space J_p consists of all sequences (α_n) of scalars such that $\lim_{n \to \infty} \alpha_n = 0$ and $\|(\alpha_n)\|_p < \infty$, with

$$\|(\alpha_n)\|_p = \sup(|\alpha_{n_1} - \alpha_{n_2}|^p + \ldots + |\alpha_{n_{k-1}} - \alpha_{n_k}|^p + |\alpha_{n_k} - \alpha_{n_1}|^p)^{1/p},$$

where the supremum is taken over all sequences $n_1 < n_2 < \ldots < n_k$ of integers. With this definition, the standard unit vectors (e_n) form a shrinking basis of J_p, and $J_p^{**} = J_p \oplus z_p^{**}$, where $z_p^{**} = (1, 1, \ldots)$. Considering $u_n = e_n - e_{n+1}$, one obtains that (u_n) is a boundedly complete basis of J_p which is not shrinking, and

$$\left\|\sum \beta_n u_n\right\|_p = \sup\left\{ \left(\sum_{j=1}^{m} \left| \sum_{n=n_j}^{n_{j+1}-1} \beta_n \right|^p \right)^{1/p} \right\},$$

where the supremum is taken over all sequences $n_1 < n_2 < \ldots < n_m$ of integers. The dual space J_p^* is spanned by the biorthogonal functionals u_n^* to u_n and the functional g_p defined by $g_p\left(\sum_n \beta_n u_n\right) = \sum_{n=1}^{\infty} \beta_n$. Moreover z_p^{**} is the w^*-limit of u_n. The original James space was defined for $p = 2$ ([10], see also [15]). Additional information on the construction of generalized James spaces can be found in [2] and also in [8], Chapter 6.

Let us prove now that there is no bounded linear operator $T\colon J_p \to J_q$, for $p > q$ such that $T^{**}(z_p^{**}) = z_q^{**}$. This will imply that $J_p^{**} \oplus J_q$ is a proper $A(J_p \oplus J_q)$-invariant subspace of $J_p^{**} \oplus J_q^{**}$.

Aiming at a contradiction, assume that there is T such that $T^{**}(z_p^{**}) = z_q^{**}$. Denote $v^{(n)} = T^{**}(u_n)$. Then the $w^* - w^*$ continuity of T^{**} implies that $w^* - \lim_{n \to \infty} v^{(n)} = z_q^{**}$. The following is a standard method using a sliding hump argument. Without loss of generality we may assume that the vectors $v^{(n)} = \sum_k \beta_k^{(n)} u_k$ have only finitely many non-zero coordinates $\beta_k^{(n)}$ for k in $[s_n, t_n]$. Now there is a subsequence $(n_k)_{k \geqslant 1}$ such that the intervals $[s_{n_k}, t_{n_k}]$ are pairwise disjoint and moreover since $g_q(z_q^{**}) = 1$, the following is true for k large enough:

$$\frac{1}{2} < g_q(v^{(n_k)}) = \sum_{n=1}^{\infty} \beta_n^{(n_k)} = \sum_{n=s_{n_k}}^{t_{n_k}} \beta_n^{(n_k)}.$$

Define elements $x^{(l)} \in J_p$ by $x^{(l)} = \sum_{k=1}^{l} (-1)^{n_k} u_{n_k}$. Then $\|x^{(l)}\|_p^p = l$ but

$$\|T^{**}\|_q^q \, l^{p/q} \geqslant \|T^{**}(x^{(l)})\|_q^q = \left\| \sum_{k=1}^{l} (-1)^k v^{(n_k)} \right\|_q^q \geqslant \frac{l}{2^q},$$

where the last inequality holds because $\{[s_{n_k}, t_{n_k}]\colon k = 1, \ldots, l\}$ is a finite collection of disjoint intervals which is suitable for the computation of the norm of $\sum_{k=1}^{l} (-1)^k v^{(n_k)}$. But this contradicts the continuity of T, since l can be chosen arbitrarily large and $p > q$.

We finish this section with an example of a separable simple Banach space with non-separable dual:

Example 4.13. The James tree space JT (see [14]) is simple. Let us sketch a proof of this: $JT^{**} = JT \oplus \ell_2(\Gamma)$, where Γ is the set of all branches of a dyadic tree. If P_γ is a projection on the branch $\gamma \in \Gamma$, then $P_\gamma(JT) \backsimeq J$, where J is the James space. Therefore there is an isomorphism $I_{\gamma_1 \gamma_2}\colon P_{\gamma_1}(JT) \to P_{\gamma_2}(JT)$. Define $T_{\gamma_1, \gamma_2} = P_{\gamma_1} \circ I_{\gamma_1, \gamma_2}$. Let $\mathbb{A}$ be the algebra generated by $\{T_{\gamma_1, \gamma_2}^{**}\colon \gamma_1, \gamma_2 \in \Gamma\}$, and denote by $\ell_2^{\mathrm{FIN}}(\Gamma)$ the set of all finitely supported vectors of $\ell_2(\Gamma)$. Then $\mathbb{A}$ is 1-transitive between $\ell_2(\Gamma)$ and $\ell_2^{\mathrm{FIN}}(\Gamma)$. Since $\ell_2^{\mathrm{FIN}}(\Gamma)$ is dense in $\ell_2(\Gamma)$, JT is simple.

5. Concluding remarks and questions

All the simple spaces we have met with in Section 4 (with the possible exception of the space JT) satisfy in fact a stronger property: the algebra $\mathbb{A}(X)$ is 1-transitive on $X^{**} \setminus X$. This can be reformulated in a convenient way by using the quotient space X^{**}/X: let us denote by $\overline{x^{**}}$ the class $x^{**} + X$ of an element x^{**} of X^{**} in the quotient space X^{**}/X. Since every operator in $S \in \mathbb{A}(X)$ leaves X invariant, $\mathbb{A}(X)$ can be canonically embedded in $\mathcal{B}(X^{**}/X)$ by the homomorphism $\varphi \colon \mathbb{A}(X) \to \mathcal{B}(X^{**}/X)$ where $\varphi(T^{**})$ is defined by

$$\varphi(T^{**}) \colon X^{**}/X \to X^{**}/X$$
$$\overline{x^{**}} \to \overline{T^{**}(x^{**})}.$$

Let us call the space X *strictly simple* when the algebra $\varphi(\mathbb{A}(X))$ is strictly transitive on X^{**}/X, i.e. for every n-tuple of linearly independent vectors $(\overline{x_1^{**}}, \ldots, \overline{x_n^{**}})$ of X^{**}/X and every n-tuple of vectors $(\overline{y_1^{**}}, \ldots, \overline{y_n^{**}})$, there exists an operator $T \in \mathcal{B}(X)$ such that for $i = 1 \ldots n$, $\varphi(T)\overline{x_i^{**}} = \overline{y_i^{**}}$.

If X is a complex vector space and $\varphi(\mathbb{A}(X))$ is 1-transitive, then $\varphi(\mathbb{A}(X))$ is also strictly transitive (see for instance [19], p. 62, for a proof of this, as well as for counterexamples to it when the underlying space is real). If X is strictly simple, then X is simple, and if $\mathbb{A}(X)$ is 1-transitive on $X^{**} \setminus X$, X is strictly simple. For instance all the spaces considered in Corollary 4.1 are strictly simple. This is clear when the underlying space is complex, and it is a consequence of the following lemma when the space is real:

Lemma 5.1. *Let X be a real Banach space and let $\widetilde{X}$ be its complexification. If $\widetilde{X}$ is strictly simple as a complex space, then X is strictly simple as a real space.*

P r o o f. The space $\widetilde{X}$ consists of vectors $x \oplus y$ where x and y are vectors of X. Let $\overline{x_1^{**}}, \ldots, \overline{x_n^{**}}$ be n $\mathbb{R}$-linearly independent vectors of X^{**}/X, and $\overline{y_1^{**}}, \ldots, \overline{y_n^{**}}$ be n other vectors of X^{**}/X. Then the vectors $\overline{x_1^{**} \oplus 0}, \ldots, \overline{x_n^{**} \oplus 0}$ are n linearly $\mathbb{C}$-independent vectors of $\widetilde{X}$, and since $\widetilde{X}$ is assumed to be strictly simple, there exists a $\mathbb{C}$-linear operator T on $\widetilde{X}$ such that for $i = 1 \ldots n$,

$$\|T^{**}(x_i^{**} \oplus 0) - y_i^{**} \oplus 0\| < \varepsilon.$$

Let P be the projection onto the first coordinate. As in the proof of Lemma 3.1, we have $\|P^{**}T^{**}x_i^{**} - y_i^{**}\| < \varepsilon$ and $T_1 = PTP$ restricted to X is a $\mathbb{R}$-linear operator on X. $\qquad \square$

We do not know of an example of a space which would be simple without being strictly simple. Potential counterexamples could be either the space JT or separable spaces such that $\mathrm{DBSC}(X)$ is norm-dense in X^{**} without being equal to X, but we do not know of any example of such a space.

Another question regarding simple spaces is the following: is simplicity a hereditary property, i.e. if X is simple and E is a closed subspace of X, does it follow that E is also simple? We have proved in Lemma 3.1 that simplicity passes to complemented subspaces. In all the examples of Corollary 4.1, simplicity passes from the space to its closed subspaces, but we do not know if this is true in general.

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