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A NOTE ON TRIANGULAR SCHEMES FOR WEAK CONGRUENCES

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Abstract. Some geometrical methods, the so called Triangular Schemes and Principles, are introduced and investigated for weak congruences of algebras. They are analogues of the corresponding notions for congruences. Particular versions of Triangular Schemes are equivalent to weak congruence modularity and to weak congruence distributivity. For algebras in congruence permutable varieties, stronger properties—Triangular Principles—are equivalent to weak congruence modularity and distributivity.

Keywords: triangular scheme, triangular principle, weak congruence, weak congruence modularity, weak congruence distributivity

MSC 2000: 08A30, 08A05

1. Introduction

Geometrical methods for investigating algebras in congruence modular varieties were introduced by H. P. Gumm [4] by means of certain schemes, the so called Shifting Lemma and Shifting Principle. He used these methods for developing commutator theory for congruence modular algebras. An analogue property—Triangular Scheme—was introduced by the first author of the present paper in [1]. In the paper [2], a property called Weak Triangular Principle is also investigated and proved to be equivalent to the distributivity of $\operatorname{Con}\mathscr{A}$ for an arbitrary algebra $\mathscr{A}$. Lattice theoretic reasons for the validity of Shifting and Triangular Lemmas were discussed in [3].

In the present paper we introduce Triangular Schemes and Principles in the framework of weak congruences. Due to the absence of reflexivity in these relations, definitions do not coincide with those for congruences. Therefore we call the new methods *Weak Congruence Triangular Scheme* and *Weak Congruence Triangular Principle*, and define them separately in the modular and in the distributive case. These schemes and principles turn out to imply some important congruence properties (e.g., the CEP). We prove the equivalence of the appropriate scheme and principle with the weak congruence modularity and distributivity.

2. Preliminaries

A *weak congruence relation* on an algebra $\mathscr{A}$ is a symmetric and transitive subalgebra of $\mathscr{A}^2$, i.e. it is a congruence on a subalgebra of $\mathscr{A}$. The set $\operatorname{Cw}\mathscr{A}$ of all weak congruences on $\mathscr{A}$ is an algebraic lattice under inclusion. The filter $\Delta\uparrow$ generated by the diagonal $\Delta = \{(x,x) \mid x \in A\}$ is the congruence lattice $\operatorname{Con}\mathscr{A}$. The ideal $\Delta\downarrow$ is isomorphic with the subalgebra lattice $\operatorname{Sub}\mathscr{A}$: every subalgebra $\mathscr{B}$ of $\mathscr{A}$ is represented by the corresponding diagonal relation Δ_B. For more details about weak congruences we refer to the book [6].

$\mathscr{A}$ satisfies the *congruence intersection property* (the CIP) if for every $\varrho, \theta \in \operatorname{Cw}\mathscr{A}$,

$$\Delta \vee (\varrho \wedge \theta) = (\Delta \vee \varrho) \wedge (\Delta \vee \theta).$$

Observe that for a congruence ϱ on a subalgebra $\mathscr{B}$ of $\mathscr{A}$, the join $\varrho \vee \Delta$ in $\operatorname{Cw}\mathscr{A}$ is a congruence on $\mathscr{A}$ obtained by

$$\varrho \vee \Delta = \bigcap(\theta \in \operatorname{Con}\mathscr{A} \mid \varrho \subseteq \theta).$$

More generally, let $\theta \in \operatorname{Con}\mathscr{B}$, $\sigma \in \operatorname{Con}\mathscr{C}$, where $\mathscr{B}$ and $\mathscr{C}$ are subalgebras of $\mathscr{A}$. Then it is easy to see that $\theta \vee \sigma$ and $\theta \vee \Delta_C$ are congruences on a subalgebra $B \vee C$ and

$$\theta \vee \Delta_C = \theta \vee \Delta_{B \vee C},$$

where $B \vee C$ is the join in $\operatorname{Sub}\mathscr{A}$.

Recall that an algebra $\mathscr{A}$ is said to possess the *Congruence Extension Property* (the CEP) if for every congruence ϱ on a subalgebra $\mathscr{B}$ of $\mathscr{A}$ there is a congruence θ on $\mathscr{A}$ such that $\varrho = B^2 \cap \theta$. Lattice-theoretic characterizations of the CEP can be found in [6].

3. Results

An algebra $\mathscr{A}$ satisfies the *Weak Congruence Triangular Scheme*, briefly the Cw *Triangular Scheme*, if for all $\varrho, \theta, \sigma \in \operatorname{Cw}\mathscr{A}$, where $\sigma \in \operatorname{Con}\mathscr{C}$, $\mathscr{C} \in \operatorname{Sub}\mathscr{A}$ we have that

$$\varrho \cap \theta \subseteq \sigma,\ (z,y) \in \theta,\ (z,x) \in \varrho \vee \Delta_C \text{ and } (x,y) \in \varrho \vee \sigma \text{ imply } (z,y) \in \sigma.$$

The Cw *Triangular Scheme* can be visualized as shown in Fig. 1.

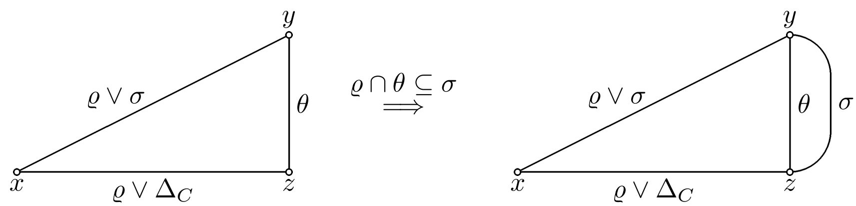

Figure 1.

In particular, if the additional condition $\sigma \subseteq \theta$ is also satisfied, we have the following special case of the above scheme.

An algebra $\mathscr{A}$ satisfies the *Weak Congruence Triangular* m-*Scheme*, briefly the Cw *Triangular* m-*Scheme*, if for all $\varrho, \theta, \sigma \in \operatorname{Cw}\mathscr{A}$, where $\sigma \in \operatorname{Con}\mathscr{C}$, $\mathscr{C} \in \operatorname{Sub}\mathscr{A}$ we have that

$$\varrho \cap \theta \subseteq \sigma \subseteq \theta,\ (z,y) \in \theta,\ (z,x) \in \varrho \vee \Delta_C \text{ and} (x,y) \in \varrho \vee \sigma \text{ imply } (z,y) \in \sigma.$$

The Cw Triangular m-Scheme can be visualized as shown in Fig. 1a.

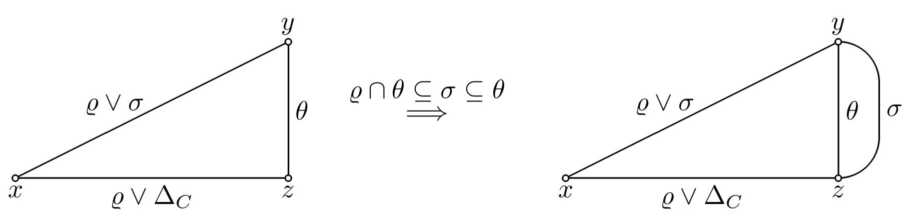

Figure 1a.

Lemma 1. *Let* $\mathbf{Cw}\mathscr{A}$ *be the lattice of weak congruences on an algebra* $\mathscr{A}$ *and let* $\varrho, \sigma \in \operatorname{Cw}\mathscr{A}$, $\varrho \in \operatorname{Con}\mathscr{B}$, $\sigma \in \operatorname{Con}\mathscr{C}$, $\mathscr{B}, \mathscr{C} \in \operatorname{Sub}\mathscr{A}$. *Then* $\varrho \vee \sigma = (\varrho \vee \Delta_C) \vee (\sigma \vee \Delta_B) = (\varrho \vee \Delta_C) \cup (\sigma \vee \Delta_B) \cup (\varrho \vee \Delta_C) \circ (\sigma \vee \Delta_B) \cup (\varrho \vee \Delta_C) \circ (\sigma \vee \Delta_B) \circ (\varrho \vee \Delta_C) \cup \ldots$

P r o o f. The proof is evident. □

Theorem 1. *Let* $\mathrm{Cw}\,\mathscr{A}$ *be the lattice of weak congruences on an algebra* $\mathscr{A}$*. Then* $\mathrm{Cw}\,\mathscr{A}$ *is modular if and only if it satisfies the* Cw *Triangular* m-*Scheme.*

P r o o f. Suppose that $\mathrm{Cw}\,\mathscr{A}$ is modular and let $\varrho \in \mathrm{Con}\,\mathscr{B}$, $\sigma \in \mathrm{Con}\,\mathscr{C}$ and $\theta \in \mathrm{Con}\,\mathscr{D}$ for $\mathscr{B}, \mathscr{C}, \mathscr{D} \in \mathrm{Sub}\,\mathscr{A}$. Let $\varrho \cap \theta \subseteq \sigma \subseteq \theta$, $(z, y) \in \theta$, $(z, x) \in \varrho \vee \Delta_C$ and $(x, y) \in \varrho \vee \sigma$. By virtue of transitivity, we have

$$(z, y) \in (\varrho \vee \Delta_C) \circ (\varrho \vee \sigma) \subseteq (\varrho \vee \sigma) \circ (\varrho \vee \sigma) \subseteq \varrho \vee \sigma.$$

Hence,

$$(z, y) \in \theta \cap (\varrho \vee \sigma) = (\theta \cap \varrho) \vee \sigma = \sigma,$$

proving that $\mathrm{Cw}\,\mathscr{A}$ satisfies the Cw Triangular m-Scheme.

Conversely, suppose that $\mathrm{Cw}\,\mathscr{A}$ satisfies the Cw Triangular m-Scheme and suppose that $\mathrm{Cw}\,\mathscr{A}$ is not modular. Then $\mathrm{Cw}\,\mathscr{A}$ contains a sublattice N_5 as shown in Fig. 2.

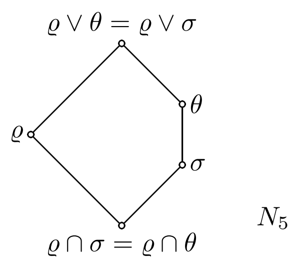

Figure 2.

Obviously, we have $\varrho \cap \theta \subseteq \sigma \subseteq \theta$. Let $(z, y) \in \theta$. Then $(z, y) \in \varrho \vee \sigma$.

Since $\varrho \vee \sigma$ and $\varrho \vee \Delta_C$ belong to the same congruence lattice, we have that $(z, z) \in \varrho \vee \Delta_C$. By $(z, y) \in \varrho \vee \sigma$, applying the Cw Triangular m-Scheme, we conclude that $(z, y) \in \sigma$. Hence $\theta \subseteq \sigma$, which is a contradiction. $\square$

If we use the general Cw Triangular Scheme (not the m-version), then we obtain the following characterization of distributivity for weak congruence lattices.

Theorem 2. *Let* $\mathrm{Cw}\,\mathscr{A}$ *be the lattice of weak congruences on an algebra* $\mathscr{A}$*. Then* $\mathrm{Cw}\,\mathscr{A}$ *is distributive if and only if it satisfies the* Cw *Triangular Scheme.*

P r o o f. The proof is similar to that of the preceding theorem. Still, in order to prove that the Scheme is sufficient for distributivity, we have to consider also the case in which the diamond M_3 is a sublattice of $\mathrm{Cw}\,\mathscr{A}$ (see Fig. 2a). A contradiction

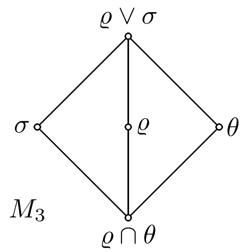

Figure 2a.

is obtained similarly to Theorem 1, for the case of the pentagon N_5. $\square$

Corollary 1. *If an algebra $\mathscr{A}$ satisfies the* Cw *Triangular* m-*Scheme, then $\mathscr{A}$ possesses both the* CEP *and the* CIP.

P r o o f. Obvious, since modularity of a weak congruence lattice is sufficient for an algebra to possess the CEP and the CIP (see [6, Theorem 2.29]). $\square$

By the same argument as above, *algebras satisfying the* Cw *Triangular Scheme* (m-*Scheme) have a distributive (modular) subalgebra lattice.*

As usual, we say that a variety satisfies some property if every algebra in the variety does. The following result is a direct consequence of Theorem 1 above, and of Theorem 3.39 in [6].

Corollary 2. *Let $\mathscr{V}$ be a variety with a nullary operation in its similarity type. Then the following are equivalent:*

(i) *$\mathscr{V}$ is weak congruence modular;*

(ii) *$\mathscr{V}$ satisfies the* Cw *Triangular* m-*Scheme;*

(iii) *$\mathscr{V}$ is polynomially equivalent to the variety of modules over a ring with unit.*

In some particular cases, in congruence permutable varieties, it is possible to apply similarly as above the new property, introduced in the sequel.

We say that an algebra $\mathscr{A}$ satisfies the Cw *Triangular Principle* if for all $\varrho, \theta, \sigma \in \operatorname{Cw}\mathscr{A}$, $\varrho \in \operatorname{Con}\mathscr{B}$, $\sigma \in \operatorname{Con}\mathscr{C}$, $\mathscr{B}, \mathscr{C} \in \operatorname{Sub}\mathscr{A}$, we have that

$$\varrho \cap \theta \subseteq \sigma,\ \ (z,y) \in \theta,\ \ (z,x) \in \varrho \vee \Delta_C \ \text{ and }\ (x,y) \in \sigma \vee \Delta_B \text{imply}\ \ (z,y) \in \sigma.$$

The Cw Triangular Principle can be visualized by a diagram as shown in Fig. 3.

Similarly as for the Scheme, we say that an algebra $\mathcal{A}$ satisfies the Cw *Triangular m-Principle* if for all $\varrho, \theta, \sigma \in \operatorname{Cw}\mathcal{A}$, $\varrho \in \operatorname{Con}\mathscr{B}$, $\sigma \in \operatorname{Con}\mathscr{C}$, $\mathscr{B}, \mathscr{C} \in \operatorname{Sub}\mathscr{A}$ we have

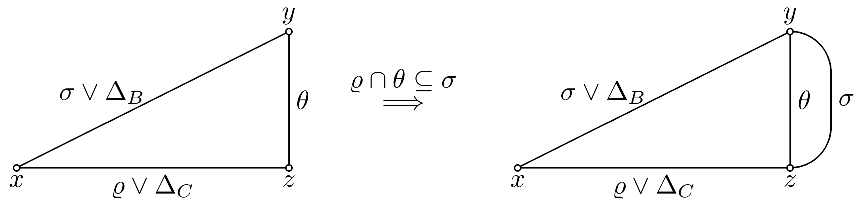

Figure 3.

that

$$\varrho \cap \theta \subseteq \sigma \subseteq \theta$$

and

$$(z, y) \in \theta, \ \ (z, x) \in \varrho \vee \Delta_C, \ \ (x, y) \in \sigma \vee \Delta_B \ \text{ imply } \ (z, y) \in \sigma.$$

The Cw Triangular m-Principle can be visualized by a diagram as shown in Fig. 3a.

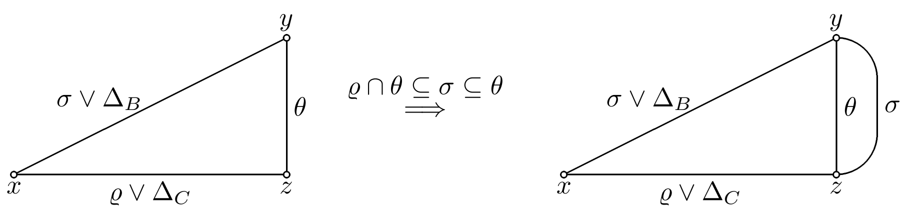

Figure 3a.

By $\sigma \vee \Delta_B \subseteq \sigma \vee \varrho$, it is obvious that Cw *Triangular Schemes imply the corresponding Principles.* Therefore we have the following propositions.

Corollary 3. *If* $\mathrm{Cw}\,\mathscr{A}$ *is distributive, then the algebra* $\mathscr{A}$ *satisfies the* Cw *Triangular Principle.*

Corollary 4. *If* $\mathrm{Cw}\,\mathscr{A}$ *is modular, then* $\mathscr{A}$ *satisfies the* Cw *Triangular* m-*Principle.*

Lemma 2. *If* $\varrho, \theta \in \mathrm{Cw}\,\mathscr{A}$, *where* $\varrho \in \mathrm{Con}\,\mathscr{B}$, $\theta \in \mathrm{Con}\,\mathscr{C}$, $\mathscr{B}, \mathscr{C} \in \mathrm{Sub}\,\mathscr{A}$ *and* $\mathrm{Con}(\mathscr{B} \vee \mathscr{C})$ *is permutable then*

$$\varrho \vee \theta = (\varrho \vee \Delta_C) \circ (\theta \vee \Delta_B).$$

P r o o f. By Lemma 1, $\varrho \vee \theta = (\varrho \vee \Delta_C) \vee (\theta \vee \Delta_B)$ and $\varrho \vee \Delta_C$ and $\theta \vee \Delta_B$ belong to the same congruence lattice $\operatorname{Con}(\mathscr{B} \vee \mathscr{C})$. Since $\operatorname{Con}(\mathscr{B} \vee \mathscr{C})$ is permutable, we conclude that

$$\varrho \vee \theta = (\varrho \vee \Delta_C) \circ (\theta \vee \Delta_B).$$

$\square$

Theorem 3. *If* $\operatorname{Con} \mathscr{B}$ *is permutable for every* $\mathcal{B} \in \operatorname{Sub} \mathcal{A}$, *then* $\operatorname{Cw} \mathcal{A}$ *is modular if and only if* $\mathcal{A}$ *satisfies the* Cw *Triangular* m-*Principle.*

P r o o f. One direction is stated in Corollary 4. To prove the converse, suppose that $\operatorname{Cw} \mathscr{A}$ is not modular. Then it contains a sublattice isomorphic to N_5 (Fig. 2).

Clearly, $\varrho \cap \theta \subseteq \sigma \subseteq \theta$.

Let $(z, y) \in \theta$. Then $(z, y) \in \varrho \vee \sigma = (\varrho \vee \Delta_C) \circ (\sigma \vee \Delta_B)$ for $\varrho \in \operatorname{Con} \mathscr{B}$, $\sigma \in \operatorname{Con} \mathscr{C}$ due to Lemma 2. Hence, there exists $x \in A$ such that $(z, x) \in \varrho \vee \Delta_C$ and $(x, y) \in \sigma \vee \Delta_B$.

By the Cw m-Triangular Principle, we conclude $(z, y) \in \sigma$. Thus $\theta \subseteq \sigma$, a contradiction. $\square$

Theorem 4. *If* $\operatorname{Con} \mathscr{B}$ *is permutable for every* $\mathscr{B} \in \operatorname{Sub} \mathscr{A}$, *then* $\operatorname{Cw} \mathscr{A}$ *is distributive if and only if* $\mathscr{A}$ *satisfies the* Cw *Triangular Principle.*

P r o o f. Since $\mathscr{A}$ satisfies the Cw Triangular Principle, it satisfies also the Cw Triangular m-Principle and, by Theorem 3, $\operatorname{Cw} \mathscr{A}$ is modular. Suppose on the contrary that $\operatorname{Cw} \mathscr{A}$ is not distributive. Then there exists a sublattice M_3 of $\operatorname{Cw} \mathscr{A}$ as shown in Fig. 2a. Suppose $\varrho \in \operatorname{Con} B$, $\sigma \in \operatorname{Con} \mathscr{C}$ for some $\mathscr{B}, \mathscr{C} \in \operatorname{Sub} \mathscr{A}$. Obviously, $\varrho \cap \theta \subseteq \sigma$. Suppose $(z, y) \in \theta$. Then $(z, y) \in \varrho \vee \sigma$ and, by Lemma 2, $(z, y) \in (\varrho \vee \Delta_C) \vee (\sigma \vee \Delta_B)$, where $\varrho \vee \Delta_C, \sigma \vee \Delta_B \in \operatorname{Con} \mathscr{B} \vee \mathscr{C}$. Since $\mathscr{B} \vee \mathscr{C} \in \operatorname{Sub} \mathscr{A}$, it has permutable congruences and thus $(z, y) \in (\varrho \vee \Delta_C) \circ (\sigma \vee \Delta_B)$. Applying the Cw Triangular Principle, we conclude $(z, y) \in \sigma$ proving $\theta \subseteq \sigma$, a contradiction. Thus $\operatorname{Cw} \mathscr{A}$ is distributive.

The converse implication is shown by Corollary 3. $\square$

Hence, *in congruence permutable varieties, the* Cw *Triangular Principle is equivalent to weak congruence distributivity and the* Cw *Triangular* m-*Principle is equivalent to weak congruence modularity.*

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