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ON SANDWICH SETS AND CONGRUENCES ON REGULAR SEMIGROUPS

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Abstract. Let S be a regular semigroup and $E(S)$ be the set of its idempotents. We call the sets $S(e,f)f$ and $eS(e,f)$ one-sided sandwich sets and characterize them abstractly where $e,f \in E(S)$. For $a, a' \in S$ such that $a = aa'a$, $a' = a'aa'$, we call $S(a) = S(a'a, aa')$ the sandwich set of a. We characterize regular semigroups S in which all $S(e,f)$ (or all $S(a)$) are right zero semigroups (respectively are trivial) in several ways including weak versions of compatibility of the natural order.

For every $a \in S$, we also define $E(a)$ as the set of all idempotets e such that, for any congruence ϱ on S, $a\varrho a^2$ implies that $a\varrho e$. We study the restrictions on S in order that $S(a)$ or $E(a) \cap D_{a^2}$ be trivial. For $\mathcal{F} \in \{\mathcal{S}, \mathcal{E}\}$, we define $\mathcal{F}$ on S by $a\ \mathcal{F}\ b$ if $F(a) \cap F(b) \neq \emptyset$. We establish for which S are $\mathcal{S}$ or $\mathcal{E}$ congruences.

Keywords: regular semigroup, sandwich set, congruence, natural order, compatibility, completely regular element or semigroup, cryptogroup

MSC 2000: 20M10

1. Introduction and summary

Let S be a regular semigroup with the set $E(S)$ of idempotents. For any $e, f \in E(S)$, the sandwich set of e and f was defined by Nambooripad [6] as

$$S(e,f) = \{p \in E(S);\ pe = p = fp, epf = ef\}.$$

As usual, for any $a \in S$, the set of inverses of a is defined by

$$V(a) = \{x \in S;\ a = axa, x = xax\}.$$

Then the sandwich set admits the important characterization

$$S(e,f) = fV(ef)e. \tag{1}$$

A simple argument shows that $S(e,f)$ is a rectangular subband of S. It plays a remarkably pervasive role in many deliberations which concern the structure of regular semigroups. Denote by $\mathcal{C}(S)$ the congruence lattice of S.

From a different point of view, congruences play a central role in the structure theory of regular semigroups. One of the modest but all important results is a lemma due to Lallement [5] which asserts that every idempotent congruence class on a regular semigroup contains an idempotent. Indeed, if $\varrho \in \mathcal{C}(S)$ and $a\varrho$ is its idempotent class, then for any $x \in V(a^2)$, axa is such an idempotent, and the lemma requires a one line proof.

Sandwich sets may be defined for arbitrary elements a and b of S by considering $S(a'a, bb')$ where $a' \in V(a)$ and $b' \in V(b)$, for, fortunately, this definition does not depend on the choice of $a' \in V(a)$ and $b' \in V(b)$. Then $S(a'a, bb') = bV(ab)a$ which is a faithful analogue of the formula (1) and in the special case, $S(a'a, aa') = aV(a^2)a$. From the preceding paragraph, we obtain that $S(a'a, aa') \subseteq a\varrho$ if $a\varrho$ is an idempotent class of $\varrho \in \mathcal{C}(S)$.

For every $a \in S$, we define

$$S(a) = S(a'a, aa')$$

where $a' \in V(a)$, which we call the *sandwich set of* a, and also

$$E(a) = \{e \in E(S);\ \varrho \in \mathcal{C}(S), a\varrho a^2 \Rightarrow a\varrho e\}$$

which we call the *idempotent neighborhood of* a. The latter definition is the end result of a search for an idempotent "nearest" to the element a. These two concepts evidently call for further notions. Define relations $\mathcal{S}$ and $\mathcal{E}$ on S by

$$a\ \mathcal{S}\ b \quad \text{if } S(a) \cap S(b) \neq \emptyset, \qquad a\ \mathcal{E}\ b \quad \text{if } E(a) \cap E(b) \neq \emptyset.$$

Both of these relations are evidently reflexive and symmetric. Equivalence relations would be obtained if we considered $S(a) = S(b)$ instead of $S(a) \cap S(b) \neq \emptyset$, and similarly for $E(a)$, but we will not pursue this line of investigation.

The very definition of a sandwich set demonstrates a deep insight into the structure of regular semigroups which is amply corroborated by its numerous applications. On the other hand, the simple Lallement lemma turns out to be a centerpiece in many investigations of regular semigroups. The purpose of this paper is to further explore the concepts and situations discussed above and to introduce some new notions.

Section 2 consists of a brief compendium of the needed terminology and notation. We characterize in Section 3 the sets $S(e,f)f$ and $eS(e,f)$ and provide new statements equivalent to the requirement that all sandwich sets be either right zero semigroups or trivial. In Section 4, we characterize the condition that all $S(a)$ be right zero semigroups or trivial in several ways. Section 5 consists of some elementary properties of the sets $E(a)$. In Section 6 we explore regular semigroups S for which $E(a) \cap D_{a^2}$ is trivial for all $a \in S$. Finally, in Section 7, we consider when the relations $\mathcal{S}$ and $\mathcal{E}$ are congruences.

2. Notation and terminology

We state here only the absolute minimum of concepts and symbolism. For the rest, we refer to the books [4], [8], [9]. In particular, D_a denotes the $\mathcal{D}$-class of a.

Throughout the paper S denotes an arbitrary regular semigroup unless specified otherwise. We retain the concepts and symbolism introduced in Section 1.

The *natural (partial) order* is defined on S by

$$a \leqslant b \quad \text{if } a = eb = bf \text{ for some } e, f \in E(S).$$

We also write

$$(a) = \{b \in S;\ b \leqslant a\}$$

for the principal order ideal generated by the element a. The natural order is *left* (respectively *right*) *compatible* if $a \leqslant b$ implies $ca \leqslant cb$ (respectively $ac \leqslant bc$); *compatible* if both. If S has a zero 0, we set $S^* = S \setminus \{0\}$.

An element a of S is *completely regular* if the $\mathcal{H}$-class H_a is a group; in such a case, a^0 denotes the identity of H_a. The semigroup S is *completely regular* if all its elements are; if in addition $\mathcal{H}$ is a congruence on S, then S is a *cryptogroup*; if also $S/\mathcal{H}$ is a normal band, then S is a *normal cryptogroup.*

If ϱ is a relation on S, ϱ^* denotes the congruence on S generated by ϱ. For $a, b \in S$, $\kappa_{a,b}$ denotes the congruence on S generated by the pair (a, b); we also write $\kappa_a = \kappa_{a,a^2}$.

For any set X, we denote by $|X|$ its cardinality and by ε the equality relation on X.

3. One-sided sandwich sets

By this we mean the sets $S(e,f)f$ and $eS(e,f)$. We first characterize these sets in two ways.

Lemma 3.1. *For any $e, f \in E(S)$ and $x \in V(ef)$, we have*

$$S(e,f)f = S(xef, f) = \{q \in S;\ eq = ef \mathrel{\mathcal{L}} q \leqslant f\} \subseteq L_{ef} \cap (f).$$

P r o o f. Let $p \in S(e,f)$, $q = pf$. Then

$$\begin{aligned} q^2 &= p(fp)f = p^2 f = q, \\ q(xef) &= pfxef = (pe)fxef = pef = pf = q, \\ fq &= f(pf) = (fp)f = pf = q, \\ (xef)qf &= xefpf = xepf = xef \end{aligned}$$

which proves that $q \in S(xef, f)$. Therefore $S(e,f)f \subseteq S(xef, f)$.

Next let $q \in S(xef, f)$. Then $q = qxef$ so that $qf = q$ which together with $q = fq$ implies that $q \leqslant f$. It follows that

$$eq = e(fqf) = ef(xefqf) = efxef = ef = (ef)q$$

and $q = (qx)ef$. Therefore $eq = ef \mathrel{\mathcal{L}} q \leqslant f$.

Now let $q \in S$ be such that $eq = ef \mathrel{\mathcal{L}} q \leqslant f$ and let $p \in S(e,f)$. We show first that $qp \in S(e,f)$. Indeed,

$$\begin{aligned} (qp)^2 &= q(pe)qp = qp(eq)p = q(pe)(fp) = qp^2 = qp, \\ (qp)e &= q(pe) = qp, \quad f(qp) = (fq)p = qp, \\ e(qp)f &= (eq)pf = e(fp)f = epf = ef \end{aligned}$$

which proves that $qp \in S(e,f)$. Further, $q = uef$ for some $u \in S$ and thus

$$q = u(ef) = uepf = ue(fp)f = (uef)pf = qpf \in S(e,f)f,$$

as required.

The inclusion in the statement of the lemma is obvious. $\square$

Dually, we get the following statement.

Lemma 3.2. *For any $e, f \in E(S)$ and $x \in V(ef)$, we have*

$$eS(e,f) = S(e, efx) = \{r \in S;\ rf = ef \mathrel{\mathcal{R}} r \leqslant e\} \subseteq R_{ef} \cap (e).$$

Parts of the following result are known. We prove it for the sake of completeness.

Theorem 3.3. *For $e, f \in E(S)$, the mappings*

$$\varphi\colon\ p \to (pf, ep), \quad \psi\colon\ (q, r) \to qxr \quad (\text{where } x \in V(ef))$$

are mutually inverse isomorphisms between $S(e, f)$ and $S(e, f)f \times eS(e, f)$. Moreover, $S(e, f)f$ is a left and $eS(e, f)$ is a right zero semigroup.

P r o o f. If $q \in S(e, f)f$, $x, y \in V(ef)$ and $r \in eS(e, f)$, then $q = uef$, $r = efv$ for some $u, v \in S$ and thus

$$qxr = (uef)x(efv) = uefv = u(efyef)v = qyr$$

and ψ is single valued. With the same notation and $p = qxr$, we get

$$xrqx = xr(fq)x = x(rf)qx = xe(fq)x = x(eq)x = xefx = x$$

which implies that $p = qxr \in E(S)$,

$$pe = qx(re) = qxr = (fq)xr = fp = p,$$
$$epf = (eq)x(rf) = efxef = ef$$

and thus $p \in S(e, f)$. Therefore ψ maps $S(e, f)f \times eS(e, f)$ into $S(e, f)$.

For $p \in S(e, f)$, we obtain with $x \in V(ef)$,

$$p\varphi\psi = (pf, ep)\psi = pfxep = (pe)fxe(fp) = (pe)(fp) = p,$$

and for $(q, r) \in S(e, f)f \times eS(e, f)$,

$$(q, r)\psi\varphi = (qxr)\varphi = (qxrf, eqxr),$$

where, since $q = uef$ for some $u \in S$,

$$qx(rf) = uefxef = uef = q$$

and dually $eqxr = r$ so that $(q, r)\psi\varphi = (q, r)$.

Now let $p, p' \in S(e, f)$. Then

$$(pp')\varphi = (pp'f, epp') = (p(ep'f), (epf)p') = (pef, efp') = (pf, ep').$$

By Lemmas 3.1 and 3.2, $S(e, f)f$ is a left and $eS(e, f)$ is a right zero semigroup so that

$$(p\varphi)(p'\varphi) = (pe, fp)(p'e, fp') = ((pe)(p'e), (fp)(fp')) = (pe, fp')$$

and φ is a homomorphism. Hence φ and ψ have all the requisite properties. □

Corollary 3.4. *For any $e, f \in E(S)$, the following conditions are equivalent.*

(i) $S(e,f)$ *is a right zero semigroup.*

(ii) $S(e,f)f$ *is trivial.*

(iii) $S(xef, f)$ *is trivial for any* $x \in V(ef)$.

P r o o f. The equivalence of (i) and (ii) follows from Theorem 3.3 and of (ii) and (iii) from Lemma 3.1. □

The next lemmas and their corollary are largely known, ([7], Lemma 1.7). Recall that for $e, f \in E(S)$,

$$e \leqslant_l f \quad \text{if } e = ef, \quad e \leqslant_r f \quad \text{if } e = fe.$$

Lemma 3.5. *For $e, f \in E(S)$, the following conditions are equivalent.*

(i) $e \leqslant_l f$.

(ii) $e \in S(f,e)$.

(iii) $S(e,f) = L_e \cap (f)$.

P r o o f. The equivalence of (i) and (ii) follows at once.

(i) *implies* (iii). Let $p \in S(e,f)$. Then $p = pe = pef = pf$ which together with $p = fp$ yields $p \leqslant f$. Hence $e = ef = epf = ep$ which together with $pe = p$ yields $p \mathrel{\mathcal{L}} e$. Therefore $p \in L_e \cap (f)$. Conversely, if $p \in L_e \cap (f)$, then $e = ef$ immediately implies that $p \in S(e,f)$.

(iii) *implies* (i). If $p \in S(e,f)$, then $p \in L_e \cap (f)$ so that $e = ep = epf$ whence $e = ef$ and $e \leqslant_l f$. □

Dually, we get the following statement.

Lemma 3.6. *For $e, f \in E(S)$, the following conditions are equivalent.*

(i) $e \leqslant_r f$.

(ii) $e \in S(e,f)$.

(iii) $S(f,e) = R_e \cap (f)$.

Corollary 3.7. *For $e, f \in E(S)$, the following conditions are equivalent.*

(i) $e \leqslant f$.

(ii) $e \in S(f,e) \cap S(e,f)$.

(iii) $S(e,f) = L_e \cap (f)$, $S(f,e) = R_e \cap (f)$.

(iv) $e \in S(ef, fe)$.

P r o o f. The equivalence of the first three conditions follows directly from Lemmas 3.5 and 3.6. If $e \leqslant f$, then $e = ef = fe$ and hence $e \in S(ef, fe)$. If $e \in S(ef, fe)$, then $exef = e = feye$ for some $x \in V(ef)$, $y \in V(fe)$ and thus $e = ef = fe$ so that $e \leqslant f$. □

Blyth and Gomes [3] characterized the regular semigroups in which all sandwich sets are right zero semigroups in various ways (including (iii) and (iv) below). We add a number of further characterizations in the next theorem.

Recall that S satisfies *θ-majorization* if for any $e, f, g \in E(S)$, $e \geqslant f, g$ and $f\theta g$ imply that $f = g$ for $\theta \in \{\mathcal{L}, \mathcal{R}\}$; S is *locally $\mathcal{P}$* if eSe has property $\mathcal{P}$ for all $e \in E(S)$; S is *right regular orthodox* if $E(S)$ is a right regular band, that is, satisfies the identity $axa = xa$; S is *$\mathcal{L}$-unipotent* if every $\mathcal{L}$-class of S contains only one idempotent.

Theorem 3.8. *The following conditions on S are equivalent.*

(i) *S satisfies $\mathcal{L}$-majorization.*

(ii) *S is locally right regular orthodox.*

(iii) *The natural order on S is right compatible.*

(iv) *For any $e, f \in E(S)$, $S(e, f)$ is a right zero semigroup.*

(v) *For any $e, f \in E(S)$ such that $e \leqslant f$, we have $S(e, f) = \{e\}$.*

Proof. Simple reflection shows that $\mathcal{L}$-majorization is equivalent to local $\mathcal{L}$-unipotency. By ([11], Theorem 1), a regular semigroup is $\mathcal{L}$-unipotent if and only if it is right regular orthodox. By ([3], Theorem 2), local $\mathcal{L}$-unipotency is equivalent to both parts (iii) and (iv). It follows that parts (i)–(iv) are equivalent.

(iv) *implies* (v). Let $e, f \in E(S)$ be such that $e \leqslant f$. By Corollary 3.7, $e \in S(e, f)$ and $S(e, f)$ is a left zero semigroup which together with the hypothesis implies that $S(e, f) = \{e\}$.

(v) *implies* (i). Let $e, f, g \in E(S)$ be such that $e \geqslant f, g$ and $f \mathrel{\mathcal{L}} g$. By Corollary 3.7 and the hypothesis, we obtain

$$\{f\} = S(f, e) = L_f \cap (e) = L_g \cap (e) = S(g, e) = \{g\}.$$

Therefore $f = g$ and S satisfies $\mathcal{L}$-majorization. □

Two further equivalent statements follow from Corollary 3.4. All this discussion is inspired by the results due to Nambooripad ([6], Theorem 7.6; [7], Theorems 3.1 and 3.3), which include the equivalence of parts (ii)–(iv) of the next corollary.

From Theorem 3.8 and its dual, we deduce the following result.

Corollary 3.9. *The following conditions on S are equivalent.*

(i) *S satisfies $\mathcal{L}$-and $\mathcal{R}$-majorization.*

(ii) *S is locally inverse.*

(iii) *The natural order on S is compatible.*

(iv) *For any $e, f \in E(S)$, $S(e, f)$ is trivial.*

(v) *For any $e, f \in E(S)$ such that $e \leqslant f$, we have $S(e, f) = S(f, e)$.*

P r o o f. Let $e, f \in E(S)$ be such that $e \leqslant f$ and $S(e,f) = S(f,e)$, and let $x \in S(e,f)$. By (1), we have $x = fye$ for some $y \in V(e)$ and by hypothesis, also $fye = ezf$ for some $z \in V(e)$. Hence

$$x = fye = ezf = ezfe = eze = e$$

so that $S(e,f) = \{e\}$. The corollary now follows from Theorem 3.8 and its dual. □

Further equivalent statements follow from Corollary 3.4 and its dual. From these statements and Corollary 3.9(v), we see that if $S(e,f)$ is trivial for some special pairs of idempotents (e,f), then $S(e,f)$ is trivial for all $e, f \in E(S)$.

4. The sandwich set of an element

By this we mean, for any $a \in S$, the set defined in Section 1 namely

$$S(a) = S(a'a, aa')$$

where the set $S(a'a, aa')$ does not depend on the choice of $a' \in V(a)$. After some preliminaries, we perform an analysis for sandwich sets of elements analogous to that in the preceding section for sandwich sets.

Lemma 4.1. *If $e, f \in E(S)$ are $\mathcal{D}$-related, then there exist $a \in S$ and $a' \in V(a)$ such that $e = a'a$ and $f = aa'$.*

P r o o f. See ([4], Proposition II.3.6). □

Corollary 4.2. *For any $a \in S$, we have $S(a) = aV(a^2)a$. For any $e, f \in E(S)$, we have: $e \leqslant f \Leftrightarrow e \in S(fef)$.*

P r o o f. The first assertion is a special case of ([10], Lemma 2.1(ii)) while the second is a consequence of the first. □

We are now ready for an analogue of Theorem 3.8.

Theorem 4.3. *The following conditions on S are equivalent.*

(i) *For any $a \in S$, $S(a)$ is a right zero semigroup.*
(ii) *For any $e, f \in E(S)$ such that $e \,\mathcal{D}\, f$, $S(e,f)$ is a right zero semigroup.*
(iii) *For any $a \in S$ and $x, y \in V(a^2)$, we have $(axa)(aya) = aya$.*
(iv) *For any $a, x, y \in S$, $a^2 = a^2xa^2 = a^2ya^2 \Rightarrow (axa)(aya) = (aya)^2$.*
(v) *For any $e, f \in E(S)$ such that $e \,\mathcal{D}\, f$ and $p, q \in S$,*

$$ep = eq = ef \,\mathcal{L}\, p \,\mathcal{L}\, q, \quad p, q \leqslant f \Rightarrow p = q.$$

(vi) *For any* $a, b, c \in S$, $x' \in V(x)$ *for* $x \in \{a, b, c, ac\}$, $e \in E(D_b)$,

$$a \leqslant b, \quad a'a = b'a, \quad ea'ac(ac)'a = eb'b \,\mathcal{L}\, a'ac(ac)'a \Rightarrow ac \leqslant bc.$$

(vii) *For any* $a, b, c, d \in S$ *and* $x' \in V(x)$ *for* $x \in \{a, b, c, d, ac\}$,

$$a \leqslant b,\ a'a = b'a,\ dd' = b'b,\ d^2d' = da'ac(ac)'a,\ a'ac \in Sda'ac \Rightarrow ac \leqslant bc.$$

Proof. (i) *implies* (ii). Let $e, f \in E(S)$ be such that $e \,\mathcal{D}\, f$. By Lemma 4.1, there exist $a \in S$ and $a' \in V(a)$ such that $e = a'a$ and $f = aa'$. The hypothesis implies that $S(e, f) = S(a'a, aa') = S(a)$ is a right zero semigroup.

(ii) *implies* (iii). Let $a \in S$ and $a' \in V(a)$. Then $a'a \,\mathcal{D}\, aa'$ and thus $S(a) = S(a'a, aa')$ is a right zero semigroup. Now let $x, y \in V(a^2)$. By Corollary 4.1, we get that axa, $aya \in S(a)$ which then implies that $(axa)(aya) = aya$.

(iii) *implies* (iv). Let $a, x, y \in S$ be such that $a^2 = a^2xa^2 = a^2ya^2$. It follows easily that xa^2x, $ya^2y \in V(a^2)$ which by hypothesis implies that

$$[a(xa^2x)a]\ [a(ya^2y)a] = a(ya^2y)a$$

whence $(axa)(aya) = (aya)^2$.

(iv) *implies* (v). Assume the antecedent of part (v). By Lemma 4.1, there exist $a \in S$ and $a' \in V(a)$ such that $e = a'a$ and $f = aa'$. By Corollary 4.2, we get

$$S(e, f) = S(a'a, aa') = S(a) = aV(a^2)a$$

and thus, by Lemma 3.1, we obtain $p, q \in S(a)f = aV(a^2)af$. Hence there exist $x, y \in V(a^2)$ such that $p = axaf$ and $q = ayaf$. Now $a^2 = a^2xa^2 = a^2ya^2$ which by hypothesis implies that $(axa)(aya) = (aya)^2 = aya$. It follows that

$$\begin{aligned} p = axaf &= ax(ae)f = axa(ef) = axa(a'a^2a') \\ &= axa'(a^2ya^2)a' = (axa)(aya)f = ayaf = q. \end{aligned}$$

(v) *implies* (vi). Assume the antecedent of part (vi). By hypothesis, we have $a = ub = bv$ for some $u, v \in E(S)$ which implies that $a = ab'b = bb'a$.

First we shall use part (v) with the following notation. Let $h = c(ac)'a$ so that $h \in S(a'a, cc')$ by ([10], Lemma 2.1). Next let $p = a'ah$, $q = b'bh$ and $f = b'b$. We now verify the conditions on these parameters. Indeed,

$$\begin{aligned}
p^2 &= (a'ah)(a'ah) = a'a(ha'a)h = a'ah = p, \\
q^2 &= (b'bh)(b'bh) = b'b(ha'a)b'bh = b'b(ha'a)h = b'bh = q, \\
pq &= (a'ah)(b'bh) = a'a(ha'a)b'bh = a'ah = p, \\
qp &= (b'bh)(a'ah) = b'b(ha'a)h = b'bp = q, \\
fp &= (b'b)(a'ah) = b'(ba'a)h = b'ah = a'ah = p, \\
pf &= (a'ah)(b'b) = a'a(ha'a)b'b = a'a(ha'a) = a'ah = p, \\
fq &= (b'b)(b'bh) = b'bh = q, \\
qf &= (b'bh)(b'b) = b'b(ha'a)b'b = b'b(ha'a) = b'bh = q
\end{aligned}$$

which proves that $p \mathcal{L} q$ and $p, q \leqslant f$. By the antecedent of part (vi), we also have $ep = ef \mathcal{L} p$ which implies that $ep = epq = efq = eq$. Now part (v) implies that $p = q$ and thus $a'ah = b'bh$. Hence

$$\begin{aligned}
bc(c'hc) &= b(cc'h)c = bhc = b(b'bh)c = b(a'ah)c = (ba'a)hc = ahc = ac, \\
(c'hc)^2 &= c'h(cc'h)c = c'hc.
\end{aligned}$$

Finally, $a = ub$ implies that $ac = ubc$ and therefore $ac \leqslant bc$.

(vi) *implies* (vii). Assume the antecedent of part (vii) and let $e = d'd$. Then $e \mathcal{D} dd' = b'b$ so that $e \in E(D_b)$. Further,

$$ea'ac(ac)'a = d'[da'ac(ac)'a] = d'd^2d' = (d'd)(dd') = eb'b.$$

Also $a'ac \in Sda'ac$ yields

$$a'ac \mathcal{L} da'ac \mathcal{L} d'da'ac = ea'ac$$

which implies that $a'ac(ac)'a \mathcal{L} ea'ac(ac)a'$. The hypothesis implies that $ac \leqslant bc$.

(vii) *implies* (i). Let $a \in S$, $a' \in V(a)$, $e = a'a$ and $f = aa'$. Then $S(a) = S(e, f)$ and hence, in view of Theorem 3.3, the desired conclusion is equivalent to $S(e, f)f$ being trivial. Thus let $p, q \in S(e, f)f$ so that, by Lemma 3.1, we have

$$ep = eq = ef \mathcal{L} p \mathcal{L} q, \quad p, q \leqslant f. \tag{2}$$

With the substitution

$$a, a' \to p, \quad b, b' \to f, \quad c, c' \to q, \quad d \to a, \quad d' \to a', \quad (ac)' \to pq$$

the antecedent of part (vii) becomes

$$(3) \qquad p \leqslant f, \quad pp = fp, \quad aa' = ff, \quad a^2a' = appq(pq)p, \quad ppq \in Sappq.$$

The first two formulae in (3) follow directly from (2); the third by the definition of f, and $p^2 = p$ by (2). By (2), we have $ep = ef$ whence $a'ap = a'af$ so that $ap = af = a^2a'$ which implies the fourth formula in (3) since $p, q \in S(e, f)f$ which by Theorem 3.3 is a left zero semigroup. Finally, by (2), we have $p \in Sep = Sa'ap = Sap$ which, similarly as before, yields the fifth formula in (3).

We have verified that (3) holds which by hypothesis implies that $pq \leqslant fq$. But $pq = p$ and $fq = q$ so that $p \leqslant q$. Now p and q are elements of the left zero semigroup $S(e, f)f$ which finally yields $p = q$. Therefore $S(e, f)f$ is trivial and Theorem 3.3 yields that $S(a) = S(e, f)$ is a right zero semigroup. □

We now discuss briefly the conditions figuring in Theorem 4.3. It is part (i) that we wished to characterize in as many ways as possible in order to better comprehend what it actually requires. Part (ii) specifies which pairs of idempotents (e, f) are affected by this condition in terms of sandwich sets. Further equivalent conditions can be obtained by using Corollary 3.4. Both parts (iii) and (iv) are the usual implications in terms of elements. Part (v) represents a weakening of $\mathcal{L}$-majorization; in fact, if we omit the requirement $e \mathrel{\mathcal{D}} f$, this condition becomes $\mathcal{L}$-majorization. Both parts (vi) and (vii) amount to a weakening of the right compatibility of the natural order; they are quite complex.

We now deduce the two-sided version.

Theorem 4.4. *The following conditions on S are equivalent.*

(i) *For any $a \in S$, $S(a)$ is trivial.*

(ii) *For any $a \in S$ and $x, y \in V(a^2)$, we have $axa = aya$.*

(iii) *For any $a, x, y \in S$, $a^2 = a^2xa^2 = a^2ya^2 \Rightarrow (axa)^2 = (aya)^2$.*

(iv) *For any $a, x, y \in S$, $a^2 = a^2xa^2 = a^2ya^2 \Rightarrow (axa)(aya) = (aya)(axa)$.*

(v) *For any $e, f \in E(S)$ such that $e \mathrel{\mathcal{D}} f$, $S(e, f)$ is trivial.*

(vi) *For any $e, f \in E(S)$ such that $e \mathrel{\mathcal{D}} f$ and $p, q \in S$,*

$$ep = eq = ef \mathrel{\mathcal{L}} p \mathrel{\mathcal{L}} q, \quad p, q \leqslant f \Rightarrow p = q,$$
$$pf = qf = ef \mathrel{\mathcal{R}} p \mathrel{\mathcal{R}} q, \quad p, q \leqslant e \Rightarrow p = q.$$

P r o o f. (i) *implies* (ii). For any $a \in S$, by Corollary 4.2, we have $S(a) = aV(a^2)a$. If $x, y \in V(a^2)$, the hypothesis yields $axa = aya$.

(ii) *implies* (iii). Let $a, x, y \in S$ be such that $a^2 = a^2xa^2 = a^2ya^2$. Then $xa^2x, ya^2y \in V(a^2)$ and the hypothesis implies that $a(xa^2x)a = a(ya^2y)a$, that is $(axa)^2 = (aya)^2$.

(iii) *implies* (iv). For $a^2 = a^2xa^2 = a^2ya^2$, we get

$$\begin{aligned}(axa)(aya) &= axa^2ya = axa^2xa^2ya = (axa)^2(aya) = (aya)^3 \\ &= (aya)(axa)^2 = ay(a^2xa^2)xa = aya^2xa = (aya)(axa).\end{aligned}$$

(iv) *implies* (v). Let $e, f \in E(S)$ be such that $e \mathcal{D} f$. By Lemma 4.1, there exist $a \in S$ and $a' \in V(a)$ such that $e = a'a$ and $f = aa'$. Let $p, q \in S(e, f)$. Then

$$pa'a = p = aa'p, \quad qa'a = q = aa'q, \quad apa = aqa = a^2.$$

It follows that

$$\begin{aligned}a^2 &= a(aa'pa'a)a = a^2(a'pa')a^2, \\ a^2 &= a(aa'qa'a)a = a^2(a'qa')a^2,\end{aligned}$$

and the hypothesis implies that $(aa'pa'a)(aa'qa'a) = (aa'qa'a)(aa'pa'a)$ and thus $pq = qp$. Since $S(e, f)$ is a rectangular band, we get $p = q$.

(v) *implies* (vi). Assume the antecedent of the first implication in part (vi). By Lemma 3.1, we get $p, q \in S(e, f)f$. The hypothesis implies that $S(e, f)$ is trivial which yields that $p = q$. The second implication in part (vi) is proved similarly using Lemma 3.2.

(vi) *implies* (i). Let $e, f \in E(S)$ be such that $e \mathcal{D} f$. The first implication in part (vi) by Lemma 3.1 yields that $S(e, f)f$ is trivial. Dually, the second implication by Lemma 3.2 yields that $eS(e, f)$ is trivial. By Theorem 3.3, we obtain that $S(e, f)$ must be trivial. For any $a \in S$ and $a' \in V(a)$, we have $a'a \mathcal{D} aa'$ and $S(a) = S(a'a, aa')$, and thus $S(a)$ is trivial. $\square$

In a regular semigroup S, for any $a \in S$ and $a' \in V(a)$, by Theorem 3.3, we have

$$S(a) = S(a'a, aa') \cong S(a)aa' \times a'aS(a),$$

where $S(a)aa'$ is a left and $a'aS(a)$ is a right zero semigroup. It follows that both $|a'aS(a)|$ and $|S(a)aa'|$ are independent of the choice of $a' \in V(a)$.

In a completely regular semigroup S, by Corollary 3.9, for any $a \in S$, we see that $S(a)$ is trivial since $S(a) \subseteq D_a$ and D_a is completely simple and thus locally inverse. But if S is not a normal cryptogroup, then S is not locally inverse, so by the same reference $S(e, f)$ is nontrivial for some $e, f \in E(S)$.

In the following example, we shall encounter infinite $S(a)$.

Example 4.5. Let $\mathbb{N} = \{1, 2, \ldots\}$ and consider the semigroup $\mathcal{T}(\mathbb{N})$ of all transformations on $\mathbb{N}$ written as right operators. Let

$$\triangle = \{\varphi \in \mathcal{T}(\mathbb{N});\ n\varphi = (n+1)\varphi \in \{n, n+1\} \quad \text{for all odd } n \in \mathbb{N}\}.$$

For any $\varphi \in \triangle$,

$$\begin{aligned} n\varphi &= (n+1)\varphi \in \{n, n+1\} \quad \text{if } n \text{ is odd,} \\ n\varphi &= (n-1)\varphi \in \{n-1, n\} \quad \text{if } n \text{ is even} \end{aligned}$$

and hence

$$\begin{aligned} n\varphi^2 &= n\varphi = (n+1)\varphi \quad \text{if } n \text{ is odd,} \\ n\varphi^2 &= (n-1)\ \varphi = n\varphi \quad \text{if } n \text{ is even} \end{aligned}$$

so that φ is idempotent, and also

$$m\varphi = n\varphi \Leftrightarrow \begin{cases} m = n & \text{if } m \text{ and } n \text{ are of the same parity,} \\ m + 1 = n & \text{if } m \text{ is odd and } n \text{ is even,} \\ m - 1 = n & \text{if } m \text{ is even and } n \text{ is odd} \end{cases}$$

and thus any two elements of $\triangle$ induce the same partition of $\mathbb{N}$ and so are $\mathcal{R}$-related.

For $\varphi \in \triangle$ and ι the identity transformation on $\mathbb{N}$, by Corollary 3.7, we get $S(\iota, \varphi) = R_\varphi \cap E(\mathcal{T}(\mathbb{N}))$ and clearly

$$\varphi\ \mathcal{R} \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 \ldots \\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4 \ldots \end{pmatrix} \mathcal{L}\ \iota$$

so that $\varphi\ \mathcal{D}\ \iota$. Hence there exists $\alpha \in \mathcal{T}(\mathbb{N})$ for which $S(\alpha) = S(\iota, \varphi) \supseteq \triangle$ so $S(\alpha)$ is infinite. Recall that $\mathcal{T}(\mathbb{N})$ is a regular semigroup.

5. Idempotent neighborhoods

These were defined in Section 1 as sets

$$E(a) = \{e \in E(S);\ \varrho \in \mathcal{C}(S), a\varrho a^2 \Rightarrow a\varrho e\}$$

for any $a \in S$. We consider first some simple properties of these sets.

Let $\varrho \in \mathcal{C}(S)$. Then

$$\ker \varrho = \{a \in S;\ a\varrho e \text{ for some } e \in E(S)\}$$

is the *kernel* of ϱ. If every ϱ-class contains at most one idempotent, ϱ is *idempotent separating*; the greatest such congruence is denoted by μ. The ϱ-class of $a \in S$ is denoted by $a\varrho$.

Lemma 5.1. *Let $a \in S$.*

(i) *Let a be completely regular. Then $a^0 \in E(a)$.*

(ii) *Let $a^n \in E(S)$ for some $n \geqslant 1$. Then $a^n \in E(a)$.*

(iii) $a \in \ker \mu \Leftrightarrow \kappa_a \subseteq \mathcal{H} \Rightarrow E(a) = \{a^0\}$.

(vi) $a \in E(S) \Leftrightarrow \kappa_a = \varepsilon$.

(v) $E(a) = a\kappa_a \cap E(S) = \{e \in E(S);\ \kappa_a = \kappa_{a,e}\}$.

Proof. (i) Let $\varrho \in \mathcal{C}(S)$ be such that $a\varrho a^2$. Then $\varrho|_{H_a}$ is a congruence having a in its kernel und thus $a\varrho a^0$. Therefore $a^0 \in E(a)$.

(ii) Let $e = a^n \in E(S)$ and $\varrho \in \mathcal{C}(S)$ be such that $a\varrho a^2$. Then $a^2\varrho a^3$, $a^3\varrho a^4, \ldots$ and thus $a\varrho e$. Hence $e \in E(a)$.

(iii) If $a \in \ker \mu$, then $a\mu a^2$ which by the definition of κ_a implies that $\kappa_a \subseteq \mu \subseteq \mathcal{H}$. If $\kappa_a \subseteq \mathcal{H}$, then $\kappa_a \subseteq \mu$ which implies that $a \in \ker \kappa_a \subseteq \ker \mu$.

Now let $a \in \ker \mu$. Since $\mu \subseteq \mathcal{H}$ and $a\mu a^2$, we get that H_a is a group and thus a is completely regular. By part (i), we see that $a^0 \in E(a)$. Conversely, if $e \in E(a)$, then $a\mu a^2$ implies that $a\mu e$ and thus $a^0\mu e$ whence $a^0 = e$. Therefore $E(a) = \{a^0\}$.

(iv) Obvious.

(v) Let $e \in E(a)$. Since $a\kappa_a a^2$, by the definition of $E(a)$, we get $a\kappa_a e$ and thus $e \in a\kappa_a \cap E(S)$.

Next let $e \in a\kappa_a \cap E(S)$. Then $a\kappa_a e$ which by the minimality of $\kappa_{a,e}$ implies that $\kappa_{a,e} \subseteq \kappa_a$. Also $a\kappa_{a,e}e$ implies that $a\kappa_{a,e}a^2$ which by the minimality of κ_a yields that $\kappa_a \subseteq \kappa_{a,e}$. Therefore $\kappa_a = \kappa_{a,e}$.

Finally let $e \in E(S)$ be such that $\kappa_a = \kappa_{a,e}$ and let $\varrho \in \mathcal{C}(S)$ be such that $a\varrho a^2$. Then $\kappa_a \subseteq \varrho$ by the minimality of κ_a and hence $\kappa_{a,e} \subseteq \varrho$. But then $a\kappa_{a,e}e$ implies that $a\varrho e$. Therefore $e \in E(a)$. □

Recall the definition of the relation $\mathcal{E}$ in Section 1.

Corollary 5.2. *On S, we have $\mathcal{E}|_{\ker \mu} = \mathcal{H}|_{\ker \mu}$ and $\mathcal{E}|_{E(S)} = \varepsilon$.*

Proof. Indeed, for any $a, b \in \ker \mu$, by Lemma 5.1(iii), we get

$$a\,\mathcal{E}\,b \Leftrightarrow E(a) \cap E(b) \neq \emptyset \Leftrightarrow a^0 = b^0 \Leftrightarrow a\,\mathcal{H}\,b.$$

This proves the first formula and implies the second. □

We now consider the interplay of the sets $S(a)$ and $E(a)$, for their definitions see Section 1.

Lemma 5.3. *For any $a \in S$, we have $S(a) \subseteq E(a) \cap D_{a^2}$.*

Proof. Let $e \in S(a)$ and $a' \in V(a)$.

First let $\varrho \in \mathcal{C}(S)$ be such that $a \varrho a^2$. Then

$$e = ea'a\varrho ea'a^2 = (ea'a)a = ea$$

and dually $e\varrho ae$ so that

$$e\varrho (ae)(ea) = aea = a^2 \varrho a$$

which shows that $e \in E(a)$. Therefore $S(a) \subseteq E(a)$.

Further, $a^2 = aea \in aeS$, $ae = a(aa'e) \in a^2 S$, $ae \in Se, e = ea'a = e(a'a)e \in Sae$ and thus $a^2 \ \mathcal{R} \ ae\mathcal{L}e$ which shows that $a^2 \ \mathcal{D} \ e$. Therefore $S(a) \subseteq D_{a^2}$. $\square$

It is easy to see that for any nonidempotent element a of a bicyclic semigroup S, we have $|S(a)| = 1$ and $E(a) \cap D_{a^2} = E(S)$.

We consider next the relationship of the relations $\mathcal{S}$ and $\mathcal{E}$; for their definitions see Section 1.

Proposition 5.4. *We have $\mathcal{H} \subseteq \mathcal{S} \subseteq \mathcal{E}$ and $\mathcal{S}^* = \mathcal{E}^*$ is the least band congruence on S.*

Proof. Let $a, b \in S$ be such that $a \ \mathcal{H} \ b$. According to Lemma 4.1, there exist $a' \in V(a)$ and $b' \in V(b)$ such that $a'a = b'b$ and $aa' = bb'$. Then

$$S(a) = S(a'a, aa') = S(b'b, bb') = S(b)$$

and thus $a \ \mathcal{S} \ b$. Therefore $\mathcal{H} \subseteq \mathcal{S}$; the inclusion $\mathcal{S} \subseteq \mathcal{E}$ follows directly from Lemma 5.3.

For any $a \in S$ and $e \in S(a)$, we get $e \in S(a) \cap S(e)$ and thus $a \ \mathcal{S} \ e$. But then $a \ \mathcal{S}^* \ e$ which shows that $\mathcal{S}^*$ is a band congruence. Since $\mathcal{S}^* \subseteq \mathcal{E}^*$, also $\mathcal{E}^*$ is a band congruence.

Let ϱ be a band congruence on S and $a \ \mathcal{E} \ b$. Then there exists $e \in E(a) \cap E(b)$. Now $a\varrho a^2$ implies that $a\varrho e$ and $b\varrho e$ by definition so that $a\varrho b$. Hence $\mathcal{E} \subseteq \varrho$ which yields $\mathcal{E}^* \subseteq \varrho$. For $\varrho = \mathcal{S}^*$, we get $\mathcal{E}^* \subseteq \mathcal{S}^*$ and equality prevails. $\square$

6. When is $E(a) \cap D_{a^2}$ trivial?

We have seen in Lemma 5.3 that $S(a) \subseteq E(a) \cap D_{a^2}$. This motivates us to ask for which regular semigroups S, do we have $|E(a) \cap D_{a^2}| = 1$ for all $a \in S$. Recall that for $a \in S$, $J(a)$ is the ideal generated by a, $I(a)$ the set of all elements of $J(a)$ which do not generate $J(a)$ and $P(a) = J(a)/I(a)$ is the principal factor of a, where $J(a)/\emptyset = J(a)$. A semigroup all of whose principal factors are completely (0−)simple is completely semisimple.

In one direction we have the following result.

Proposition 6.1. *Let S be a regular semigroup in which $E(a) \cap D_{a^2}$ is trivial for all $a \in S$. Then S is completely semisimple.*

Proof. The argument is by contrapositive. Hence assume that S is not completely semisimple. By ([9], Corollary IX.4.13), S has a bicyclic subsemigroup B. Let $a \in B \setminus E(B)$. Then κ_a is a congruence on S for which $\kappa_a|_B$ is a nonidentity congruence on B. It follows that $\kappa_a|_B$ is a group congruence which implies that $E(B) \subseteq E(a)$. Since $a \mathcal{D} a^2$, we also have $E(B) \subseteq D_{a^2}$. But then $E(B) \subseteq E(a) \cap D_{a^2}$ and $E(B)$ is infinite. $\square$

The following example shows that the converse of Proposition 6.1 does not hold.

Example 6.2. Let $X = \{1, 2, 3\}$ and S be the semigroup of all partial transformations on X written as right operators and S_0 be the 0-minimal ideal of S. Next let $\alpha = \binom{1\ 2}{2\ 3}$. Then $\alpha^2 = \binom{1}{3}$ and $\alpha^3 = \emptyset$, the empty transformation. Hence $\alpha\kappa_\alpha = \alpha^2\kappa_\alpha = \alpha^3\kappa_\alpha = \emptyset\kappa_\alpha$ where $\alpha^2 \in S_0^*$ and $\emptyset$ is the zero of S so that $S_0 \subseteq \alpha\kappa_\alpha$ since S_0 is completely 0-simple. By Lemma 5.1(v), we get

$$E(\alpha) \cap D_{\alpha^2} = \alpha\kappa_\alpha \cap E(S_0^*) = E(S_0^*)$$

which is the set of all partial constants on X. Therefore $|E(\alpha) \cap D_{\alpha^2}| > 1$ and S is completely semisimple since it is finite and regular.

It is instructive to also compute $S(\alpha)$. First

$$S(\alpha) = \{\beta \in S;\ \beta\alpha = \beta = \alpha\beta,\ \alpha\beta\alpha = \alpha^2,\ \beta^2 = \beta\}$$

and

$$E(S_0^*) = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}, \begin{pmatrix}1&2\\1&1\end{pmatrix}, \begin{pmatrix}1&3\\1&1\end{pmatrix}, \begin{pmatrix}1&2\\2&2\end{pmatrix}, \begin{pmatrix}2&3\\2&2\end{pmatrix}, \begin{pmatrix}1&3\\3&3\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}2&3\\3&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&3&3\end{pmatrix} \right\}.$$

So by inspection, we get

$$S(\alpha) = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}.$$

□

In the opposite direction, we shall show that the classes of completely regular and strict regular semigroups satisfy our condition. We start with a more general situation which will provide some additional information, see [2] for related results.

Lemma 6.3. *Let S be a completely semisimple semigroup and $a \in S$. Let λ be an idempotent separating congruence on $P(a) = J(a)/I(a)$. Define a relation ϱ on S by*

$$x \varrho y \Leftrightarrow \begin{cases} x = y & \text{if } \; J(x) = J(y) \nsubseteq J(a) \\ x \lambda y & \text{if } \; J(x) = J(y) = J(a), \\ J(x), J(y) \subset J(a). \end{cases}$$

Then ϱ is a congruence on S.

P r o o f. It is easily seen that ϱ is an equivalence relation on S. In order to prove that ϱ is a congruence, it suffices to consider the following case.

Let $J(x) = J(y) = J(a) \subset J(b)$, $x\lambda y$. First note that $P(a)$ is either completely 0-simple or completely simple. We assume the former; the latter is even simpler to treat. Hence we may set $P(a) = \mathcal{M}^0(I, G, \Lambda; P)$ and $x = (i, g, \mu), y = (i, h, \mu)$ since λ is idempotent separating. Considering the semigroup $S/I(a)$ as an ideal extension of the Rees matrix semigroup $P(a)$ by the semigroup $S/J(a)$, we may apply the results of ([8], Sections III.2 and V.3) as follows. The element b acts by left multiplication on $P(a)$ as a left translation and hence either $bx = (\alpha i, (\varphi i)g, \mu)$ or $bx = 0$ in $S/I(a)$, where α is a partial transformation on I and $\varphi\colon I \to G$ is a function. Similarly $by = (\alpha i, (\varphi i)h, \mu)$ or $by = 0$. Hence $bx \neq 0$ if and only if $by \neq 0$. We now represent the congruence λ by an admissible triple (r, N, π) see ([4], Sections III.4, III.5). The hypothesis $x\lambda y$ implies that $gh^{-1} \in N$. This yields

$$(\varphi i)g((\varphi i)h)^{-1} = (\varphi i)gh^{-1}(\varphi i)^{-1} \in N$$

since N is a normal subgroup of G. But then $bx\lambda by$ if $bx \neq 0$. Therefore in any case $bx\lambda by$ in $S/I(a)$. Now returning to S, we conclude that in both cases, that is $bx, by \in J_a$ or $bx, by \in I(a)$, we get $bx\varrho by$.

Consequently ϱ is a congruence on S. □

Corollary 6.4. *Let S be a completely semisimple semigroup. Then the nonzero part of any idempotent separating congruence on a principal factor of S extends to a congruence on S. If $a \in S$ is completely regular, then $E(a) = \{a^0\}$.*

P r o o f. The first assertion follows directly from Lemma 6.3. Let λ be the congruence on $P(a)$ generated by the pair (a, a^2). Since $a \,\mathcal{H}\, a^2$ in $P(a)$ and $\mathcal{H}$ is a congruence on $P(a)$, it follows that $\lambda \subseteq \mathcal{H}$. Hence λ is idempotent separating and by Lemma 6.3, $\lambda|_{J_a}$ extends to a congruence ϱ on S. Since $a\varrho a^2$, we obtain that $\kappa_a \subseteq \varrho$. But then $a\kappa_a \subseteq H_a$ which by Lemma 5.1(v) yields that $E(a) = \{a^0\}$. □

Theorem 6.5. *Let S be a completely regular semigroup.*

(i) *Every idempotent separating congruence on a $\mathcal{D}$-class of S extends to a congruence on S.*

(ii) *For any $a \in S$, $E(a) = \{a^0\}$.*

(iii) *$\mathcal{H} = \mathcal{S} = \mathcal{E}$.*

P r o o f. Parts (i) and (ii) follow from Corollary 6.4 and part (iii) from Proposition 5.4 and part (ii). □

Regular semigroups which are subdirect products of completely 0-simple semigroups are *strict regular semigroups*. They are given by parameters $(X; S_\alpha, \varphi_{\alpha,\beta})$, see ([1], Theorem 2.3) where the notation $(X; I_\alpha, f_{\alpha,\beta})$ is used; for their properties, consult ([1], Theorem 2.1). The following result follows from [2].

Lemma 6.6. *Let S be a strict regular semigroup with the parameters $(X; S_\alpha, \varphi_{\alpha,\beta})$. Fix $\zeta \in X$ and let $\lambda \in \mathcal{C}(S_\zeta)$. Define a relation ϱ on S by: for $a \in S^*_\alpha$, $b \in S^*_\beta$,*

$$a\varrho b \Leftrightarrow \begin{cases} a\varphi_{\alpha,\zeta}\lambda b\varphi_{\beta,\zeta} & \text{if} \quad \alpha, \beta \geqslant \zeta, \\ \alpha, \beta \ngeqslant \zeta. \end{cases}$$

Then ϱ is a congruence on S.

P r o o f. It is easily seen that ϱ is an equivalence relation on S. In order to prove that ϱ is a congruence, it suffices to consider the following case.

Let $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$ be such that $\alpha, \beta, \gamma \geqslant \zeta$ and $a\varphi_{\alpha,\zeta}\lambda b\varphi_{\beta,\zeta}$. Since $c\varphi_{\alpha,\zeta} \in S^*_\zeta$,

either $(c\varphi_{\gamma,\zeta})(a\varphi_{\alpha,\zeta})$, $(c\varphi_{\gamma,\zeta})(b\varphi_{\beta,\zeta}) \in S^*_\zeta$

or $(c\varphi_{\gamma,\zeta})(a\varphi_{\alpha,\zeta})$, $(c\varphi_{\gamma,\zeta})(b\varphi_{\beta,\zeta}) \notin S^*_\zeta$

and hence it remains to consider the former case. Again using the congruence property of λ, we get $(c\varphi_{\gamma,\zeta})(a\varphi_{\alpha,\zeta})\lambda(c\varphi_{\gamma,\zeta})(b\varphi_{\beta,\zeta})$. We also have $ca \in S^*_\delta$ for some $\delta \geqslant \zeta$ and hence $(ca)\varphi_{\delta,\zeta} = (c\varphi_{\alpha,\zeta})(a\varphi_{\alpha,\zeta})$. Analogously we have $(cb)\varphi_{\varepsilon,\zeta} = (c\varphi_{\gamma,\zeta})(b\varphi_{\beta,\zeta})$

for some $\varepsilon \geqslant \zeta$ with $cb \in S^*_\varepsilon$ and thus $(ca)\varphi_{\delta,\zeta}\lambda(cb)\varphi_{\varepsilon,\zeta}$. We have proved that $a\varrho b$ implies $ca\varrho cb$ in this particular instance.

A similar argument will show that also $ac\varrho bc$ under the same hypothesis. The remaining cases follow without difficulty. □

The second principal result of this section follows.

Theorem 6.7. *In a strict regular semigroup S, the nonzero part of any congruence on a principal factor of S extends to a congruence on S and $E(a) \cap D_{a^2} = \{(a^2)^0\}$ for every $a \in S$.*

Proof. The first assertion follows directly from Lemma 6.6. We represent S by the parameters $(X; S_\alpha, \varphi_{\alpha,\beta})$ and let $a \in S^*_\alpha$, $a^2 \in S^*_\beta$. Then $a^2 = (a\varphi_{\alpha,\beta})^2 \in S^*_\beta$ and hence the element $b = a\varphi_{\alpha,\beta}$ is completely regular. Let λ be the congruence on S_β generated by the pair (b, b^2). For this λ, let ϱ be as constructed in Lemma 6.6. Then $a\varrho b\varrho b^2 = a^2$ and thus $\kappa_a \subseteq \varrho$ by the minimality of κ_a. Since $\varrho|_{S^*_\beta}$ separates idempotents, we get that $a\kappa_a \cap E(S^*_\beta) = \{b^0\}$. Therefore $E(a) \cap D_{a^2} = \{(a^2)^0\}$. □

7. When is $\mathcal{S}$ or $\mathcal{E}$ a congruence?

We answer these two questions in the next theorem. It is remarkable that in this way we encounter a familiar class of regular semigroups and that $\mathcal{S}$ and $\mathcal{E}$ are simultaneously congruences or not. For definitions, see Section 1.

Theorem 7.1. *The following conditions on S are equivalent.*

(i) *$\mathcal{E}$ is a congruence.*

(ii) *$\mathcal{S}$ is a congruence.*

(iii) *S is a cryptogroup.*

Proof. (i) *implies* (ii). By Proposition 5.4, we have $\mathcal{H} \subseteq \mathcal{E}$ and by Corollary 5.2, we have $\mathcal{E}|_{E(S)} = \varepsilon$. The latter together with the hypothesis implies that $\mathcal{E}$ is an idempotent separating congruence and thus $\mathcal{E} \subseteq \mathcal{H}$. Therefore $\mathcal{H} = \mathcal{E}$ which by Proposition 5.4 yields that $\mathcal{H} = \mathcal{S} = \mathcal{E}$. In particular, $\mathcal{S}$ is a congruence.

(ii) *implies* (iii). By Proposition 5.4, we have $\mathcal{H} \subseteq \mathcal{S} \subseteq \mathcal{E}$ and by Corollary 5.2, we get $\mathcal{E}|_{E(S)} = \varepsilon$. Hence $\mathcal{S}|_{E(S)} = \varepsilon$ and $\mathcal{S}$ is an idempotent separating congruence and thus $\mathcal{S} \subseteq \mathcal{H}$. Therefore $\mathcal{H} = \mathcal{S}$. By Proposition 5.4, $\mathcal{S}$ is a band congruence, hence each $\mathcal{H}$-class contains an idempotent and so S is completely regular and thus a cryptogroup.

(iii) *implies* (i). By Theorem 6.5, we have $\mathcal{H} = \mathcal{E}$ and $\mathcal{H}$ being a congruence implies that $\mathcal{E}$ is too. □

References

[1] *K. Auinger*: Free objects in joins of strict inverse and completely simple semigroups. J. London Math. Soc. *45* (1992), 491–507. Zbl 0774.20039

[2] *K. Auinger*: The congruence lattice of a strict regular semigroup. J. Pure Appl. Algebra *81* (1992), 219–245. Zbl 0809.20054

[3] *T. S. Blyth and M. G. Gomes*: On the compatibility of the natural order on a regular semigroup. Proc. Royal Soc. Edinburgh *A94* (1983), 79–84. Zbl 0511.20043

[4] *J. M. Howie*: An introduction to semigroup theory. Academic Press, London, 1976. Zbl 0355.20056

[5] *G. Lallement*: Congruences et équivalences de Green sur un demi-groupe régulier. C.R. Acad. Sci., Paris *262* (1966), 613–616. Zbl 0136.26603

[6] *K. S. S. Nambooripad*: Structure of regular semigroups. Mem. Amer. Math. Soc. *224* (1979). Zbl 0457.20051

[7] *K. S. S. Nambooripad*: The natural partial order on a regular semigroup. Proc. Edinburgh Math. Soc. *23* (1980), 249–260. Zbl 0459.20054

[8] *M. Petrich*: Introduction to semigroups. Merrill, Columbus, 1973. Zbl 0321.20037

[9] *M. Petrich*: Inverse semigroups. Wiley, New York, 1984. Zbl 0546.20053

[10] *P. G. Trotter*: Congruence extensions in regular semigroups. J. Algebra *137* (1991), 166–179. Zbl 0714.20053

[11] *P. S. Venkatesan*: Right (left) inverse semigroups. J. Algebra *31* (1974), 209–217. Zbl 0301.20058

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