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ON FINITELY GENERATED MULTIPLICATION MODULES

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Abstract. We shall prove that if M is a finitely generated multiplication module and $\mathrm{Ann}(M)$ is a finitely generated ideal of R, then there exists a distributive lattice $\overline{M}$ such that $\mathrm{Spec}(M)$ with Zariski topology is homeomorphic to $\mathrm{Spec}(\overline{M})$ to Stone topology. Finally we shall give a characterization of finitely generated multiplication R-modules M such that $\mathrm{Ann}(M)$ is a finitely generated ideal of R.

Keywords: prime submodules, multiplication modules, distributive lattices, spectral spaces

MSC 2000: 13C13, 13C99

1. Introduction

Throughout this note all rings are commutative with identity and all modules are unital.

For any submodule N of an R-module M, we define $(N : M) = \{r \in R\colon\ rM \subseteq N\}$ and denote $(0 : M)$ by $\mathrm{Ann}(M)$. A submodule P of M is called prime if $P \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in P$, then $m \in P$ or $r \in (P : M)$. It is easy to show that if P is a prime submodule of an R-module M, then $(P : M)$ is a prime ideal of R. The set of all prime submodules of M is denoted by $\mathrm{Spec}(M)$. As defined in [4] the radical of a submodule N of an R-module M is given by $\mathrm{rad}(N) = \bigcap P$, where the intersection is over all prime submodules of M containing N. If there is no prime submodule containing N, then we define $\mathrm{rad}(N) = M$. The radical of an ideal I of R is denoted by $\sqrt{I}$.

An R-module M is called a multiplication module provided for any submodule N of M there exists an ideal I of R such that $N = IM$. It is easy to check that M is a multiplication module if and only if $N = (N : M)M$ for every submodule N of M (see [8]).

In this paper at first we shall construct a distributive lattice $\overline{M}$ and discuss some properties of $\operatorname{Spec}(\overline{M})$, where $\operatorname{Spec}(\overline{M})$ is the set of all prime ideals in the lattice $\overline{M}$. We shall then prove that if M is a finitely generated multiplication module and $\operatorname{Ann}(M)$ is a finitely generated ideal of R, then $\operatorname{Spec}(M)$ and $\operatorname{Spec}(\overline{M})$ are homeomorphic. Finally we shall generalize the notion of reticulated and semi-reticulated rings for modules and characterize some classes of semi-reticulated modules.

2. On the lattice $\overline{M}$ and its prime spectrum

Let R be a ring and let $\mathrm{FI}(R)$ be the set of all finitely generated ideals of R. Now let M be an R-module and $\mathrm{su}(M)$ the $\mathrm{FI}(R)$-semimodule generated by the principal R-submodules of M and M under the operations $N+K$, IN, where $N, K \in \mathrm{su}(M)$ and $I \in \mathrm{FI}(R)$. Hence

$$\mathrm{su}(M) = \left\{ \sum_{i=1}^{k} I_i R m_i + J_i M \colon\ I_i, J_i \in \mathrm{FI}(R),\ \ m_i \in M,\ \ k \in \mathbb{N} \right\}.$$

It is clear that if M is a finitely generated R-module then $\mathrm{su}(M)$ is the set of all finitely generated submodules of M.

Define the equivalence relation on $\mathrm{su}(M)$, "$\sim$" by $N \sim L$ if and only if $\mathrm{rad}(N) = \mathrm{rad}(L)$ [6, p. 1470], and denote the resulting set of equivalence classes by $\overline{M}$; i.e., $\overline{M} = \{[K] \colon\ K \in \mathrm{su}(M)\}$.

Lemma 2.1. *Let $N, N', K, K' \in \mathrm{su}(M)$ and $I, I' \in \mathrm{FI}(R)$. If $N \sim N'$ and $K \sim K'$, then we have*
(i) *$(N+K) \sim (N'+K')$;*
(ii) *if $\sqrt{I} = \sqrt{I'}$ then $IN \sim I'N'$.*

Proof. (i) By [6, Lemma 1.5].
(ii) Let $P \in \operatorname{Spec}(M)$ and $IN \subseteq P$. Hence $N \subseteq P$ or $I \subseteq (P : M)$. If $N \subseteq P$ then $I'N' \subseteq N' \subseteq \mathrm{rad}(N') = \mathrm{rad}(N) \subseteq P$. Suppose that $I \subseteq (P : M) \in \operatorname{Spec}(R)$. Hence $I' \subseteq \sqrt{I'} = \sqrt{I} \subseteq (P : M)$. Thus $I'N' \subseteq I'M \subseteq P$. Therefore $\mathrm{rad}(I'N') \subseteq \mathrm{rad}(IN)$. Similarly $\mathrm{rad}(IN) \subseteq \mathrm{rad}(I'N')$ and hence $IN \sim I'N'$. □

Let $[N]$, $[K]$ belong to $\overline{M}$ and $I \in \mathrm{FI}(R)$.

We define $[N]+[K] := [N+K]$ and $I[N] := [IN]$. Then by Lemma 2.1, $\overline{M}$ becomes an $\mathrm{FI}(R)$-semimodule. Furthermore we define $[N] \leqslant [K]$ if for each $P \in \operatorname{Spec}(M)$, $K \subseteq P$ implies that $N \subseteq P$. Therefore $(\overline{M}, \leqslant)$ is a partially ordered set.

Let N be a subset of M.

We define $\overline{M}(N) = \{[L] \in \overline{M} \colon L \sim K$, for some $K \subseteq N\}$. If $0 \in N$ then $[0] \in \overline{M}(N)$ and hence $\overline{M}(N) \neq \emptyset$.

Now let N be a subset of $\overline{M}$. We define $M[N] = \{x \in M \colon [Rx] \in N\}$. If $[0] \in N$ then $0 \in M[N]$ and hence $M[N] \neq \emptyset$.

Lemma 2.2. *Let $P \in \operatorname{Spec}(M)$. Then $M[\overline{M}(P)] = P$.*

Proof. Let $x \in P$. Then $Rx \subseteq P$ and hence $[Rx] \in \overline{M}(P)$. Therefore $x \in M[\overline{M}(P)]$. Now let $x \in M[\overline{M}(P)]$. Then $[Rx] \in \overline{M}(P)$ and so $Rx \sim L$ for some $L \subseteq P$. Thus $Rx \subseteq \operatorname{rad}(Rx) = \operatorname{rad}(L) \subseteq P$. Hence $x \in P$ □

For the remainder of this section we let M be a finitely generated multiplication R-module and $\operatorname{Ann}(M)$ a finitely generated ideal of R.

Proposition 2.3. *Suppose that M is an R-module. Then $(\overline{M}, \leqslant)$ is a distributive lattice.*

Proof. Put $0 := [0_M]$ and $1 := [M]$.

Define for any $N, K \in \operatorname{su}(M)$; $[N] \vee [K] := [N + K]$ and $[N] \wedge [K] = [(N : M)K]$. Since M, N and $\operatorname{Ann}(M)$ are finitely generated, by [8, Proposition 13], $(N : M)$ is finitely generated. Therefore $(N : M) \in \operatorname{FI}(R)$ and so $(N : M)K \in \operatorname{su}(M)$. Since M is a multiplication module, the infimum of $[N]$ and $[K]$ is well-defined.

We now show that $\overline{M}$ is a distributive lattice. It is enough to show that

$$[(N : M)K + L] = [(N + L : M)(K + L)].$$

Let $P \in \operatorname{Spec}(M)$ be such that $(N : M)K + L \subseteq P$. Then $(N : M)K \subseteq P$ and $L \subseteq P$. Hence $K \subseteq P$ or $(N : M) \subseteq (P : M)$. If $K \subseteq P$ then $(N + L : M)K \subseteq P$ and since $(N+L : M)L \subseteq P$, we get $(N+L : M)(K+L) \subseteq P$. If $(N : M) \subseteq (P : M)$, then since M is a multiplication module, $N = (N : M)M \subseteq P$. Hence $(N + L) \subseteq P$ and so $(N + L : M)K \subseteq P$. Therefore

$$[(N : M)K + L] \leqslant [(N + L : M)(K + L)].$$

Similarly $[(N + L : M)(K + L)] \leqslant [(N : M)K + L]$. □

Let N be an ideal of $\overline{M}$. Since $R(x + y) \subseteq Rx + Ry$ and $R(rx) \subseteq Rx$, where $x, y \in M$ and $r \in R$, we see that $M(N)$ is an R-submodule of M.

Lemma 2.4. *If M is a finitely generated R-module, then $\overline{M}(M[N]) = N$, for all ideals N of $\overline{M}$.*

P r o o f. It is clear that $N \subseteq \overline{M}(M[N])$. Let $[L] \in \overline{M}(M[N])$. Then for some finitely generated $K \in \mathrm{su}(M)$, $K \in [L]$ and $K \subseteq M[N]$. Suppose that $K = \sum_{i=1}^{n} m_i R$. Therefore we have $[L] = [K] = \sum_{i=1}^{n} [m_i R] \in N$. We conclude that $\overline{M}(M[N]) = N$ and the proof is complete. □

Lemma 2.5. *Let M be an R-module and $N \in \mathrm{Spec}(\overline{M})$. Then $M[N] \in \mathrm{Spec}(M)$.*

P r o o f. If $M[N] = M$ then by Lemma 2.4, $N = \overline{M}(M[N]) = \overline{M}(M) = \overline{M}$, which is a contradiction. Suppose that $N \in \mathrm{Spec}(\overline{M})$ and $rm \in M[N]$, $r \in R$, $m \in M$. Then $[Rrm] \in \overline{M}(M[N]) = N$. Since M is a multiplication module, so $(Rm : M)M = Rm$. Hence $[(Rm : M)rM] = [Rm] \wedge [rM] = [Rrm] \in N$, and so $[Rm] \in N$ or $[rM] \in N$. If $[Rm] \in N$ then $m \in M[N]$. Now if $[rM] \in N$, then $rM \subseteq M[N]$. □

Proposition 2.6.

(i) *If $N \in \mathrm{Spec}(\overline{M})$ then $\overline{M}(M[N]) = N$.*

(ii) *For every ideal N of $\overline{M}$,*

$$N \subseteq \overline{M}(M[N]) \subseteq \mathrm{rad}(N) = \bigcap\{P \in \mathrm{Spec}(\overline{M})\colon\ N \subseteq P\}.$$

P r o o f. (i) Clearly $N \subseteq \overline{M}(M[N])$. Let $[K] \in \overline{M}(M[N])$. Hence there exists $L \subseteq M[N]$ such that $L \sim K$. By Lemma 2.5, $M[N] \in \mathrm{Spec}(M)$ and so $K \subseteq M[N]$. Since $K \in \mathrm{su}(M)$, we have $K = \sum_{i=1}^{t} I_i m_i R + J_i M$, where $I_i, J_i \in \mathrm{FI}(R)$ and $m_i \in M$. Therefore $I_i m_i R \subseteq M[N]$ and $J_i M \subseteq M[N]$, for all i. Thus $m_i \in M[N]$ or $I_i \subseteq (M[N] : M)$. If $m_i \in M[N]$ then $[m_i R] \in N$. Since $[I_i m_i R] \leqslant [m_i R]$, we get $[I_i m_i R] \in N$. Now if $I_i \subseteq (M[N] : M)$ then $[I_i m_i R] \in N$. Therefore $[I_i m_i R] \in N$, for all i. By a similar proof $[J_i M] \in N$. We conclude that $[K] \in N$.

(ii) Let N be any ideal of $\overline{M}$. If $N = \overline{M}$ then clearly $N = \overline{M}(M[N])$. Therefore assume that $N \neq \overline{M}$. Let $[K] \in \overline{M}(M[N])$. Hence $K \sim L$, for some $L \subseteq M[N]$. Choose a $P \in \mathrm{Spec}(\overline{M})$, with $N \subseteq P$, then $M[N] \subseteq M[P]$. By Lemma 2.5, $M[P] \in \mathrm{Spec}(M)$ and hence $K \subseteq \mathrm{rad}(L) \subseteq \mathrm{rad}(M[N]) \subseteq M[P]$. Thus $[K] \in \overline{M}(M[P]) = P$ (by (i)). So $[K] \in \mathrm{rad}(N)$. Therefore $\overline{M}(M[N]) \subseteq \mathrm{rad}(N)$. The proof is complete. □

Lemma 2.7. *Let M be an R-module and N a submodule of M. Then $\overline{M}(N)$ is an ideal in the lattice $\overline{M}$.*

Proof. Let $[L_1], [L_2] \in \overline{M}(N)$. Then there exist $K_1 \subseteq N$ and $K_2 \subseteq N$ such that $K_1 \sim L_1$ and $K_2 \sim L_2$. By Lemma 2.1, $(K_1 + K_2) \sim (L_1 + L_2)$ and so $[L_1] \vee [L_2] = [L_1 + L_2] \in \overline{M}(N)$. Now assume that $[L] \in \overline{M}(N)$, $[K] \in \overline{M}$ and $[K] \leqslant [L]$. We must show that $[K] \in \overline{M}(N)$. There exists $L' \subseteq N$, $L' \sim L$. Put $L_1 = (K : M)L'$. It is clear that $L_1 \subseteq N$. Let $Q \in \operatorname{Spec}(M)$ and $L_1 \subseteq Q$. Then $L' \subseteq Q$ or $(K : M) \subseteq (Q : M)$. If $L' \subseteq Q$ then $L \subseteq \operatorname{rad}(L) = \operatorname{rad}(L') \subseteq Q$ and hence $K \subseteq Q$, because $[K] \leqslant [L]$. Now if $(K : M) \subseteq (Q : M)$ then $K \subseteq Q$. Clearly $(K : M)L' \subseteq K \subseteq Q$. Thus $K \sim L_1$ and so $[K] \in \overline{M}(N)$. □

Let N be a submodule of $\overline{M}$.

Put $(N : \overline{M}) = \{J \in \mathrm{FI}(R)$: for all $[K] \in \overline{M}$, there exists $[L] \in N$; $J[K] \leqslant [L]\}$. It is easy to show that $(N : \overline{M})$ is an ideal of $\mathrm{FI}(R)$, i.e. $J_1 + J_2 \in (N : \overline{M})$, $IJ \in (N : \overline{M})$, where $J_1, J_2, J \in (N : \overline{M})$ and $I \in \mathrm{FI}(R)$.

Proposition 2.8. *Let M be an R-module. Then $P \in \operatorname{Spec}(\overline{M})$ if and only if $(P : \overline{M}) \in \operatorname{Spec}(\mathrm{FI}(R))$.*

Proof. Let $(P : \overline{M}) = \mathrm{FI}(R)$. By assumption $P \neq \overline{M}$, so there exists $[K] \in \overline{M} \setminus P$. Since $R \in (P : \overline{M})$, we have $R[K] = [K] \leqslant [L]$, for some $[L] \in P$. So $[K] \in P$, which is a contradiction. Therefore $(P : \overline{M}) \neq \mathrm{FI}(R)$. Assume that $I, J \in \mathrm{FI}(R)$ are such that $IJ \in (P : \overline{M})$. Let $[K] \in \overline{M}$. Then there exists $[L] \in P$ such that $IJ[K] \leqslant [L]$ and so $[IJK] \in P$. Clearly $[(IK : M)JM] \leqslant [IJK]$. Hence $[IK] \wedge [JM] = [(IK : M)JK] = [IJK] \in P$, and so $[IK] \in P$ or $[JM] \in P$. We conclude that $I \in (P : \overline{M})$ or $J \in (P : \overline{M})$ and $(P : \overline{M}) \in \operatorname{Spec}(\mathrm{FI}(R))$. Conversely, let $[K] \wedge [L] = [(K : M)L] \in P$ and $[T] \in \overline{M}$. Since $[(K : M)(L : M)T] \leqslant [(K : M)L]$, we have $(K : M) \in (P : \overline{M})$ or $(L : M) \in (P : \overline{M})$. If $(K : M) \in (P : \overline{M})$ then $[(K : M)M] \in P$. Since M is a multiplication module, $[K] = [(K : M)M] \in P$. Similarly $[L] \in P$ and hence $P \in \operatorname{Spec}(\overline{M})$. □

Lemma 2.9. *Let M be an R-module. If $P \in \operatorname{Spec}(M)$ then $\overline{M}(P) \in \operatorname{Spec}(\overline{M})$.*

Proof. Assume that $\overline{M}(P) = \overline{M}$. By Lemma 2.2, $P = M[\overline{M}] = M$, which is a contradiction. Now let $[K] \wedge [L] = [(K : M)L] \in \overline{M}(P)$. Then there exists $L' \subseteq P$ such that $(K : M)L \sim L'$. Therefore $(K : M)L \subseteq \operatorname{rad}(L') \subseteq P$. So $L \subseteq P$ or $K = (K : M)M \subseteq P$. Thus $[K] \in \overline{M}(P)$ or $[L] \in \overline{M}(P)$. □

3. Topologies on Spec(M) and Spec($\overline{M}$)

We begin this section by introducing a topology called the Zariski topology on Spec(M) for any R-module M, in which closed sets are varieties

$$V(N) = \{P \in \operatorname{Spec}(M)\colon\ (N : M) \subseteq (P : M)\}$$

of all submodules N of M [2, Proposition 1.1]. Similarly, for any ideal L of $\overline{M}$, put

$$\overline{V}(L) = \{Q \in \operatorname{Spec}(\overline{M})\colon\ L \subseteq Q\}.$$

For the remainder of this section we let M be a finitely generated multiplication R-module and Ann(M) a finitely generated ideal of R.

Lemma 3.1. *Let M be an R-module. Put $\overline{T} = \{\overline{V}(L) \mid L$ is an ideal of $\overline{M}\}$. Then $\overline{T}$ is the collection of closed sets of the Stone topology on* Spec($\overline{M}$).

Proof. It is easy to show that $\overline{V}([0]) = \operatorname{Spec}(\overline{M})$ and $\overline{V}(\overline{M}) = \emptyset$. Let L and N be ideals of $\overline{M}$. We show that $\overline{V}(L) \cup \overline{V}(N) = \overline{V}(L \cap N)$. Suppose that $Q \in \operatorname{Spec}(\overline{M})$ is such that $L \cap N \subseteq Q$ and $L \not\subseteq Q$. Then there exists $[K] \in L \setminus Q$. Let $[K_1] \in N$. Clearly $[K] \wedge [K_1] \in L \cap N$. Therefore $[K_1] \in Q$. Hence $\overline{V}(L \cap N) \subseteq \overline{V}(L) \cup \overline{V}(N)$. It is clear that $\overline{V}(L) \cup \overline{V}(N) \subseteq \overline{V}(L \cap N)$. Let $\{N_i \mid i \in I\}$ be a family of ideals of $\overline{M}$. Then $\bigcap\limits_{i\in I} \overline{V}(N_i) = \overline{V}\Bigl(\sum\limits_{i\in I} N_i\Bigr)$. □

For any subset $X \subseteq \operatorname{Spec}(M)$, let $\overline{X} = \{\overline{M}(P)\colon\ P \in X\}$. Since M is a finitely generated multiplication R-module and Ann(M) is a finitely generated ideal of R, by Lemma 2.9 $\overline{X} \subseteq \operatorname{Spec}(\overline{M})$.

Lemma 3.2. *Let M be an R-module. Then for each submodule N of M, $\overline{V(N)} = \overline{V}(\overline{M}(N))$.*

Proof. Let $\overline{M}(P) \in \overline{V(N)}$, so $P \in V(N)$. Thus $(N : M) \subseteq (P : M)$. Let $[L] \in \overline{M}(N)$. Then $L \sim L'$, for some $L' \subseteq N$. But $(N : M)M = N$ and hence $L' \subseteq P$. Therefore $[L] \in \overline{M}(P)$. We conclude that $\overline{M}(N) \subseteq \overline{M}(P)$ and so $\overline{V(N)} \subseteq \overline{V}(\overline{M}(N))$. Now let $Q \in \overline{V}(\overline{M}(N))$, then $\overline{M}(N) \subseteq Q$. By Lemma 2.5, $M[Q] = P \in \operatorname{Spec}(M)$. Hence by Lemma 2.4, $\overline{M}(P) = \overline{M}(M[Q]) = Q$. We claim that $(N : M) \subseteq (P : M)$. If $rM \subseteq N$ then $[rM] \in \overline{M}(N) \subseteq Q$ and so $rR[M] \in Q$. Hence $rM \subseteq M[Q] = P$. We conclude that $Q \in \overline{V(N)}$. □

Put $T = \{V(N) \mid N$ is a submodule of $M\}$.

Theorem 3.3. *Let M be a finitely generated multiplication R-module and* $\mathrm{Ann}(M)$ *a finitely generated ideal of R. Then the topological spaces* $(\mathrm{Spec}(M),T)$ *and* $(\mathrm{Spec}(\overline{M}),\overline{T})$ *are homeomorphic.*

Proof. Define

$$\varphi\colon \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(\overline{M}); \quad \varphi(P) = \overline{M}(P)$$

and

$$\psi\colon \mathrm{Spec}(\overline{M}) \longrightarrow \mathrm{Spec}(M); \quad \psi(L) = M[L].$$

By Lemmas 2.9 and 2.5, φ and ψ are well-defined. By Lemmas 2.2 and 2.4, we have

$$\psi \circ \varphi(P) = \psi(\overline{M}(P)) = M[\overline{M}(P)] = P$$

and

$$\varphi \circ \psi(L) = \varphi(M[L]) = \overline{M}(M[L]) = L.$$

Hence the two mappings φ and ψ are inverses of each other. The bijection φ induces a map $\overline{\varphi}\colon T \longrightarrow \overline{T}$ by $\overline{\varphi}(V(N)) = \overline{V(N)}$. By Lemma 3.2, $\overline{V(N)} = \overline{V}(\overline{M}(N))$ and so $\overline{\varphi}$ is well-defined. We claim that this induced map is also a bijection. Suppose $\overline{V(N)} = \overline{V(L)}$. By Lemma 3.2, we have $\overline{V}(\overline{M}(N)) = \overline{V}(\overline{M}(L))$. We must show that $V(N) = V(L)$. Let $P \in \mathrm{Spec}(M)$ and $(N : M) \subseteq (P : M)$. Suppose that $rM \subseteq L$. Hence $[rM] \in \overline{M}(L)$. Since $\overline{M}(P) \in \overline{V}(\overline{M}(N)) = \overline{V}(\overline{M}(L))$, we get $[rM] \in \overline{M}(P)$. Therefore $rM \subseteq P$. We conclude that $P \in V(L)$ and so $V(N) \subseteq V(L)$. By symmetry we infer that $V(L) = V(N)$. Hence $\overline{\varphi}$ is one-to-one. Now let $\overline{V}(L) \in \overline{T}$. Since L is an ideal of $\overline{M}$, we have $\overline{\varphi}(V(M[L])) = \overline{V}(\overline{M}(M[L])) = \overline{V}(L)$ and so $\overline{\varphi}$ is onto. □

Following M. Hochster [3], we say that a topological space W is a spectral space if W is homeomorphic to $\mathrm{Spec}(S)$ with the Zariski topology, for some ring S.

Definition. A semi-reticulation for an R-module M is a pair $(\overline{M},\lambda)$ where $\overline{M}$ is a distributive lattice with 0, 1 and $\lambda\colon M \longrightarrow \overline{M}$ is a mapping such that

(I) $\lambda(x+y) \leqslant \lambda(x) \vee \lambda(y)$;

(II) $\lambda(rx) \leqslant \lambda(x) \wedge \lambda(y)$, for some $y \in rM$;

(III) $\lambda(0) = 0$;

(IV) the inverse image map induced by λ is a homeomorphism between $\mathrm{Spec}(\overline{M})$ and $\mathrm{Spec}(M)$ (with the Stone and the Zariski topologies respectively).

Moreover, if $\lambda(m) = 1$, for some $m \in M$, then we say that M has a reticulation (this generalizes [7]).

Theorem 3.5. *Let M be a finitely generated R-module and* $\operatorname{Ann}(M)$ *be a finitely generated ideal of R. Then the following are equivalent.*

(i) *M is a multiplication module;*

(ii) *there exists a semi-reticulation for M;*

(iii) $\operatorname{Spec}(M)$ *is spectral.*

P r o o f. (i) → (ii) Define $\lambda\colon M \longrightarrow \overline{M}$ by $\lambda(x) = [Rx]$, where $x \in M$. Clearly (I), (II) and (III) are satisfied. By Theorem 3.3, we have $\lambda^{-1}(Q) = \psi(Q)$. Hence the inverse image map induced by λ is a homeomorphism between $\operatorname{Spec}(\overline{M})$ and $\operatorname{Spec}(M)$.

(ii) → (iii) It is well known that the prime ideal space of a distributive lattice with 0, 1, is spectral under the Stone topology (see [1]). By Proposition 2.3 and Theorem 3.3, $\operatorname{Spec}(M)$ is spectral.

(iii) → (i) By [5, Corollary 6.6]. □

Corollary 3.6. *Let M be a finitely generated R-module. Suppose that R is a Noetherian ring or M is a faithful module (i.e.* $\operatorname{Ann}(M) = 0$*). Then M is multiplication if and only if M has a semi-reticulation.*

Corollary 3.7. *Let M be a cyclic R-module. Suppose that R is a Noetherian ring or M is a faithful module, then M has a reticulation.*

P r o o f. By Corollary 3.6, M has a semi-reticulation. Since M is a cyclic R-module, there exists $m \in M$ such that $Rm = M$. Therefore $\lambda(m) = [Rm] = [M] = 1$. We conclude that M has a reticulation. □

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