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ON HARMONIC MAJORIZATION OF THE MARTIN FUNCTION AT INFINITY IN A CONE

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Abstract. This paper shows that some characterizations of the harmonic majorization of the Martin function for domains having smooth boundaries also hold for cones.

Keywords: harmonic majorization, cone, minimally thin

MSC 2000: 31B05, 31B20

1. Introduction

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\mathbb{R}^n$ $(n \geqslant 2)$ the n-dimensional Euclidean space. A point in $\mathbb{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbb{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbb{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbb{R}^n$ are denoted by ∂S and $\overline{S}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbb{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, y)$ by

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin\theta_j\right) \quad (n \geqslant 2), \quad y = r\cos\theta_1,$$

and if $n \geqslant 3$, then

$$x_{n+1-k} = r\left(\prod_{j=1}^{k-1} \sin\theta_j\right)\cos\theta_k \quad (2 \leqslant k \leqslant n-1),$$

where $0 \leqslant r < +\infty$, $-\frac{1}{2}\pi \leqslant \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geqslant 3$, then $0 \leqslant \theta_j \leqslant \pi$ $(1 \leqslant j \leqslant n-2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbb{S}^{n-1}$ and $\mathbb{S}_+^{n-1}$, respectively. For simplicity, a point $(1,\Theta)$ on $\mathbb{S}^{n-1}$ and the set $\{\Theta\colon\ (1,\Theta)\in\Omega\}$ for a set Ω, $\Omega\subset\mathbb{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda\subset\mathbb{R}_+$ and $\Omega\subset\mathbb{S}^{n-1}$, the set

$$\{(r,\Theta)\in\mathbb{R}^n\colon\ r\in\Lambda,\ \ (1,\Theta)\in\Omega\}$$

in $\mathbb{R}^n$ is simply denoted by $\Lambda\times\Omega$. In particular, we denote by $C_n(\Omega)$ the set $\mathbb{R}_+\times\Omega$ in $\mathbb{R}^n$ with the domain Ω on $\mathbb{S}^{n-1}$ $(n\geqslant 2)$. We call it a cone. Then the half-space $\mathbb{T}_n=\{(X,y)\in\mathbb{R}^n\colon\ y>0\}$ is a cone obtained by putting $\Omega=\mathbb{S}_+^{n-1}$.

To extend a result of Beurling [7] for n=2, Armitage and Kuran [4] said that a sequence $\{P_m\}$ of points $P_m=(X_m,y_m)\in\mathbb{T}_n$, $|P_m|\to+\infty$ $(m\to+\infty)$ *characterizes the positive harmonic majorization of* y, if every positive harmonic function h in $\mathbb{T}_n$ which majorizes the function y on the set $\{P_m\colon\ m=1,2,\dots\}$ majorizes y everywhere in $\mathbb{T}_n$, i.e.

$$\inf_{P\in\mathbb{T}_n}\frac{h(P)}{y}=\inf_{m}\frac{h(P_m)}{y_m}\quad (P=(X,y)\in\mathbb{T}_n).$$

They proved

Theorem A (Beurling [7] for $n=2$, Armitage and Kuran [4, Theorem 1] for $n\geqslant 2$)**.** *Let $\{P_m\}$ be a sequence of points,*

$$P_m=(r_m,\Theta_m)\in\mathbb{T}_n,\quad \Theta_m=(\theta_{1,m},\theta_{2,m},\dots,\theta_{(n-1),m})$$

in $\mathbb{T}_n$ satisfying

$$r_{m+1}\geqslant a\,r_m\quad (m=1,2,\dots) \tag{1.1}$$

for a certain $a>1$. Then the sequence $\{P_m\}$ characterizes the positive harmonic majorization of y if and only if

$$\sum_{m=1}^{\infty}(\cos\theta_{1,m})^n=+\infty. \tag{1.2}$$

Theorem A was also extended by Maz'ya [15] to positive solutions of a second order elliptic differential equation in an n-dimensional bounded domain with smooth boundary of class $C^{1,\alpha}$ $(0<\alpha<1)$.

Let D be a domain in $\mathbb{R}^n$ and $\Delta(D)$ the Martin boundary of D. The Martin function at $Q\in\Delta(D)$ is denoted by $K_Q(P)$ $(P\in D)$ (for these definitions see e.g. Helms [14, pp. 243–245], Armitage and Gardiner [5, pp. 235–237]). Following Armitage and Kuran [4], we say that a subset E of D *characterizes the positive harmonic majorization of* $K_Q(P)$, if every positive harmonic function h in D which majorizes $K_Q(P)$ on E majorizes $K_Q(P)$ everywhere in D, i.e.

$$\inf_{P\in D}\frac{h(P)}{K_Q(P)}=\inf_{P\in E}\frac{h(P)}{K_Q(P)}. \tag{1.3}$$

We set

$$B(P,r)=\{P'\in\mathbb{R}^n:\ |P'-P|<r\}\quad (r>0)$$

and

$$d(P)=\inf_{Q\notin D}|P-Q|$$

for any $P\in D$. For a subset E of D and a number ϱ $(0<\varrho<1)$ we put

$$E_\varrho=\bigcup_{P\in E}B(P,\varrho d(P)). \tag{1.4}$$

Dahlberg proved

Theorem B (Dahlberg [10, Theorem 1]). *Let D be a Liapunov-Dini domain in $\mathbb{R}^n$ and $Q\in\partial D$. If $E\subset D$, then the following conditions on E are equivalent:*

(i) *E characterizes the positive harmonic majorization of $K_Q(P)$;*

(ii) *for every ϱ, $0<\varrho<1$*

$$\int_{E_\varrho}|P-Q|^{-n}\,\mathrm{d}P=+\infty;$$

(iii) *for some ϱ, $0<\varrho<1$*

$$\int_{E_\varrho}|P-Q|^{-n}\,\mathrm{d}P=+\infty.$$

Since (1.3) is closely related to the notion of minimal thinness of E_ϱ in (1.4) (see Sjögren [18], Ancona [3] and Zhang [21]), which will be also seen in Theorem 2 of this paper, Aikawa and Essén [2, Corollary 7.4.7] also proved Theorem B in a way different from Dahlberg's.

By using a suitable Kelvin transformation which maps $\mathbb{T}_n$ onto a ball, the following Theorem C follows from Theorem B.

Theorem C (Dahlberg [10, Theorem 3]). *If $E \subset \mathbb{T}_n$, then the following conditions on E are equivalent:*

(i) *E characterizes the positive harmonic majorization of y;*

(ii) *for every ϱ, $0 < \varrho < 1$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty;$$

(iii) *for some ϱ, $0 < \varrho < 1$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty.$$

All proofs of Theorems A and B are based on the smoothness of the boundary having no wedges, e.g. a ball. For a domain having rougher boundary, e.g. a Lipschitz domain, Ancona [3, Theorem 7.4] and Zhang [21, Theorem 3] gave more complicated results which generalize Theorem A.

In this paper we shall prove that Theorems A and C can be still extended in the similar form to a result at a corner point of a wedge, i.e. to a result at ∞ of a cone (Theorem 3). We remark that a half-space is one of cones. To prove this result, we need a result (Theorem 2) which is a specialized version of that due to Aikawa [1, Theorem 1]. Since his proof is too complicated we give a simple proof based on an example of positive harmonic functions (Theorem 1).

For a Lipschitz domain and an NTA domain D, Zhang [21, Corollary 1] and Aikawa [1, Remark and Theorem 1] gave a necessary and sufficient qualitative condition for a subset E of D to characterize the positive harmonic majorization of $K_Q(P)$ by connecting it with minimal thinness of E_ϱ in (1.4). On the other hand, with respect to the quantitative Theorem B Aikawa said in his paper [1] that since a general NTA domain may have wedges, Theorem B does not hold for an NTA domain. However, if we observe in this paper that a cone has a wedge, at the corner point of which Theorem B still holds, against Aikawa's opinion we may ask whether Theorem B can be extended in the similar form to a result for a Lipschitz domain or an NTA domain.

2. Statements of results

Let Ω be a domain on $\mathbb{S}^{n-1}$ ($n \geqslant 2$) with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \tau) f = 0 \quad \text{on } \Omega,$$
$$f = 0 \quad \text{on } \partial\Omega,$$

where Λ_n is the spherical part of the Laplace operator Δ_n:

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$; hence

$$\int_\Omega f_\Omega^2(\Theta)\,\mathrm{d}\sigma_\Theta = 1,$$

where $\mathrm{d}\sigma_\Theta$ is the surface element on $\mathbb{S}^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by α_Ω, $-\beta_\Omega$ ($\alpha_\Omega, \beta_\Omega > 0$). If $\Omega = \mathbb{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$ and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1,$$

where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbb{S}^{n-1}$.

To simplify our next consideration, we shall assume that if $n \geqslant 3$, then Ω is a $C^{2,\alpha}$-domain ($0<\alpha<1$) on $\mathbb{S}^{n-1}$ (see e.g. Gilbarg and Trudinger [12, pp. 88–89] for the definition of a $C^{2,\alpha}$-domain). It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each point of which is a minimal Martin boundary point, and the Martin kernel at ∞ with respect to a reference point chosen suitably is $K_\infty(P) = r^{\alpha_\Omega} f_\Omega(\Theta)$ ($P = (r,\Theta) \in C_n(\Omega)$) (see e.g. Yoshida [20, pp. 276–277]). In particular, y is the Martin function at ∞ of $\mathbb{T}_n$.

A subset E of a domain D in $\mathbb{R}^n$ is said to be *minimally thin* at $Q \in \Delta(D)$ (Brelot [8, p. 122], Doob [11, p. 208]), if there exists a point $P \in D$ such that

$$\hat{R}^E_{K_Q(\cdot)}(P) \neq K_Q(P),$$

where $\hat{R}^E_{K_Q(\cdot)}(P)$ is the regularized reduced function of $K_Q(P)$ relative to E (Helms [14, p. 134]).

The following results are conical versions of Dahlberg's results [10, p. 239].

Theorem 1. *Let E be a set in $C_n(\Omega)$ satisfying $\overline{E} \cap \partial C_n(\Omega) = \emptyset$. If E_ϱ with a positive number ϱ ($0<\varrho<1$) is minimally thin at ∞, then there exists a positive harmonic function $h(P)$ on $C_n(\Omega)$ such that*

$$\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P \in E} \frac{h(P)}{K_\infty(P)}.$$

Theorem 2. *Let E be a subset of $C_n(\Omega)$. The following conditions on E are equivalent:*
(i) *E characterizes the positive harmonic majorization of $K_\infty(P)$;*
(ii) *for any ϱ, $0 < \varrho < 1$, E_ϱ is not minimally thin at ∞;*
(iii) *for some ϱ, $0 < \varrho < 1$, E_ϱ is not minimally thin at ∞.*

The following Theorem 3 extends Theorem C.

Theorem 3. *Let E be a subset of $C_n(\Omega)$. Then the following conditions on E are equivalent:*
(i) *E characterizes the positive harmonic majorization of $K_\infty(P)$;*
(ii) *for every ϱ $(0 < \varrho < 1)$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty;$$

(iii) *for some ϱ $(0 < \varrho < 1)$*

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P = +\infty.$$

A sequence $\{P_m\}$ of points $P_m \in D$ is said to be *separated*, if there exists a positive constant c such that

$$|P_i - P_j| \geqslant c d(P_i) \quad (i, j = 1, 2, \dots, \ \ i \neq j)$$

(see e.g. Ancona [3, p. 18], Aikawa and Essén [2, p. 156]).

From Theorem 3 we immediately obtain the following Corollary which extends Theorem A.

Corollary. *Let $\{P_m\}$, $P_m \in C_n(\Omega)$ be a separated sequence satisfying*

$$\inf_m |P_m| > 0.$$

The sequence $\{P_m\}$ characterizes the positive harmonic majorization of $K_\infty(P)$ if and only if

$$\sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|} \right)^n = +\infty.$$

3. Proofs of theorems and corollary

Let f and g be two positive real valued functions defined on a set S. Then we shall write $f \approx g$, if there exist two constants A_1, A_2, $0 < A_1 \leqslant A_2$ such that $A_1 g \leqslant f \leqslant A_2 g$ everywhere on S. For a subset S in $\mathbb{R}^n$, the interior of S and the diameter of S are denoted by $\operatorname{int} S$ and $\operatorname{diam} S$, respectively. For two subsets S_1 and S_2 in $\mathbb{R}^n$, the distance between S_1 and S_2 is denoted by $\operatorname{dist}(S_1, S_2)$. A cube $\mathcal{M}_k$ $(k = 0, \pm 1, \pm 2, \ldots)$ is of the form

$$[l_1 2^{-k}, (l_1+1)2^{-k}] \times \ldots \times [l_n 2^{-k}, (l_n+1)2^{-k}]$$

where $l_1, \ldots, l_n$ are integers. Let ϱ be a number satisfying $0 < \varrho \leqslant \frac{1}{2}$. A family of the Whitney cubes of $C_n(\Omega)$ with ϱ is the set of cubes having the following properties:
(i) $\bigcup\limits_i W_i = C_n(\Omega)$,
(ii) $\operatorname{int} W_i \cap \operatorname{int}\ W_j = \emptyset$ $(i \neq j)$,
(iii) $[8/(3\varrho)] \operatorname{diam} W_i \leqslant \operatorname{dist}(W_i, \mathbb{R}^n \setminus C_n(\Omega)) \leqslant 2([8/(3\varrho)] + 1) \operatorname{diam} W_i$,
where [a] denotes the integer satisfying $[a] \leqslant a < [a] + 1$ (Stein [19, p. 167, Theorem 1]).

The following Lemma 1 is fundamental in this paper.

Lemma 1 (I. Miyamoto, M. Yanagishita and H. Yoshida [16, Theorems 2 and 3]). *Let a Borel subset E of $C_n(\Omega)$ be minimally thin at ∞. Then we have*

$$\int_E \frac{\mathrm{d}P}{(1+|P|)^n} < +\infty. \tag{3.1}$$

If E is a union of cubes from a family of the Whitney cubes of $C_n(\Omega)$ with ϱ $(0 < \varrho \leqslant \frac{1}{2})$, then (3.1) *is also sufficient for E to be minimally thin at ∞.*

For a set $E \subset C_n(\Omega)$ and a number ϱ $(0 < \varrho \leqslant \frac{1}{2})$, define E_ϱ and $E_{\varrho/4}$ as in (1.4).

Lemma 2. *Let $\{W_i\}_{i\geqslant 1}$ be a family of the Whitney cubes of $C_n(\Omega)$ with ϱ. Let E be a subset of $C_n(\Omega)$. Then there exists a subsequence $\{W_{i_j}\}_{j\geqslant 1}$ of $\{W_i\}_{i\geqslant 1}$ such that*
(i) $\bigcup\limits_j W_{i_j} \subset E_\varrho$,
(ii) $W_{i_j} \cap E_{\varrho/4} \neq \emptyset$ $(j = 1, 2, \ldots)$, $E_{\varrho/4} \subset \bigcup\limits_j W_{i_j}$.

P r o o f. Let k be an integer. Let $c = [8/(3\varrho)] + 1$ and set

$$I_k = \{P \in C_n(\Omega)\colon\ c\sqrt{n}2^{-k} < \operatorname{dist}(P, \partial C_n(\Omega)) \leqslant c\sqrt{n}2^{-k+1}\}.$$

Let $\{W_{i_j}\}_{j\geqslant 1}$ be a subsequence of all Whitney cubes from $\{W_i\}_{i\geqslant 1}$ such that

$$W_{i_j}\cap E_{\varrho/4}\neq\emptyset\quad (j=1,2,\ldots).$$

Then it is evident that (ii) holds. We shall also show that this $\{W_{i_j}\}_{j\geqslant 1}$ satisfies (i), i.e. $W_{i_j}\subset E_\varrho$ $(j=1,2,\ldots)$.

Take any W_{i_j} $(j=1,2,\ldots)$. Since $W_{i_j}\cap E_{\varrho/4}\neq\emptyset$, there exists a point P_j in E such that

$$B(P_j,\frac{\varrho}{4}d(P_j))\cap W_{i_j}\neq\emptyset. \tag{3.2}$$

We can easily see that $W_{i_j}\in\mathcal{M}_{m+1}\cup\mathcal{M}_m\cup\mathcal{M}_{m-1}$, if there is a point $P\in I_m$ such that $W_{i_j}\cap B(P,\frac{\varrho}{4}d(P))\neq\emptyset$. Hence, for an integer k satisfying $W_{i_j}\in\mathcal{M}_k$, P_j taken above satisfies $P_j\in I_{k+1}\cup I_k\cup I_{k-1}$. So, if $P_j\in I_{k+1}$, then

$$\varrho d(P_j)-\frac{\varrho}{4}d(P_j)=\frac{3}{4}\varrho d(P_j)>\frac{3}{4}\varrho\Big(\Big[\frac{8}{3\varrho}\Big]+1\Big)\sqrt{n}2^{-(k+1)}>\sqrt{n}2^{-k}.$$

Since the diameter of W_{i_j} is $\sqrt{n}2^{-k}$, we have from (3.2) that $W_{i_j}\subset B(P_j,\varrho d(P_j))$ and hence $W_{i_j}\subset E_\varrho$. If $P_j\in I_k$ or $P_j\in I_{k-1}$, then we similarly have $W_{i_j}\subset E_\varrho$. □

Proof of Theorem 1. If E is a bounded subset of $C_n(\Omega)$, then let h be a constant function. When E is unbounded, we shall follow Dahlberg [10, p. 240] to make the required function.

We can assume $\varrho\leqslant\frac{1}{2}$. Let $\{P_j\}$ be a sequence of points P_j which are the central points of cubes W_{i_j} in Lemma 2. Then by our assumption on E, $\{P_j\}$ can not accumulate to any finite boundary point of $C_n(\Omega)$ and hence $|P_j|\to+\infty$, because $P_j\in E_\varrho$ due to (i) of Lemma 2. Since E_ϱ is minimally thin at ∞ and

$$\int_{W_{i_j}}\frac{\mathrm{d}P}{(1+|P|)^n}\approx\Big(\frac{d(P_j)}{|P_j|}\Big)^n\quad (j=1,2,\ldots),$$

Lemma 1 and (i) of Lemma 2 give

$$\sum_{j=1}^{\infty}\Big(\frac{\mathrm{d}(P_j)}{|P_j|}\Big)^n<+\infty. \tag{3.3}$$

Hence we can take a positive integer J such that $d(P_j)\leqslant\frac{1}{2}|P_j|$ for every $j\geqslant J$.

Now, take a point $Q_j=(t_j,\Phi_j)\in\partial C_n(\Omega)\setminus\{O\}$ satisfying

$$|P_j-Q_j|=d(P_j)\quad (j=J,J+1,\ldots).$$

Then we also see $|Q_j| \geqslant \frac{1}{2}|P_j|$ and hence $|Q_j| \to +\infty$ $(j \to +\infty)$. We define $h_1(P)$ by

$$h_1(P) = \sum_{j=J}^{\infty} \mathbb{P}_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}}, \quad \mathbb{P}_{Q_j}(P) = \frac{\partial G(P, Q_j)}{\partial n_{Q_j}} \quad (P \in C_n(\Omega)),$$

where $G(P_1, P_2)$ $(P_1, P_2 \in C_n(\Omega))$ is the Green function of $C_n(\Omega)$ and $\partial/\partial n_Q$ denotes the differentiation at $Q \in \partial C_n(\Omega)$ along the inward normal into $C_n(\Omega)$. Then h_1 is well-defined and hence is a positive harmonic function on $C_n(\Omega)$, because at any fixed $P = (r, \Theta) \in C_n(\Omega)$ we have

$$\mathbb{P}_{Q_j}(P) \approx r^{\alpha_\Omega} f_\Omega(\Theta) t_j^{-\beta_\Omega - 1} \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

for every Q_j satisfying $t_j \geqslant 2r$ (see Azarin [6, Lemma 1]).

First, to see

$$\inf_{P \in E} \frac{h_1(P)}{K_\infty(P)} > 0, \tag{3.4}$$

denote the Poisson kernel of the ball $B_j = B(P_j, d(P_j))$ by $\mathbb{P}_j(P, Q)$ $(P \in B_j, Q \in \partial B_j)$. Then we have

$$\mathbb{P}_{Q_j}(P) \geqslant \mathbb{P}_j(P, Q_j) \quad (P \in B_j;\ j = J, J+1, \ldots)$$

and hence

$$\mathbb{P}_{Q_j}(P_j) \geqslant \mathbb{P}_j(P_j, Q_j) = s_n^{-1}\{d(P_j)\}^{1-n} \quad (j = J, J+1, \ldots).$$

Since

$$f_\Omega(\Theta) \approx d(P') \quad (P' = (1, \Theta),\ \Theta \in \Omega),$$

we obtain

$$h_1(P_j) \geqslant \mathbb{P}_{Q_j}(P_j) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}} \geqslant A K_\infty(P_j) \quad (j = J, J+1, \ldots) \tag{3.5}$$

with some positive constant A. Now, take any $P \in E$. Then by (ii) of Lemma 2 there exists a point P_j such that

$$|P - P_j| < \frac{1}{2} \operatorname{diam}(W_{i_j}) \leqslant \delta d(P_j) \quad \left(\delta = \frac{1}{2}\left[\frac{8}{3\varrho}\right]^{-1}\right).$$

From Harnack's inequalities (see Armitage and Gardiner [5, Theorem 1.4.1]) we have

$$h_1(P) \geqslant \frac{1-\delta}{(1+\delta)^{n-1}} h_1(P_j), \quad K_\infty(P) \leqslant \frac{1+\delta}{(1-\delta)^{n-1}} K_\infty(P_j).$$

These inequalities and (3.5) immediately give (3.4).

Next, for a fixed ray L which is inside $C_n(\Omega)$ and starts from O, we shall show

$$\lim_{|P|\to+\infty,\, P\in L} \frac{h_1(P)}{K_\infty(P)} = 0. \tag{3.6}$$

Put

$$g_j(P) = \frac{\mathbb{P}_{Q_j}(P)}{K_\infty(P)} |P_j|^{\beta_\Omega+1} \quad (P \in C_n(\Omega);\ j = J, J+1, \ldots).$$

Then we have

$$\frac{h_1(P)}{K_\infty(P)} = \sum_{j=J}^{\infty} g_j(P) \left(\frac{d(P_j)}{|P_j|} \right)^n.$$

Since

$$\mathbb{P}_{Q_j}(P) \approx t_j^{\alpha_\Omega - 1} r^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j) \quad (P = (r,\Theta) \in C_n(\Omega),\ r \geqslant 2t_j) \tag{3.7}$$

(see Azarin [6, Lemma 1]), we see that

$$\lim_{|P|\to+\infty,\, P\in L} g_j(P) = 0$$

for any fixed $j \geqslant J$. Hence if we can show that

$$|g_j(P)| \leqslant M \quad (P \in L;\ j = J, J+1, \ldots) \tag{3.8}$$

for some constant M independent of j, then we shall have (3.6) from (3.3) and Lebesgue's dominated convergence theorem.

Now we shall prove (3.8) by dividing the proof into three cases. If $r \leqslant \frac{t_j}{2}$, then we have

$$\mathbb{P}_{Q_j}(P) \approx r^{\alpha_\Omega} t_j^{-\beta_\Omega - 1} f_\Omega(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

and hence

$$|g_j(P)| \leqslant M \quad (P = (r,\Theta) \in C_n(\Omega);\ j = J, J+1, \ldots).$$

If $r \geqslant 2t_j$, then we have

$$|g_j(P)| \leqslant M \quad (P = (r,\Theta) \in C_n(\Omega);\ j = J, J+1, \ldots)$$

from (3.7). Finally, put $R_1 = r/t_j$, $u = t_j$ and $\Theta_1 = \Theta$ in

$$u^{n-2}G((uR_1,\Theta_1),(uR_2,\Theta_2)) = G((R_1,\Theta_1),(R_2,\Theta_2)),$$
$$((R_1,\Theta_1),(R_2,\Theta_2) \in C_n(\Omega)).$$

When (R_2,Θ_2) approaches $(1,\Phi_j)$ along the inward normal, we obtain

$$\frac{\partial G(P,Q_j)}{\partial n_{Q_j}} = \frac{1}{t_j^{n-1}}\frac{\partial G}{\partial n_{Q'_j}}\Big(\Big(\frac{r}{t_j},\Theta\Big),(1,\Phi_j)\Big).$$

If $\frac{1}{2}t_j \leqslant r \leqslant 2t_j$, then

$$t_j^{n-1}\mathbb{P}_{Q_j}(P) \leqslant M' \quad (P=(r,\Theta)\in L;\ j=J,J+1,\dots)$$

for some constant M' and hence

$$|g_j(P)| \leqslant M \quad (P\in L;\ j=J,J+1,\dots).$$

Finally, put $\gamma = \max_{1\leqslant j<J} K_\infty(P_j)$ and $h(P) = h_1(P)+\gamma$ for any $P\in C_n(\Omega)$. Then we easily see from (3.4) and (3.6) that $h(P)$ is also a positive harmonic function on $C_n(\Omega)$ required in Theorem 1.

P r o o f of Theorem 2. (i) $\Rightarrow$ (ii). Let c be a positive constant and put $E_1 = \{P\in E\colon K_\infty(P) > c\}$. Then E_1 is a set satisfying $\overline{E_1}\cap\partial C_n(\Omega)=\emptyset$. Since E characterizes the harmonic majorization of $K_\infty(P)$, E_1 also characterizes the harmonic majorization of $K_\infty(P)$. Indeed, otherwise there would exist a positive harmonic function $h(P)$ on $C_n(\Omega)$ satisfying

$$a = \inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)} < \inf_{P\in E_1}\frac{h(P)}{K_\infty(P)} = b.$$

If we put $u(P) = h(P) + bc$ $(P\in C_n(\Omega))$, then $u(P)\geqslant bK_\infty(P)$ for all $P\in E$ and hence

$$\inf_{P\in C_n(\Omega)}\frac{u(P)}{K_\infty(P)} = a < b \leqslant \inf_{P\in E}\frac{u(P)}{K_\infty(P)},$$

which contradicts (i).

If we can show that for any ϱ $(0<\varrho<1)$ $(E_1)_\varrho$ is not minimally thin at ∞, then for any ϱ $(0<\varrho<1)$ E_ϱ is not minimally thin at ∞ either, which is (ii).

So, suppose that for some number ϱ $(0<\varrho<1)$ $(E_1)_\varrho$ is minimally thin at ∞. Then by Theorem 1 there exists a positive harmonic function $h(P)$ on $C_n(\Omega)$ satisfying

$$\inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)} < \inf_{P\in E_1}\frac{h(P)}{K_\infty(P)},$$

which contradicts the fact that E_1 characterizes the harmonic majorization of $K_\infty(P)$.

(iii) $\Rightarrow$ (i). Suppose that E does not characterize the positive harmonic majorization of $K_\infty(P)$. Then there exists a positive harmonic function $h(P)$ in $C_n(\Omega)$ such that

$$a = \inf_{P\in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P\in E} \frac{h(P)}{K_\infty(P)} = b.$$

If we put $v(P) = h(P) - aK_\infty(P)$ $(P \in C_n(\Omega))$, then $v(P)$ is a positive harmonic function on $C_n(\Omega)$ satisfying

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} = 0. \tag{3.9}$$

Let ϱ be any positive number satisfying $0 < \varrho < 1$. For any $P \in E_\varrho$, there exists a point $P' \in E$ such that $|P - P'| < \varrho d(P')$ and hence

$$\Big(\frac{1-\varrho}{1+\varrho}\Big)^n \frac{v(P')}{K_\infty(P')} \leqslant \frac{v(P)}{K_\infty(P)}$$

by Harnack's inequality. Hence we have

$$\inf_{P\in E_\varrho} \frac{v(P)}{K_\infty(P)} \geqslant \Big(\frac{1-\varrho}{1+\varrho}\Big)^n \inf_{P\in E} \frac{v(P)}{K_\infty(P)} = \Big(\frac{1-\varrho}{1+\varrho}\Big)^n (b-a) > 0. \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} < \inf_{P\in E_\varrho} \frac{v(P)}{K_\infty(P)}$$

for the positive superharmonic function $v(P)$. Hence, from Miyamoto, Yanagishita and Yoshida [16, Theorem 1] it follows that E_ϱ is minimally thin at ∞. This contradicts (iii). $\square$

P r o o f of Theorem 3. (i) $\Rightarrow$ (ii). Suppose that

$$\int_{E_\varrho} (1+|P|)^{-n}\,\mathrm{d}P < +\infty$$

for some ϱ $(0 < \varrho < 1)$. We can assume that this ϱ satisfies $0 < \varrho \leqslant \frac{1}{2}$. Let $\{W_{i_j}\}_{j\geqslant 1}$ be the subsequence of $\{W_i\}_{i\geqslant 1}$ from Lemma 2. Then from (i) of Lemma 2 we also have

$$\int_{\bigcup_j W_{i_j}} \frac{\mathrm{d}P}{(1+|P|)^n} < +\infty.$$

Since $\bigcup_j W_{i_j}$ is a union of cubes from the Whitney cubes of $C_n(\Omega)$ with ϱ, we see from the second part of Lemma 1 that $\bigcup_j W_{i_j}$ is minimally thin at ∞, and hence from (ii) of Lemma 2 that $E_{\varrho/4}$ is minimally thin at ∞.

On the other hand, since E characterizes the positive harmonic majorization of $K_\infty(P)$, it follows from Theorem 2 that $E_{\varrho/4}$ is not minimally thin at ∞, which contradicts the conclusion obtained above.

(iii) $\Rightarrow$ (i). Suppose that E does not characterize the positive harmonic majorization of $K_\infty(P)$. Then we see from Theorem 2 that for any ϱ $(0 < \varrho < 1)$ E_ϱ is minimally thin at ∞. Lemma 1 gives that for any ϱ $(0 < \varrho < 1)$

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P < +\infty.$$

This contradicts (iii). $\square$

P r o o f of Corollary. It is easy to see that if $\{P_m\}$ is a separated sequence, then

$$B(P_i, \varrho d(P_i)) \cap B(P_j, \varrho d(P_j)) = \emptyset \quad (i, j = 1, 2, \ldots ; \ i \neq j)$$

for a sufficiently small ϱ $(0 < \varrho < 1)$ and hence

$$\int_{E_\varrho} (1 + |P|)^{-n} \, \mathrm{d}P \approx \sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|} \right)^n.$$

Hence the corollary immediately follows from (iii) of Theorem 3.

References

[1] *H. Aikawa*: Sets of determination for harmonic functions in an NTA domain. J. Math. Soc. Japan *48* (1996), 299-315.

[2] *H. Aikawa, M. Essén*: Potential Theory-Selected Topics. Lecture Notes in Math. Vol. 1633. Springer-Verlag, 1996.

[3] *A. Ancona*: Positive Harmonic Functions and Hyperbolicity. Lecture Notes in Math. Vol. 1344. Springer-Verlag, 1987, pp. 1–23.

[4] *D. H. Armitage, Ü. Kuran*: On positive harmonic majorization of y in $\mathbb{R}^n \times (0, +\infty)$. J. London Math. Soc. Ser. II *3* (1971), 733–741.

[5] *D. H. Armitage, S. J. Gardiner*: Classical Potential Theory. Springer-Verlag, 2001.

[6] *V. S. Azarin*: Generalization of a theorem of Hayman on subharmonic functions in an m-dimensional cone Am. Math. Soc. Transl. II. Ser..

[7] *A. Beurling*: A minimum principle for positive harmonic functions. Ann. Acad. Sci. Fenn. Ser. AI. Math. *372* (1965).

[8] *M. Brelot*: On Topologies and Boundaries in Potential Theory. Lect. Notes in Math. Vol. 175. Springer-Verlag, 1971.

[9] *R. Courant, D. Hilbert*: Methods of Mathematical Physics, 1st English edition. Interscience, New York, 1954.

[10] *B. E. J. Dahlberg*: A minimum principle for positive harmonic functions. Proc. London Math. Soc. *33* (1976), 2380–250.

[11] *J. L. Doob*: Classical Potential Theory and its Probabilistic Counterpart. Springer-Verlag, 1984.

[12] *D. S. Jerison, C. E. Kenig*: Boundary behavior of harmonic functions in non-tangentially accessible domains. Adv. Math. *46* (1982), 80–147.
[13] *D. Gilbarg, N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, 1977.
[14] *L. L. Helms*: Introduction to Potential Theory. Wiley, New York, 1969.
[15] *V. G. Maz'ya*: Beurling's theorem on a minimum principle for positive harmonic functions. Zapiski Nauchnykh Seminarov LOMI *30* (1972); (English translation) J. Soviet. Math. *4* (1975), 367–379.
[16] *I. Miyamoto, M. Yanagishitam, and H. Yoshida*: Beurling-Dahlberg-Sjögren type theorems for minimally thin sets in a cone. Canad. Math. Bull. *46* (2003), 252–264.
[17] *I. Miyamoto, H. Yoshida*: Two criteria of Wiener type for minimally thin sets and rarefied sets in a cone. J. Math. Soc. Japan *54* (2002), 487–512.
[18] *P. Sjögren*: Une propriété des fonctions harmoniques positives d'après Dahlberg, Séminaire de théorie du potentiel. Lecture Notes in Math. Vol. 563. Springer-Verlag, 1976, pp. 275–282.
[19] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, 1970.
[20] *H. Yoshida*: Nevanlinna norm of a subharmonic function on a cone or on a cylinder. Proc. London Math. Soc. Ser. III *54* (1987), 267–299.
[21] *Y. Zhang*: Ensembles équivalents a un point frontière dans un domaine lipshitzien, Séminaire de théorie du potentiel. Lecture Note in Math. Vol. 1393. Springer-Verlag, 1989, pp. 256–265.

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