Czechoslovak Mathematical Journal, 55 (130) (2005), 803–808

A NOTE ON $\aleph$-SPACES AND g-METRIZABLE SPACES

Zhaowen Li, Changsha

(Received December 30, 2002)

Abstract. In this paper, we give the mapping theorems on $\aleph$-spaces and g-metrizable spaces by means of some sequence-covering mappings, mssc-mappings and π-mappings.

Keywords: $\aleph$-spaces, g-metrizable spaces, strong sequence-covering mappings, sequence-covering mappings, mssc-mappings, π-mappings

MSC 2000: 54E99, 54C10

$\aleph$-spaces and g-metrizable spaces are two classes of generalized metric spaces, and they play an important role in metrization theory. Papers [4], [7], [16], [17], [18], [19], [20] have done wonderful work on $\aleph$-spaces or g-metrizable spaces, but they have only investigated the internal characterizations of $\aleph$-spaces or g-metrizable spacs.

In 1965, R. W. Heath [12] proved that a space is developable if and only if it is an open π-image of a metric space. In 1969, J. A. Kofner [13] proved that a space is a symmetric space satisfying the weak Cauchy condition if and only if it is a quotient π-image of a metric space. In 1972, D. K. Burke [14] proved that a space is semimetrizable if and only if it is a countably bi-quotient (or pseudo-open) π-image of a metric space. In 1976, K. B. Lee [15] proved that every g-metrizable space is a quotient π-image of a metric space. In this paper, the relationships between metric spaces and $\aleph$-spaces, or g-metrizable spaces are established by means of some sequence-covering mappings, mssc-mappings and π-mappings.

In this paper, all spaces are regular and T_1, all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. ω denotes $\mathbb{N} \cup \{0\}$. For a collection $\mathcal{P}$ of subsets of a space X and a mapping $f\colon X \to Y$, denote $f(\mathscr{P}) = \{f(P)\colon P \in \mathscr{P}\}$. For two collections $\mathscr{A}$ and $\mathscr{B}$ of subsets of X, denote $\mathscr{A} \wedge \mathscr{B} = \{A \cap B\colon A \in \mathscr{A}$ and $B \in \mathscr{B}\}$. For the usual product space $\prod_{i\in\mathbb{N}} X_i$, p_i denotes the projection from $\prod_{i\in\mathbb{N}} X_i$ onto X_i. Let us recall some basic definitions.

Definition 1. Let $f\colon X \to Y$ be a mapping.

(1) f is a mssc-mapping [2] (i.e., metrizably stratified strong compact mapping) if there exists a subspace X of the usual product space $\prod_{i\in\mathbb{N}} X_i$ of a collection $\{X_i\colon i \in \mathbb{N}\}$ of metric spaces satisfying the following condition: for each $y \in Y$, there exists an open neighborhood sequence $\{V_i\}$ of y in Y such that each $\mathrm{cl}(p_i f^{-1}(V_i))$ is compact in X_i.

(2) f is a strong sequence-covering mapping [6] if each convergent sequence (including its limit point) of Y is the image of some convergent sequence (including its limit point) of X.

(3) f is a sequence-covering mapping [9] if each convergent sequence (including its limit point) of Y is the image of some compact subset of X.

(4) f is a π-mapping [10] if (X, d) is a metric space and for each $y \in Y$ and each open neighborhood V of y in Y, $d(f^{-1}(y), X - f^{-1}(V)) > 0$.

Definition 2. Let $\mathscr{P}$ be a cover of a space X.

(1) $\mathscr{P}$ is a k-network [11] for X if for each compact subset K of X and its open neighborhood V, there exists a finite subcollection $\mathscr{P}'$ of $\mathscr{P}$ such that $K \subset \bigcup \mathscr{P}' \subset V$.

(2) $\mathscr{P}$ is a cs-network for X if for each $x \in X$, its open neighborhood V and a sequence $\{x_n\}$ converging to x, there exists $P \in \mathscr{P}$ such that $\{x_n\colon n \geqslant m\} \cup \{x\} \subset P \subset V$ for some $m \in \mathbb{N}$.

(3) $\mathscr{P}$ is a cs*-network for X if for each $x \in X$, its open neighborhood V and a sequence $\{x_n\}$ converging to x, there exist $P \in \mathscr{P}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\colon k \in \mathbb{N}\} \cup \{x\} \subset P \subset V$.

A space X is called an $\aleph$-space if X has a σ-locally finite k-network.

Definition 3. Let $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$ be a collection of subsets of a space X satisfying that for each $x \in X$,

(1) $\mathscr{P}_x$ is a network of x in X. i.e., $x \in \bigcap \mathscr{P}_x$ and for $x \in U$ with U open in X, $P \subset U$ for some $P \in \mathscr{P}_x$.

(2) If $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$.

$\mathscr{P}$ is called a weak-base for X [1] if whenever $G \subset X$ is such that for each $x \in G$, there exists $P \in \mathscr{P}_x$ satisfying $P \subset G$, then G is open in X.

A space X is called a g-metrizable space [8] if X has a σ-locally finite weak-base.

We have the following implications for a space [7]:

$$\text{metrizable} \Longrightarrow g\text{-metrizable} \Longleftrightarrow \text{symmetrizable} + \aleph\text{-space}.$$

Theorem 4. *The following are equivalent for a space X:*

(1) *X is an $\aleph$-space;*

(2) *X is the strong sequence-covering mssc-image of a metric space;*

(3) *X is the sequence-covering mssc-image of a metric space.*

P r o o f. (1) $\Longrightarrow$ (2) Suppose X is an $\aleph$-space, then X has a σ-locally finite cs-network by Theorem 4 of [4]. Let $\mathscr{P} = \bigcup\{\mathscr{P}_i\colon\ i \in \mathbb{N}\}$ be a σ-locally finite cs-network for X, where each $\mathscr{P}_i = \{P_\alpha\colon\ \alpha \in A_i\}$ is a locally finite collection of subsets of X which is closed under finite intersections and $X \in \mathscr{P}_i \subset \mathscr{P}_{i+1}$. For each $i \in \mathbb{N}$, endow A_i with discrete topology, then A_i is metrizable. Put

$$M = \Big\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} A_i\colon\ \{P_{\alpha_i}\colon\ i \in \mathbb{N}\} \subset \mathscr{P} \text{ forms a network at some point} x(\alpha) \in X\Big\},$$

and endow M with the subspace topology induced from the usual product topology of the collection $\{A_i\colon\ i \in \mathbb{N}\}$ of metric spaces, then M is a metric space. Since X is Hausdroff, $x(\alpha)$ is unique in X for each $\alpha \in M$. We define $f\colon M \to X$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in M$. Because $\mathscr{P}$ is a σ-locally finite cs-network, f is surjective. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x(\alpha)$. Suppose V is an open neighborhood of $x(\alpha)$ in X, there exists $n \in \mathbb{N}$ such that $x(\alpha) \in P_{\alpha_n} \subset V$, set $W = \{c \in M\colon$ the n-th coordinate of c is $\alpha_n\}$, then W is an open neighborhood of α in M, and $f(W) \subset P_{\alpha_n} \subset V$. Hence f is continuous. We will show that f is a strong sequence-covering mssc-mapping.

(i) f is a mssc-mapping. For each $x \in X$ and each $i \in \mathbb{N}$, there exists an open neighborhood V_i of x in X such that $\{\alpha \in A_i\colon\ P_\alpha \cap V_i \neq \emptyset\}$ is finite. Put

$$B_i = \{\alpha \in A_i\colon\ P_\alpha \cap V_i \neq \emptyset\},$$

then $p_i f^{-1}(V_i) \subset B_i$. So $\mathrm{cl}(p_i f^{-1}(V_i)) \subset \mathrm{cl}(B_i) = B_i$. Thus $\mathrm{cl}(p_i f^{-1}(V_i))$ is compact in A_i. Hence f is a mssc-mapping.

(ii) f is a strong sequence-covering mapping. For each sequence $\{x_n\}$ converging to x_0, we can assume that all x_n's are distinct, and that $x_n \neq x_0$ for each $n \in \mathbb{N}$. Set $K = \{x_m\colon\ m \in \omega\}$. Suppose V is an open neighborhood of K in X. A subcollection $\mathscr{A}$ of $\mathscr{P}_i$ is said to have the property $F(K, V)$ if:

(a) $\mathscr{A}$ is finite;

(b) for each $P \in \mathscr{A}$, $\emptyset \neq P \cap K \subset P \subset V$

(c) for each $z \in K$, exists unique $P_z \in \mathscr{A}$ such that $z \in \mathscr{P}_z$

(d) if $x_0 \in P \in \mathscr{A}$, then $K \setminus P$ is finite.

Since $\mathscr{P}$ is a σ-locally finite cs-network, the above construction can be realized, and we can assume that $\{\mathscr{A} \subset \mathscr{P}_i\colon \mathscr{A}$ has the property $F(K,X)\} = \{\mathscr{A}_{ij}\colon j \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, put

$$\mathscr{P}'_n = \bigwedge_{i,j\leqslant n} \mathscr{P}_{ij},$$

then $\mathscr{P}'_n \subset \mathscr{P}_n$ and $\mathscr{P}'_n$ also has the property $F(K,X)$.

For each $i \in \mathbb{N}$, $m \in \omega$ and $x_m \in K$, there is $\alpha_{im} \in A_i$ such that $x_m \in P_{\alpha_{im}} \in \mathscr{P}'_i$. Let $\beta_m = (\alpha_{im}) \in \prod\limits_{i\in\mathbb{N}} A_i$. It is easy to prove that $\{P_{\alpha_{im}}\colon i \in \mathbb{N}\}$ is a network of x_m in X. Then there is a $\beta_m \in M$ such that $f(\beta_m) = x_m$ for each $m \in \omega$. For each $i \in \mathbb{N}$, there is $n(i) \in \mathbb{N}$ such that $\alpha_{in} = \alpha_{i0}$ when $n \geqslant n(i)$. Hence the sequence $\{\alpha_{in}\}$ converges to α_{i0} in A_i. Thus the sequence $\{\beta_n\}$ converges to β_0 in M. This show that f is a strong sequence-covering mapping.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) Suppose X is the image of a metric space M under a sequence-covering mssc-mapping f. Since f is a mssc-mapping, there exists the collection $\{M_i\colon i \in \mathbb{N}\}$ of metric spaces satisfying Definition 1 (1). For each $i \in \mathbb{N}$, M_i has a σ-locally finite base $\mathscr{P}_i = \bigcup\limits_{k\in\mathbb{N}} \mathscr{B}_k^{(i)}$, where each $\mathscr{B}_k^{(i)}$ is locally finite in M_i. For each $i \in \mathbb{N}$ and $k \in \mathbb{N}$, put

$$\mathscr{R}_i^{(k)} = \Big\{M \cap \Big(\bigcap_{j\leqslant i} p_j^{-1}(P_j)\Big)\colon\ P_j \in \mathscr{B}_k^{(i)} \text{ and } j \leqslant i\Big\},$$

$$\mathscr{R}_i = \bigcap_{k\in\mathbb{N}} \mathscr{R}_i^{(k)},$$

$$\mathscr{R} = \bigcap_{i\in\mathbb{N}} \mathscr{R}_i,$$

then $\mathscr{R}$ is a base for M. For each $x \in X$, there exists an open neighborhood sequence $\{V_i\colon i \in \mathbb{N}\}$ of x in X such that each $\mathrm{cl}(p_i f^{-1}(V_i))$ is compact in M_i. For each $n \in \mathbb{N}$ and $k \in \mathbb{N}$, put

$$V = \bigcap_{i\leqslant n} V_i,$$

then $\{Q \in f(\mathscr{R}_n^{(k)})\colon V \cap Q \neq \emptyset\}$ is finite. Thus $f(\mathscr{R}_n^{(k)})$ is locally finite in X. So $f(\mathscr{R})$ is σ-locally finite in X.

Because sequence-covering mappings preserve cs*-networks by Proposition 2.7.3 of [3], $f(\mathscr{R})$ is a σ-locally finite cs*-network for X. Hence X is an $\aleph$-space by [5, Lemma 1.17, Theorem 1.4].

Lemma 5 [2]**.** *Suppose f is a quotient mapping from a k-space X onto a space Y. If $\mathscr{B}$ is a k-network for X and $f(\mathcal{B})$ is point-countable in Y, then $f(\mathscr{B})$ is a k-network for Y.*

Lemma 6 [7]**.** *Suppose f is a quotient mapping from a metric space X onto a space Y. Then Y is a symmetric space if and only if f is a π-mapping.*

Theorem 7. *The following are equivalent for a space X:*

(1) *X is a g-metrizable space;*
(2) *X is a strong sequence-covering, quotient, π, mssc-image of a metric space;*
(3) *X is a sequence-covering, quotient, π, mssc-image of a metric space;*
(4) *X is a quotient, π, mssc-image of a metric space.*

P r o o f. (1) $\Longrightarrow$ (2) Suppose X is a g-metrizable space, then X is an $\aleph$-space. By Theorem 4, X is the image of a metric space M under a strong sequence-covering mssc-mapping f. Thus f is a quotient mapping by Proposition 2.1.16 (2) of [3]. Since X is symmetrizable, f is a π-mapping by Lemma 6. Hence X is a strong sequence-covering, quotient, π, mssc-image of a metric space.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1) Suppose X is the image of a metric space M under a quotient, π, mssc-mapping f. According to the proof of Theorem 4 (3) $\Longrightarrow$ (1), we can prove that there exists a base $\mathscr{R}$ for M such that $f(\mathscr{R})$ is σ-locally finite in X. By Lemma 5, $f(\mathscr{R})$ is a k-network for X. Hence X is an $\aleph$-space. By Lemma 6, X is symmetrizable. Therefore X is a g-metrizable space.

Remark 8. (1) Theorem 7 affirmatively answers the following problem posed in [8]:

Problem [8, problem 1.19]**.** Is every g-metrizable space a quotient π-image of a metric space?

(2) D. K. Burke has pointed out that an open compact image of a metric space need not be g-metrizable. For let Y be the space of Example B in [21]. Y is metacompact, developable, and not metrizable, so not g-metrizable. But Y is an open compact image of a metric space. Since a compact mapping defined on a metric space is a π-mapping, then Y is an open π-image of a metric space. Thus an open π-image of a metric space need not be g-metrizable. Hence the condition "mssc" in Theorem 7 cannot be omitted.

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Author's address: Department of Information, Hunan Business College, Changsha, Hunan 410205, P.R. China, e-mail: Lizhaowen8846@163.com.