Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 3, 781–790

Persistent URL: `http://dml.cz/dmlcz/128021`

Czechoslovak Mathematical Journal, 55 (130) (2005), 781–790

REMARKS ON SPECTRAL RADIUS AND LAPLACIAN EIGENVALUES OF A GRAPH

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(Received December 16, 2002)

Abstract. Let G be a graph with n vertices, m edges and a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. The spectral radius and the largest Laplacian eigenvalue are denoted by $\varrho(G)$ and $\mu(G)$, respectively. We determine the graphs with

$$\varrho(G) = \frac{d_n - 1}{2} + \sqrt{2m - nd_n + \frac{(d_n+1)^2}{4}}$$

and the graphs with $d_n \geqslant 1$ and

$$\mu(G) = d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \left(d_n - \frac{1}{2}\right)^2}.$$

We also present some sharp lower bounds for the Laplacian eigenvalues of a connected graph.

Keywords: spectral radius, Laplacian eigenvalue, strongly regular graph

MSC 2000: 05C50

1. Introduction

Let $G = (V, E)$ be a simple finite undirected graph with a vertex set V and an edge set E. Let $\delta(G) = \delta$ be the minimal degree of vertices of G. Let $A(G)$ be the $(0,1)$ adjacency matrix of G and $D(G)$ the diagonal matrix of vertex degrees. An eigenvalue of G is an eigenvalue of $A(G)$. The spectral radius $\varrho(G)$ of G is the largest

The work was supported by National Nature Science Foundation of China (10201009), Guangdong Provincial Natural Science Foundation of China (021072) and Com^{2}MaC-KOSEF

eigenvalue of G. It turns out that the Laplacian matrix of G is $L(G) = D(G) - A(G)$ and $L(G)$ is positive semidefinite and singular. A Laplacian eigenvalue of G is an eigenvalue of $L(G)$. Denote the Laplacian eigenvalues of G by $\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_{n-1}(G) \geqslant \mu_n(G) = 0$. We also write $\mu(G)$ for $\mu_1(G)$. It is well known [3] that $\mu_{n-1}(G) > 0$ if and only if G is connected.

Let G be a graph with n vertices, m edges and a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. We determine the graphs with

$$\varrho(G) = \frac{d_n - 1}{2} + \sqrt{2m - nd_n + \frac{(d_n + 1)^2}{4}}$$

and the graphs with $d_n \geqslant 1$ and

$$\mu(G) = d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \Bigl(d_n - \frac{1}{2}\Bigr)^2}.$$

We also present some lower bounds for the Laplacian eigenvalues of a connected graph.

2. Spectral radius

Recall that a bidegreed graph is a graph with two different vertex degrees. Hong, Shu and Fang [6] proved

Theorem 1 [6]**.** *Let G be a connected graph with n vertices and m edges and let δ be the minimal degree of vertices of G. Then*

$$\varrho(G) \leqslant \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}},$$

and equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.

Recently, Nikiforov [10] proved the above inequality independently by a quite different method for a (not necessarily connected) graph, and mentioned that equality holds for regular graphs, the maximally irregular n-vertex graph which is the complement of K_{n-k}, and the disjoint union of K_p and K_{n-p}. Based on Theorem 1, we can characterize the extreme case for not necessarily connected graphs.

Lemma 1 [6, 10]**.** *For nonnegative integers p and q with $2q \leqslant p(p - 1)$ and $0 \leqslant x \leqslant p - 1$, the function $f(x) = (x - 1)/2 + \sqrt{2q - px + (1 + x)^2/4}$ is decreasing with respect to x.*

Theorem 2. *Let G be a graph with n vertices and m edges and let δ be the minimal degree of vertices of G. Then*

$$\varrho(G) \leqslant \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}, \tag{1}$$

and equality holds if and only if in one component of G each vertex is either of degree δ or adjacent to all other vertices, and all other components are regular with degree δ.

P r o o f. The case that G is connected is proved in [6]. Suppose G is not connected. Then there is a component G_1 of G such that $\varrho(G) = \varrho(G_1)$. Suppose G_1 has n_1 vertices, m_1 edges and a minimal vertex degree δ_1. Let G_2 be the graph obtained from G by deleting the component G_1. Suppose G_2 has n_2 vertices and m_2 edges. Then by Theorem 1,

$$\varrho(G) = \varrho(G_1) \leqslant \frac{\delta_1-1}{2} + \sqrt{2m_1 - n_1\delta_1 + \frac{(\delta_1+1)^2}{4}}.$$

Note that $\delta_1 \geqslant \delta$ and $2m - n\delta = (2m_1 - n_1\delta) + (2m_2 - n_2\delta) \geqslant 2m_1 - n_1\delta$. By Lemma 1, we have

$$\varrho(G) \leqslant \frac{\delta-1}{2} + \sqrt{2m_1 - n_1\delta + \frac{(\delta+1)^2}{4}} \leqslant \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Suppose the equality holds in (1). Then all inequalities in the above argument are equalities. In particular, $2m_2 - n_2\delta = 0$, which implies that G_2 is regular with vertex degree δ. We also have $\delta_1 = \delta$ and

$$\varrho(G_1) = \frac{\delta_1-1}{2} + \sqrt{2m_1 - n_1\delta_1 + \frac{(\delta_1+1)^2}{4}},$$

and hence by Theorem 1 we know that G_1 is either a regular graph with a vertex degree δ or $n_1 - 1$ or a bidegreed graph in which each vertex is of a degree either δ or $n_1 - 1$. So if the equality holds in (1), then G_1 is either a regular graph with a vertex degree δ or $n_1 - 1$ or a bidegreed graph in which each vertex is of a degree either δ or $n_1 - 1$ and G_2 is a regular graph with the vertex degree δ.

Conversely, suppose one component of G, say, G_1 is a graph with n_1 vertices and m_1 edges, in which each vertex is either of a degree δ or $n_1 - 1$, and all other components are regular with the vertex degree δ. Since $\varrho(G) \geqslant \delta$ and $2(m - m_1) - (n - n_1)\delta = 0$, we have by Theorem 1

$$\begin{aligned}\varrho(G) = \varrho(G_1) &= \frac{\delta-1}{2} + \sqrt{2m_1 - n_1\delta + \frac{(\delta+1)^2}{4}} \\ &= \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.\end{aligned}$$

$\square$

Remark. Equality in (1) holds if and only if G is a graph of one of the following four types:

(i) a regular graph with the vertex degree δ;

(ii) the disjoint union of a complete graph with at least $\delta+2$ vertices and a regular graph with the vertex degree δ;

(iii) a bidegreed graph in which every vertex is either of a degree δ or $n-1$ ($\delta < n-1$);

(iv) the disjoint union of a connected bidegreed graph in which every vertex is either of the degree δ or adjacent to all other vertices, and a regular graph with the vertex degree δ.

Let G be a graph with n vertices, m edges and let δ be the minimal degree of vertices of G. Clearly $\delta \geqslant 0$. If G has no isolated vertices, then $\delta \geqslant 1$. By Lemma 1 and Theorem 1 we have

Corollary 1 [8]. *Let G be a graph with n vertices and m edges. Then*

$$\varrho(G) \leqslant -\frac{1}{2} + \sqrt{2m + \frac{1}{4}},$$

and equality holds if and only if one component of G is a complete graph with m edges, and all other components are isolated vertices.

Corollary 2 [5]. *Let G be a graph with n vertices and m edges. If G has no isolated vertices, then*

$$\varrho(G) \leqslant \sqrt{2m - n + 1},$$

and equality holds if and only if one component of G is a star or a complete graph with at least 2 vertices, and all other components are K_2's.

3. Largest Laplacian eigenvalue

Recently Shu, Hong and Kai [9], using Theorem 1, provided an upper bound for the largest Laplacian eigenvalue of a connected graph in terms of the vertex degree sequence: Let G be a connected graph with a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. Then

$$\mu(G) \leqslant d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \left(d_n - \frac{1}{2}\right)^2}.$$

They also pointed out that the equality holds if G is a regular bipartite graph. It is mentioned in [1, p. 283] that the equality also holds if G is the star graph. Let L_G be the line graph of a graph G.

Lemma 2 [7, 9]**.** *If G is a connected graph, then $\mu(G) \leqslant 2 + \varrho(L_G)$, and equality hols if and only if G is a bipartite graph.*

For (not necessarily connected) graphs, we have

Theorem 3. *Let G be a graph with a vertex degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n \geqslant 1$. Then*

$$\mu(G) \leqslant d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \Big(d_n - \frac{1}{2}\Big)^2}, \tag{2}$$

and equality holds if and only if G is a regular graph with at least one bipartite component, or G is the disjoint union of a star graph and (possibly) K_2's.

Proof. First suppose that $d_n \geqslant 2$. If G is connected, then by the proof in [9, p. 128], we have (2), and equality holds in and only if G is a regular gipartite graph. Suppose G is not connected. Then there is a component G_1 of G such that $\mu(G) = \mu(G_1)$. Suppose G_1 has n_1 vertices, m_1 edges and a minimal vertex degree δ_1. Suppose L_G has n' vertices, m' edges and a minimal vertex degree δ', and L_{G_1} has n_1' vertices, m_1' edges and a minimal vertex degree δ_1'. We have

$$n' = m = \frac{1}{2}\sum_{i=1}^{n} d_i, \quad 2m' = \sum_{i=1}^{n} d_i(d_i - 1) \quad \text{and} \quad \delta' \geqslant 2d_n - 2.$$

Note that $\delta_1' \geqslant \delta' \geqslant 2d_n - 2$ and $2m' - n'\delta' \geqslant 2m_1' - n_1'\delta'$. By Theorem 1 and Lemmas 1 and 2,

$$\begin{aligned}
\mu(G) = \mu(G_1) &\leqslant 2 + \varrho(L_{G_1}) \\
&\leqslant 2 + \frac{\delta_1' - 1}{2} + \sqrt{2m_1' - n_1'\delta_1' + \frac{(\delta_1' + 1)^2}{4}} \\
&\leqslant 2 + \frac{\delta' - 1}{2} + \sqrt{2m_1' - n_1'\delta' + \frac{(\delta' + 1)^2}{4}} \\
&\leqslant 2 + \frac{\delta' - 1}{2} + \sqrt{2m' - n'\delta' + \frac{(\delta' + 1)^2}{4}} \\
&\leqslant d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \Big(d_n - \frac{1}{2}\Big)^2}.
\end{aligned}$$

This proves (2).

Suppose the equality holds in (2). Then all inequalities in the above argument are equalities. In particular, we have

$$\delta' = 2d_n - 2 \quad \text{and} \quad 2(m' - m_1') - (n' - n_1')\delta' = 0.$$

So any component of L_G except L_{G_1} is regular with the vertex degree δ', and hence any component H of G except G_1 is either a regular graph with the vertex degree d_n or a semi-regular bipartite graph. If H is a semi-regular bipartite graph with p_1 independent vertices of degree r_1 and p_2 independent vertices of degree r_2, then $r_1+r_2 = 2d_n$, which implies $r_1 = r_2 = d_n$ since $r_1, r_2 \geqslant d_n$. Hence any component H of G except G_1 is a regular graph with the vertex degree d_n. Note that $\mu(G) = \mu(G_1) \leqslant 2 + \varrho(L_{G_1})$. We also have $\delta_1' = \delta' = 2d_n - 2$, which implies the minimal vertex degree of G_1 is d_n. Let $(d_{11}, d_{12}, \ldots, d_{1n_1})$ be the vertex degree sequence of G_1 with $d_{11} \geqslant d_{12} \geqslant \ldots \geqslant d_{1n_1} = d_n$. Then

$$\mu(G_1) = d_{1n_1} + \frac{1}{2} + \sqrt{\sum_{i=1}^{n_1} d_{1i}(d_{1i} - d_{1n_1}) + \Bigl(d_{1n_1} - \frac{1}{2}\Bigr)^2}.$$

It follows that G_1 is a regular bipartite graph with the vertex degree d_n. Hence G is regular with the vertex degree d_n and a bipartite component G_1.

Conversely, suppose G is a regular graph with at least one bipartite component. Then $d_i = d_n = r$ for all i. For any non-bipartite component H of G, the smallest eigenvalue of H is $> -r$, and hence $\mu(H) < 2r$. For any bipartite component G_1 of G, $\mu(G) = 2r$. Hence

$$\mu(G) = 2r = d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i - d_n) + \Bigl(d_n - \frac{1}{2}\Bigr)^2}.$$

Now suppose that $d_n = 1$ and $\mu(G) = \mu(G_1)$ where G_1 is a component of G with vertex degree sequence $(d_{11}, d_{12}, \ldots, d_{1n_1})$, where $d_{11} \geqslant d_{12} \geqslant \ldots d_{1n_1}$. Let m_1' be the number of edges of L_{G_1}. Then by Corollary 1,

$$\varrho(L_{G_1}) \geqslant -\frac{1}{2} + \sqrt{2m_1' + \frac{1}{4}}, \tag{3}$$

and equality holds if and only if one component of L_{G_1} is a complete graph. Note that $2m_1' = \sum_{j=1}^{n_1} d_{1j}(d_{1j}-1)$, $(d_n = 1)$. Then by Lemma 2,

$$\begin{aligned}
\mu(G) = \mu(G_1) &\leqslant 2 + \varrho(L_{G_1}) \\
&\leqslant \frac{3}{2} + \sqrt{2m_1' + \frac{1}{4}} \\
&= \frac{3}{2} + \sqrt{\sum_{j=1}^{n_1} d_{1j}(d_{1j}-1) + \frac{1}{4}} \\
&\leqslant \frac{3}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i-1) + \frac{1}{4}} \\
&= d_n + \frac{1}{2} + \sqrt{\sum_{i=1}^{n} d_i(d_i-d_n) + (d_n - \frac{1}{2})^2}.
\end{aligned} \tag{4}$$

This proves (2) if $d_n = 1$.

Suppose the equality holds in (2) and $d_n = 1$. Then all inequalities in (3) and (4) are equalities. Hence G_1 is bipartite, L_{G_1} is a complete graph, and the minimal vertex degree of any component of G is 1. If G is connected, then clearly G is the star graph. If G is not connected, then one component of G is a star graph, and all other components are K_2's.

Conversely, it can be easily checked that if one component of G is a star graph, and all other components (if exist) are K_2's, then the equality holds in (2).

□

4. The k-th Laplacian eigenvalues with $k \geqslant 2$

Various lower bounds for μ_k $(1 \leqslant k \leqslant n-1)$ of a graph G have been established, some in terms of the order, the degree sequence or the number of spannning trees of G (see [4], [11]). In the following we suppose that G is a connected graph with n vertices and m edges. Zhang and Li [11] have recently obtained a lower bound for $\mu_1(G)$ in terms of n and m in the form

$$\mu_1(G) \geqslant \frac{1}{n-1}\left(2m + \sqrt{\frac{2(n^2-n-2m)m}{n(n-2)}}\right),$$

where equality holds if and only if $G = K_n$.

We present lower bounds for $\mu_k(G)$ $(2 \leqslant k \leqslant n-1)$ in terms of n and m.

Lemma 3 [11]. *Let G be a graph with n vertices, m edges and a vertex degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

$$\sum_{i=1}^{n} d_i^2 \leqslant \frac{nm^2}{n-1},$$

where equality holds if and only if $G = K_{1,n-1}$.

The following lemma is well known [2].

Lemma 4. *A connected graph with two distinct eigenvalues is complete, a regular connected graph with three distinct eigenvalues is strongly regular.*

Theorem 4. *Let G be a connected graph with n vertices and m edges. Write $M(G) = \min\{(nm - 4m + 2n - 2)m, 2(n^2 - n - 2m)m\}$. Then for $2 \leqslant k \leqslant n-1$ we have*

$$\mu_k(G) \geqslant \frac{1}{n-1}\left(2m - \sqrt{\frac{k-1}{n-k} M(G)}\right), \tag{5}$$

and equality holds for some k with $2 \leqslant k \leqslant n-1$ if and only if $G = K_n$.

Proof. Write μ_k for $\mu_k(G)$ and L for $L(G)$. Let $\operatorname{Tr}(B)$ be the trace of a square matrix B. Denote $N_k = \sum\limits_{i=k}^{n-1} \mu_i$. Note that $\sum\limits_{i=1}^{n-1} \mu_i = \sum\limits_{i=1}^{n} d_i = 2m$. We have

$$\begin{aligned} \operatorname{Tr}(L^2) = \sum_{i=1}^{k-1} \mu_i^2 + \sum_{i=k}^{n-1} \mu_i^2 &\geqslant \frac{1}{k-1}\left(\sum_{i=1}^{k-1} \mu_i\right)^2 + \frac{1}{n-k}\left(\sum_{i=k}^{n-1} \mu_i\right)^2 \\ &= \frac{(2m - N_k)^2}{k-1} + \frac{N_k^2}{n-k}. \end{aligned} \tag{6}$$

Hence

$$N_k \geqslant \frac{1}{n-1}\left(2m(n-k) - \sqrt{(n-k)(k-1)((n-1)\operatorname{Tr}(L^2) - 4m^2)}\right).$$

Since $N_k \leqslant (n-k)\mu_k$, we have

$$\mu_k \geqslant \frac{1}{n-1}\left(2m - \sqrt{\frac{k-1}{n-k}((n-1)\operatorname{Tr}(L^2) - 4m^2)}\right). \tag{7}$$

By Lemma 3,

$$(n-1)\operatorname{Tr}(L^2) - 4m^2 = (n-1)\sum_{i=1}^{n} d_i(d_i+1) - 4m^2 \leqslant (nm - 4m + 2n - 2)m.$$

By virtue of the inequality $d_i \leqslant n-1$ for $1 \leqslant i \leqslant n$ we obtain

$$(n-1)\operatorname{Tr}(L^2) - 4m^2 \leqslant (n-1)\sum_{i=1}^{n} d_i n - 4m^2 = 2(n^2-n-2m)m.$$

Hence

$$(n-1)\operatorname{Tr}(L^2) - 4m^2 \leqslant M(G)$$

and (5) follows from (7).

Suppose that the equality in (5) holds for some k_0 with $2 \leqslant k_0 \leqslant n-1$. Then $(n-1)\operatorname{Tr}(L^2) - 4m^2 = M(G)$, and hence $\sum\limits_{i=1}^{n} d_i^2 = nm^2/(n-1)$ or $d_i = n-1$ for $1 \leqslant i \leqslant n$. In the former case, we have $G = K_{1,n-1}$ by Lemma 3, and hence $\mu_{k_0} = 1$, which is impossible. In the latter case, we have $G = K_n$.

If $G = K_n$, then $M(G) = 0$ and hence the equality in (5) holds for each k with $2 \leqslant k \leqslant n-1$. $\square$

Note that the bound in (6) is trivial if $2m \leqslant \sqrt{(k-1)/(n-k)M(G)}$. For a regular graph we give a finer lower bound for $\mu_k(G)$.

Theorem 5. *Let G be a connected regular graph with n vertices and a vertex degree δ. Then for $2 \leqslant k \leqslant n-1$ we have*

$$\mu_k(G) \geqslant \frac{1}{n-1}\left(n\delta - \sqrt{\frac{k-1}{n-k} n\delta(n-\delta-1)}\right), \tag{8}$$

where equality holds for some k with $2 \leqslant k \leqslant n-1$ if and only if G is K_n or a strongly regular graph.

P r o o f. Note that $(n-1)\operatorname{Tr}(L(G)^2) - 4m^2 = (n-1)n\delta(\delta+1) - n^2\delta^2 = nd(n-\delta-1)$. (8) follows from (7).

Suppose that equality in (8) holds for some k_0 with $2 \leqslant k_0 \leqslant n-1$. Then the equality in (6) holds for $k = k_0$. It follows that G has only two or three distinct Laplacian eigenvalues and hence has only two or three distinct eigenvalues. By Lemma 4, G is K_n or a strongly regular graph.

Conversely, if $G = K_n$, then equality in (8) holds for each k with $2 \leqslant k \leqslant n-1$; if G is a strongly regular graph, then G has three distinct eigenvalues δ, ϱ and σ ($\delta > \varrho > \sigma$) with multiplicities 1, r and s, and hence G has three distinct Laplacian eigenvalues $\delta - \sigma$, $\delta - \varrho$ and 0 with multiplicities s, r and 1, which implies that (6), (7) and hence (8) become equalities for $k = s+1$. $\square$

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