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ON NONREGULAR IDEALS AND z°-IDEALS IN $C(X)$

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Abstract. The spaces X in which every prime z°-ideal of $C(X)$ is either minimal or maximal are characterized. By this characterization, it turns out that for a large class of topological spaces X, such as metric spaces, basically disconnected spaces and one-point compactifications of discrete spaces, every prime z°-ideal in $C(X)$ is either minimal or maximal. We will also answer the following questions: When is every nonregular prime ideal in $C(X)$ a z°-ideal? When is every nonregular (prime) z-ideal in $C(X)$ a z°-ideal? For instance, we show that every nonregular prime ideal of $C(X)$ is a z°-ideal if and only if X is a ∂-space (a space in which the boundary of any zeroset is contained in a zeroset with empty interior).

Keywords: z°-ideal, prime z-ideal, nonregular ideal, almost P-space, ∂-space, m-space

MSC 2000: 54C40

1. Introduction

Important ideals concerning primes in $C(X)$ are z-ideals. A special case of z-ideals consisting entirely of zero divisors are z°-ideals which play a fundamental role in studying nonregular prime ideals. We will investigate the relations between ideals consisting entirely of zero divisors, such as z°-ideals, nonregular prime ideals, prime z°-ideals and so on. We will also characterize the topological *spaces* X for which some of these ideals in $C(X)$ coincide. In a commutative ring R, an ideal I consisting entirely of zero divisors is called a nonregular ideal. For each $a \in R$, let P_a be the intersection of all minimal prime ideals containing a. A proper ideal I is called a z°-ideal if for each $a \in I$ we have $P_a \subseteq I$, see [3] and [4]. Clearly P_a itself is a z°-ideal. In $C(X)$, the ideal P_f, $f \in C(X)$ is both an algebraic and a topological object which is presented in Propositions 2.2 and 2.3 in [2] as follows:

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Proposition 1.1. *For every $f \in C(X)$, we have*

$$P_f = \{g \in C(X) \colon \operatorname{Ann}(f) \subseteq \operatorname{Ann}(g)\} = \{g \in C(X) \colon \operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)\}.$$

It is easy to see that an ideal I in $C(X)$ is a z°-ideal if and only if $f \in I$ and $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$ imply that $g \in I$. For the other equivalent definitions of z°-ideals in $C(X)$, see Proposition 2.2 in [3]. Important z°-ideals in any ring are minimal prime ideals. For every $f \in C(X)$, $\operatorname{Ann}(f)$ and $\forall x \in X$, O_x are z°-ideals in $C(X)$. If $S \subseteq X$ is a regular closed set in X, i.e., if $\operatorname{cl}(\operatorname{int} S) = S$, then $M_S = \{f \in C(X) \colon S \subseteq Z(f)\}$ is also a z°-ideal in $C(X)$. In particular, whenever $Z(f)$ is regular closed, then $M_f = \{g \in C(X) \colon Z(f) \subseteq Z(g)\}$, the intersection of all maximal ideals containing f, is a z°-ideal. We recall that I is a z-ideal in a ring R if $a \in I$ implies that $M_a \subseteq I$, where M_a is the intersection of all maximal ideals containing a. Equivalently, I is a z-ideal in $C(X)$ if $f \in I$ and $Z(f) \subseteq Z(g)$ imply that $g \in I$. It is easy to see that every z°-ideal is a z-ideal but not convesely, see [3], Remark 2.4.

Nonregular ideals and z°-ideals are investigated in [3] and [4] in an arbitrary reduced commutative rings and in $C(X)$ and it is shown that every nonregular ideal (in a reduced ring with some property, see [4] and in $C(X)$, see [3]) is contained in a z°-ideal. We give a short proof for this result in $C(X)$.

Proposition 1.2. *If I is a nonregular ideal in $C(X)$, then I is contained in a z°-ideal.*

Proof. $J = \sum\limits_{f \in I} P_f$ is a z°-ideal and $I \subseteq J$. To see this, we note that each element of J is a zero divisor, i.e., J is a proper ideal. Now let $h = h_1 + \ldots + h_n$, where $h_i \in P_{f_i}$, $i = 1, 2, \ldots, n$; then $h \in P_f$, where $f = f_1^2 + \ldots + f_n^2$, i.e., $h \in J$. $\square$

The proof of the following proposition is similar to that of Theorem 14.7 in [7] and hence we leave it to the reader, see also [3] and [4].

Proposition 1.3. *If I is a z°-ideal and P is a prime ideal in $C(X)$ minimal over I, then P is also a z°-ideal.*

Corollary 1.4. *Every nonregular ideal in $C(X)$ is contained in a prime z°-ideal. In particular, every nonregular maximal ideal is z°-ideal.*

In [3], the spaces X in which every prime z°-ideal in $C(X)$ is minimal are investigated. By Proposition 1.26 and Theorem 1.28 in [4] and Corollary 5.5 in [8], the equivalence of the first two parts of the following proposition is immediate.

Proposition 1.5. *The following statements are equivalent:*

(i) *Every prime z°-ideal in $C(X)$ is minimal.*

(ii) *For any zeroset Z in X there exists a zeroset F in X such that $Z \cup F = X$ and $\operatorname{int} Z \cap \operatorname{int} F = \emptyset$.*

(iii) *For any zeroset Z in X, $\operatorname{cl}(\operatorname{int} Z)$ is the support of some zeroset in X, i.e., there exists $g \in C(X)$ such that $\operatorname{cl}(\operatorname{int} Z(g) = \operatorname{cl}(X \setminus Z(g)$.*

Proof. (ii) $\Leftrightarrow$ (iii) Suppose that $\forall f \in C(X)$, $\exists g \in C(X)$ such that $\operatorname{cl}(\operatorname{int} Z(f)) = \operatorname{cl}(X \setminus Z(g))$. Then $\operatorname{cl}(\operatorname{int} Z(f)) = X \setminus \operatorname{int} Z(g))$ implies that $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$ and $Z(f) \cup Z(g) \supseteq Z(f) \cup \operatorname{int} Z(g) = Z(f) \cup (X \setminus \operatorname{cl}(\operatorname{int} Z(f)) \supseteq Z(f) \cup (X \setminus Z(f)) = X$. Conversely, suppose $\forall f \in C(X)$, $\exists g \in C(X)$ such that $Z(f) \cup Z(g) = X$ and $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$. Therefore $\operatorname{int} Z(f) \subseteq X \setminus \operatorname{int} Z(g) = \operatorname{cl}(X \setminus Z(g)) \subseteq \operatorname{cl}(\operatorname{int} Z(f))$ implies that $\operatorname{cl}(\operatorname{int} Z(f)) \subseteq \operatorname{cl}(X \setminus Z(g)) \subseteq \operatorname{cl}(\operatorname{int} Z(f))$ and hence $\operatorname{cl}(\operatorname{int} Z(f)) = \operatorname{cl}(X \setminus Z(g))$. □

By the above proposition, whenever X is a metric space or a basically disconnected space, then every prime z°-ideal of $C(X)$ is minimal. In this case, in fact for every zeroset Z, $F = X \setminus \operatorname{int} Z$ is also a zeroset and clearly $Z \cup F = X$ and $\operatorname{int} Z \cap \operatorname{int} F = \emptyset$. Existence of spaces X in which every prime z°-ideal in $C(X)$ is minimal or maximal is shown in [3]. This kind of spaces are also investigated in [9] for prime z-ideals in $C(X)$. In [3], it is also shown that there exist spaces X with a prime z°-ideal in $C(X)$ which is neither a minimal nor a maximal ideal. Our aim in Section 3 is characterization of the spaces X in which every prime z°-ideal in $C(X)$ is either minimal or maximal.

We observe that every z°-ideal is a nonregular ideal, but every nonregular ideal need not be even a z-ideal. Clearly the first natural question concerning nonregular ideals, z-ideals and z°-ideals in $C(X)$ are as follows: When is every nonregular ideal (z-ideal) a z°-ideal? In [3], Proposition 2.12, it is shown that X is P-space if and only if every nonregular ideal in $C(X)$ is z°-ideal. In [3], Theorem 2.14, it is also proved that X is an almost P-space if and only if every z-ideal of $C(X)$ is z°-ideal. Now there are three other natural questions which are not answered in [3]. We present these questions as follows:

1. When is every nonregular z-ideal a z°-ideal?
2. When is every nonregular prime z-ideal a z°-ideal?
3. When is every nonregular prime ideal a z°-ideal?

We are going to answer these questions in Section 4. It turns out that for any metric space X, every nonregular prime ideal in $C(X)$ is z°-ideal. By our characterizations, it is also easy to see that for the non-almost P-space $Y = \{0, 1, \frac{1}{2}, \frac{1}{3} \ldots\}$, there is a nonregular z-ideal in $C(Y)$ which is not a z°-ideal.

In the next section, we will study the extension of ideals of $C^*(X)$ in $C(X)$ for later use. Throughout, X will denote a completely regular Hausdorff space and $C(X)$ ($C^*(X)$) is the ring of all (bounded) real valued continuous functions on X. Ideals in $C(X)$ and $C^*(X)$ are considered proper ideals and we refer the readers to [3] and [7] for undefined terms, notations and general information about $C(X)$.

2. Extension of an ideal of $C^*(X)$ in $C(X)$

In [11] Lemma 0.2, it is shown that $C(X)$ is the ring of fractions of $C^*(X)$ with respect to the multiplicatively closed set $S = \{f \in C^*(X) \colon Z(f) = \emptyset\}$. In this section we will investigate the extension of nonregular ideals of $C^*(X)$ in $C(X)$. The *extension* of an ideal I of $C^*(X)$ in $C(X)$ is denoted by $I^{\rm e} = IC(X)$. For an ideal I of $C^*(X)$, we have $I^{\rm e} \neq C(X)$ if and only if $I \cap S = \emptyset$. We denote $I^{\rm e} \cap C^*(X)$ by $I^{\rm ec}$ and call an ideal I in $C^*(X)$ with $I \cap S = \emptyset$ is *contracted* if $I = I^{\rm ec}$. In commutative rings, it is well-known that prime ideals, semiprime ideals and primary ideals disjoint from S are contracted, see [1]. Since for every nonregular ideal I in $C^*(X)$, we have $I \cap S = \emptyset$ and every z°-ideal (minimal prime ideal) in $C^*(X)$ is a nonregular semiprime ideal, see [4], Remark 1.6, the following result is evident.

Proposition 2.1. *z°-ideals and minimal prime ideals of $C^*(X)$ are contracted.*

Proposition 2.2. *If $S^{-1}R$ is the ring of fractions of a commutative ring R with respect to a saturated multiplicatively closed set $S \subseteq R$, and $S^{-1}R \setminus R$ has nonunits, then each ideal I with $I \cap S = \emptyset$ is contracted if and only if $R = S^{-1}R$.*

Proof. If $R = S^{-1}R$, then we are through. Conversely, let $a/s \in S^{-1}R$ with $a/s \notin R$ and also we may assume that $a \notin S$. Now we must have $(as)^{\rm ec} = (as)$. But $a = as/s$ shows that $a \in (as)^{\rm ec} = (as)$, i.e., $a = ast$, $t \in R$. Hence $a/s = at \in R$, which is impossible. $\square$

Now the above fact implies the following corollary.

Corollary 2.3. *Every ideal I in $C^*(X)$ with $I \cap S = \emptyset$ is contracted if and only if X is pseudocompact. (Note that $S = \{f \in C^*(X) \colon Z(f) = \emptyset\}$.)*

Proposition 2.4. *Let I be an ideal in $C^*(X)$ and suppose $S = \{f \in C^*(X) \colon Z(f) = \emptyset\}$. Then the following statements hold.*

(i) *If I is a z°-ideal, then $I^{\rm e}$ is also a z°-ideal. Whenever I is contracted, the converse is also true.*

(ii) *If $I \cap S = \emptyset$ and I is prime, then $I^{\rm e}$ is. The converse is true if I is contracted.*

(iii) *If I is a minimal prime ideal, then I^{e} is also a minimal prime ideal. The converse is true if I is contracted.*

(iv) *If I is a nonregular prime ideal, then I^{e} is. The converse is true if I is contracted.*

(v) *If $I \cap S = \emptyset$ and I is maximal, then I^{e} is. The converse is true if I is contracted.*

Proof. Parts (ii), (iii), (iv) and (v) are true in any commutative ring of fractions. We will prove part (i) and for the other parts we refer the reader to [1]. If I is a z°-ideal, then it is contracted by Proposition 2.1 and hence $I = I^{\mathrm{e}} \cap C^*(X)$. Let $f \in I^{\mathrm{e}}$, $g \in C(X)$ and $\mathrm{Ann}_{C(X)}(f) = \mathrm{Ann}_{C(X)}(g)$. Therefore $\mathrm{Ann}_{C^*(X)}\big(\frac{f}{1+|f|}\big) = \mathrm{Ann}_{C^*(X)}\big(\frac{g}{1+|g|}\big)$ and $\frac{f}{1+|f|} \in I$ implies that $\frac{g}{1+|g|} \in I$, see Proposition 1.4 in [4]. Hence $g \in I^{\mathrm{e}}$ implies that I^{e} is a z°-ideal in $C(X)$. Conversely, let I^{e} be a z°-ideal in $C(X)$, $f \in I$, $g \in C^*(X)$ and $\mathrm{Ann}_{C^*(X)}(f) = \mathrm{Ann}_{C^*(X)}(g)$. Clearly $\mathrm{Ann}_{C(X)}(f) = \mathrm{Ann}_{C(X)}(g)$ and since $f \in I \subseteq I^{\mathrm{e}}$, then $g \in I^{\mathrm{e}} \cap C^*(X)$, implies that $g \in I$, i.e., I is a z°-ideal in $C^*(X)$. □

3. Spaces X in which every prime z°-ideal in $C(X)$ is either minimal or maximal

In Proposition 1.5, we observed that every prime z°-ideal in $C(X)$ is minimal if and only if for every zeroset Z in X there exists a zeroset F in X such that $Z \cup F = X$ and $\operatorname{int} Z \cap \operatorname{int} F = \emptyset$. By Corollary 5.5 in [8], this is equivalent to compactness of the space of minimal prime ideals of $C(X)$. Let us call a space X *m-space* if every prime z°-ideal of $C(X)$ is minimal. We will also call a space X *quasi m-space* if every prime z°-ideal of $C(X)$ is either minimal or maximal. Clearly every m-space is a quasi m-space, but a quasi m-space need not be an m-space, see Examples 3.3. Our aim in this section is to recognize most of these spaces by a topological characterization. To prove the main result of this section, we shall need the following lemma.

Lemma 3.1. *Let $f \in C(X)$, then $\sum\limits_{h \in \mathrm{Ann}(f)} P_{f^2+h^2} = \bigcup\limits_{h \in \mathrm{Ann}(f)} P_{f^2+h^2}$ is a z°-ideal in $C(X)$.*

Proof. Clearly $\bigcup\limits_{h \in \mathrm{Ann}(f)} P_{f^2+h^2} \subseteq \sum\limits_{h \in \mathrm{Ann}(f)} P_{f^2+h^2}$. Now we let

$$g \in \sum_{h \in \mathrm{Ann}(f)} P_{f^2+h^2},$$

then $g = g_1 + g_2 + \ldots + g_n$, where $g_i \in P_{f^2+h_i^2}$ for $h_i \in \mathrm{Ann}(f)$ and $i = 1, 2, \ldots, n$. If we define $h = h_1^2 + \ldots + h_n^2$, then $h \in \mathrm{Ann}(f)$ and $\operatorname{int} Z(f^2 + h^2) = \Big(\bigcap\limits_{i=1}^{n} \operatorname{int} Z(h_i)\Big) \cap$

$\operatorname{int} Z(f) \subseteq \bigcap_{i=1}^{n} \operatorname{int} Z(g_i) \subseteq \operatorname{int} Z(g)$ imply that $g \in P_{f^2+h^2}$ by Proposition 1.1. This means that $\sum_{h\in \operatorname{Ann}(f)} P_{f^2+h^2} \subseteq \cup_{h\in\operatorname{Ann}(f)} P_{f^2+h^2}$. Finally, since every $P_{f^2+h^2}$ is a z°-ideal, clearly $\bigcup_{h\in\operatorname{Ann}(f)} P_{f^2+h^2}$ is also a z°-ideal. □

Next we prove the main theorem of this section.

Theorem 3.2. *The following statements are equivalent:*

(i) *X is quasi m-space.*

(ii) *$\forall p \in \beta X$ and $\forall f, g \in M^p$, $\exists h \in \operatorname{Ann}(f)$ and $k \notin M^p$ such that $\operatorname{Ann}(f^2 + h^2) \subseteq \operatorname{Ann}(gk)$.*

(iii) *$\forall p \in \beta X$ and every two zerosets Z and F in X with $p \in \operatorname{cl}_{\beta X} Z \cap \operatorname{cl}_{\beta X} F$, there exist zerosets Z' and F' such that $Z \cup Z' = X$, $p \notin \operatorname{cl}_{\beta X} F'$ and $\operatorname{int}_X Z \cap \operatorname{int}_X Z' \subseteq \operatorname{int}_X(F \cup F')$.*

P r o o f. The equivalence of parts (ii) and (iii) is evident by Lemma 2.1 in [3]. We will show that (i) and (ii) are equivalent. First suppose that (ii) holds and P is a prime z°-ideal, $P \subseteq M^p$ for some $p \in \beta X$ and $P \neq M^p$. Then $\exists g \in M^p$ such that $g \notin P$. If P is not minimal, then $\exists f \in C(X)$ such that $(f, \operatorname{Ann}(f)) \subseteq P$. Now by part (ii), $\exists h \in \operatorname{Ann}(f)$ and $k \notin M^p$ such that $\operatorname{Ann}(f^2 + h^2) \subseteq \operatorname{Ann}(gk)$. Since $f^2 + h^2 \in P$ and P is z°-ideal, then $gk \in P$ (note that for $u, v \in C(X)$, $\operatorname{Ann}(u) \subseteq \operatorname{Ann}(v)$ if and only if $\operatorname{int} Z(u) \subset \operatorname{int} Z(v)$, see also Proposition 2.2 in [3]). But $k \notin P$, for $k \notin M^p$, hence $g \in P$, a contradiction. Conversely, let every prime z°-ideal of $C(X)$ be minimal or maximal. Assume that part (ii) does not hold; then $\exists p \in \beta X$ and $\exists f, g \in M^p$ such that $\forall h \in \operatorname{Ann}(f)$ and $k \notin M^p$, $\operatorname{Ann}(f^2 + h^2) \not\subset \operatorname{Ann}(gk)$. Consider $S = \{g^n k\colon\ k \notin M^p,\ n = 0, 1, 2, \ldots\}$ and $I = \bigcup_{h\in\operatorname{Ann}(f)} P_{f^2+h^2}$. Obviously S is closed under multiplication. We also have $I \cap S = \emptyset$, for if $g^n k \in P_{f^2+h^2}$ for some n and $h \in \operatorname{Ann}(f)$, then by Proposition 1.1, $\operatorname{Ann}(f^2 + h^2) \subseteq \operatorname{Ann}(gk)$ which is impossible by our hypothesis. So there exists a prime ideal P which $I \subseteq P$ and $P \cap S = \emptyset$. We have already observed in Lemma 3.1 that I is a z°-ideal and hence by Proposition 1.3, P is also a z°-ideal, for we may assume that P is minimal over I. Now $P \cap S = \emptyset$ and $C(X) \setminus M^p \subseteq S$ imply that $P \subseteq M^p$. On the other hand, since $(f, \operatorname{Ann}(f)) \subseteq P$, then P is not minimal and hence it must be maximal, i.e., $P = M^p$. This implies that $g \in M^p = P$, a contradiction. □

Examples 3.3. We observed in Section 1 that every metric space and every basically disconnected space is an m-space and the space Σ (see [7], 4M for details) is an m-space which is not metrizable. By the following proposition, βX is also an m-space, whenever X is an m-space. In particular $\beta\mathbb{R}$ is an m-space. If X is the one-point compactification of an uncountable discrete space, then X is a quasi

m-space which is not an m-space. To see this, let $p \in X$ be the only nonisolated point of X, then $\forall f \in M_p$, $X \setminus Z(f)$ is countable, for $Z(f)$ is a G_δ-set. Since $\forall f \in M_p$, $\operatorname{int} Z(f) \neq \emptyset$, then M_p is a nonregular ideal and according to Corollary 1.4, M_p is a z°-ideal. On the other hand, M_p is not minimal, then X is not an m-space. Now we show that X is a quasi m-space. Let $f, g \in M_p$, since $X \setminus Z(g)$ is countable, then $Z(f) \setminus Z(g)$ is also countable. We define $h \in C(X)$ such that $X \setminus Z(h) = Z(f) \setminus Z(g)$. Hence $h \in \operatorname{Ann}(f)$ and $Z(f^2 + h^2) \subseteq Z(g)$ implies that $\operatorname{int} Z(f^2 + h^2) \subseteq \operatorname{int} Z(gk)$, $\forall k \in C(X)$. Therefore by Theorem 3.2, X is a quasi m-space. For an example which is not a quasi m-space, let D be the one-point compactification of an uncountable discrete space X with the only nonisolated point δ. For every $n \in \mathbb{N}$, suppose D_n is a copy of D with nonisolated point δ_n. Let Y be the quotient space of the free union $\bigcup_{n=1}^{\infty} D_n \cup \mathbb{R}$ by identifying each point $\frac{1}{n}$ with the point δ_n. Since $\mathbb{R}$ and D_n, $\forall n \in \mathbb{N}$ are normal, clearly Y is also a normal space. To see that Y is not a quasi m-space, suppose it is. Consider $f, g \in C(Y)$ where $Z(f) = \bigcup_{n=1}^{\infty} D_n \cup \{0\}$, $Z(g) = \{0\}$ and there exists $h \in \operatorname{Ann}(f)$ and $k \notin M_0$ such that $\operatorname{int} Z(f) \cap \operatorname{int} Z(h) \subseteq \operatorname{int}[Z(g) \cup Z(k)]$. Since $\mathbb{R} \setminus \{\frac{1}{n} \colon n \in \mathbb{N}\} \subseteq Z(h)$, then $\mathbb{R} \subseteq \operatorname{cl}(\mathbb{R} \setminus \{\frac{1}{n} \colon n \in \mathbb{N}\}) \subseteq Z(h)$ and hence $\{\frac{1}{n} \colon n \in \mathbb{N}\} \subseteq \operatorname{int} Z(h)$. On the other hand, $\bigcup_{n=1}^{\infty} D_n \subseteq \operatorname{int} Z(f)$ implies that $\{\frac{1}{n} \colon n \in \mathbb{N}\} \subseteq \operatorname{int} Z(f) \cap \operatorname{int} Z(h)$. Hence $\{\frac{1}{n} \colon n \in \mathbb{N}\} \subseteq Z(k)$ implies that $0 \in Z(k)$ which is a contradiction. For another example which is not a quasi m-space, see [3].

By Proposition 2.4 and the fact that $C(X)$ is a ring of fractions of $C^*(X)$, the following result is clear.

Proposition 3.4.
(i) *X is an m-space if and only if βX is.*
(ii) *X is a quasi m-space if and only if βX is.*

Remark 3.5. It is easy to check that X is basically disconnected if and only if $\forall f \in C(X)$, $\exists g \in C(X)$ such that $\operatorname{int} Z(f) \cup \operatorname{int} Z(g) = X$ and $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$. Therefore every basically disconnected space is an m-space and hence a quasi m-space. Since every metric space is an m-space, not every m-space is basically disconnected.

Remark 3.6. A point $p \in X$ is said to be an *almost P-point* if $\forall f \in M_p$, $\operatorname{int}_X Z(f) \neq \emptyset$, and X is called an almost P-space if every point of X is an almost P-point. Now if the compact space X has no almost P-point, then every maximal ideal in $C(X)$ is regular and hence $C(X)$ has no maximal z°-ideal. In fact, if X is a quasi m-space but not an m-space, then X has at least one almost P-point.

4. Nonregular ideals and z°-ideals

In this section we are going to answer the questions which are mentioned in Section 1. It is easy to see that a space X is an almost P-space if every zeroset in X is a regular closed. We refer the reader to [2], [5], [10] and [12] for more details and properties of almost P-spaces. Now we want to define a weak almost P-space, namely *w.almost P-space*. A w.almost P-space is a topological space X in which for every two zerosets Z and F, whenever $\operatorname{int} Z \subseteq \operatorname{int} F$, then there exists a zeroset E in X with empty interior such that $Z \subseteq F \cup E$. Clearly every almost P-space is w.almost P-space, for if $\operatorname{int} Z \subseteq \operatorname{int} F$, then $Z = \operatorname{cl}(\operatorname{int} Z) \subseteq \operatorname{cl}(\operatorname{int} F) = F$ and hence we consider $E = \emptyset$. But every w.almost P-space is not necessarily an almost P-space, for example consider $\alpha\mathbb{N} = \{0, 1, 2, \ldots, \frac{1}{n}, \ldots\}$. More generally, any space in which every closed set (boundary of any zeroset) is contained in a zeroset with empty interior (for example a metric space) is a w.almost P-space. To see this let $f, g \in C(X)$ and $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$. Then $Z(f) \setminus Z(g) \subseteq Z(f) \setminus \operatorname{int} Z(g) \subseteq Z(f) \setminus \operatorname{int} Z(f)$ and the closed set $Z(f) \setminus \operatorname{int} Z(f)$ is contained in a zeroset with empty interior, say $Z(h)$. Hence $Z(f) \setminus Z(g) \subseteq Z(h)$ with $\operatorname{int} Z(h) = \emptyset$ which implies that $Z(f) \subseteq Z(gh)$, i.e., X is a w.almost P-space.

To prove the first theorem of this section, we need the following lemma.

Lemma 4.1. *If every zeroset in X with nonempty interior is open (regular closed), then every zeroset in X is open (regular closed).*

Proof. Let $0 \neq f \in C(X)$; then $\exists g \in C(X)$ such that $\operatorname{int} Z(g) \neq \emptyset$ and $Z(f) \cap Z(g) = \emptyset$. First suppose that every zeroset with nonempty interior is open. Since $Z(g)$ and $Z(fg) = Z(f) \cup Z(g)$ are open sets, then $Z(f)$ is also open, for $Z(f)$ and $Z(g)$ are disjoint. Now let every zeroset with nonempty interior be regular closed and suppose that $Z(f)$ is not empty but $\operatorname{int} Z(f) = \emptyset$. Since $Z(f) \cap Z(g) = \emptyset$, it is easy to see that $\operatorname{int}(Z(f) \cup Z(g)) = \operatorname{int} Z(g)$. Now we have $Z(f) \cup Z(g) = Z(fg) = \operatorname{cl}(\operatorname{int} Z(fg)) = \operatorname{cl}(\operatorname{int}(Z(f) \cup Z(g))) = \operatorname{cl}(\operatorname{int} Z(g)) = Z(g)$. This implies that $Z(f) \subseteq Z(g)$ which is impossible for $Z(f)$ and $Z(g)$ are disjoint. Therefore $\operatorname{int} Z(f) \neq \emptyset$ and hence $Z(f)$ is also regular closed by our hypothesis. $\square$

Theorem 4.2.

(i) *Every nonregular z-ideal in $C(X)$ is a z°-ideal if and only if X is an almost P-space.*

(ii) *Every nonregular prime z-ideal in $C(X)$ is a z°-ideal if and only if X is a w.almost P-space.*

Proof. (i) Let every nonregular z-ideal in $C(X)$ be a z°-ideal. By Lemma 4.1, it is enough to show that every zeroset with nonempty interior is a regular closed. Hence suppose that $f \in C(X)$ and $\operatorname{int} Z(f) \neq \emptyset$. Since M_f is a nonregular z-ideal in $C(X)$, then by our hypothesis it is a z°-ideal. Suppose that $\operatorname{cl}(\operatorname{int} Z(f)) \neq Z(f)$, then $\exists x \in Z(f) \setminus \operatorname{cl}(\operatorname{int} Z(f))$. Define $h \in C(X)$ such that $h(x) = 1$ and $h(\operatorname{cl}(\operatorname{int} Z(f)) = 0$. Since $Z(h)$ does not contain $Z(f)$, then $h \notin M_f$, but $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(h)$, a contradiction for M_f is a z°-ideal. Therefore $Z(f)$ is a regular closed, i.e., X is an almost P-space. Conversely, if X is an almost P-space, then every z-ideal in $C(X)$ is a z°-ideal; see [3], Theorem 2.14.

(ii) First suppose that every nonregular prime z-ideal in $C(X)$ is a z°-ideal. To the contrary, suppose that $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$ and for every $h \in C(X)$ with $\operatorname{int} Z(h) = \emptyset$, $Z(gh)$ does not contain $Z(f)$. Therefore $gh \notin M_f$, $\forall h \in C(X)$ with $\operatorname{int} Z(h) = \emptyset$. Now consider $S = \{g^n h\colon \operatorname{int} Z(h) = \emptyset,\ n = 0, 1, \ldots\}$. Clearly S is closed under multiplication and $M_f \cap S = \emptyset$, for M_f is a z-ideal and $Z(g^n h) = Z(gh)$, $\forall n \in \mathbb{N}$. Now by Theorem 14.7 in [7], there exists a prime z-ideal P such that $M_f \subseteq P$ and $P \cap S = \emptyset$. $P \cap S = \emptyset$ implies that P is also a nonregular ideal and hence by our hypothesis, P must be a z°-ideal. But $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$, $f \in P$ and $g \notin P$, a contradiction. Conversely, let X be a w.almost P-space, P be a nonregular z-ideal in $C(X)$, $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$ and $f \in P$. By our hypothesis, $\exists h \in C(X)$ with $\operatorname{int} Z(h) = \emptyset$ and $Z(f) \subseteq Z(gh)$. Since P is a z-ideal, then $gh \in P$. But $h \notin P$, for h is not a zero divisor, hence $g \in P$, i.e., P is a z°-ideal. □

Corollary 4.3. *X is a w.almost P-space if and only if $\forall f, g \in C(X)$, whenever $\operatorname{int} Z(f) = \operatorname{int} Z(g)$, then there exists a regular $h \in C(X)$ such that $Z(fh) = Z(gh)$.*

Proof. Let X be a w.almost P-space and $\operatorname{int} Z(f) = \operatorname{int} Z(g)$; then by the above theorem, there exist regular functions $h, k \in C(X)$ such that $Z(f) \subseteq Z(gk)$ and $Z(g) \subseteq Z(fh)$. Hence $Z(fhk) \subseteq Z(ghk) \subseteq Z(fhk)$, i.e., $Z(fhk) = Z(ghk)$, where hk is regular. Conversely, if $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$, then $\operatorname{int} Z(f) = \operatorname{int} Z(f^2 + g^2)$ implies that $Z(fh) = Z((f^2 + g^2)h)$ for some regular $h \in C(X)$ and hence $Z(f) \subseteq Z(fh) = Z((f^2 + g^2)h) \subseteq Z(gh)$, i.e., X is a w.almost P-space. □

Next we prove the main theorem of this section.

First, let us call the space X a *∂-space* if the boundary of any zeroset in X is contained in a zeroset with empty interior. The class of topological ∂-spaces includes metric spaces and more generally, the perfectly normal spaces. We have already shown that every ∂-space X is a w.almost P-space; see the introduction of Section 4. But every w.almost P-space, even every (compact) almost P-space is not necessarily a ∂-space. For example let X be an uncountable discrete space and $Y = X \cup \{p\}$ be the one-point compactification of the space X. Then clearly Y is an almost P-space, but $\forall f \in C(Y)$ with $f(p) = 0$ and infinite cozeroset, we have $\partial Z(f) = Z(f) \setminus \operatorname{int} Z(f) = \{p\}$ which is not contained in a zeroset in Y with empty interior; i.e., Y is not a ∂-space. More generally, it is easy to see that the space X is an almost P-space and a ∂-space if and only if X is P-space. This shows that there are almost P-spaces which are not ∂-spaces and there are ∂-spaces which are not almost P-spaces. $\square$

Theorem 4.4. *Every nonregular prime ideal of $C(X)$ is a z°-ideal if and only if X is a ∂-space.*

Proof. We first suppose that there exists $f \in C(X)$ such that $\partial Z(f) = Z(f) \setminus \operatorname{int} Z(f)$ is not contained in a zeroset in X with empty interior. We will show that there is a nonregular prime ideal in $C(X)$ which is not even a z-ideal. To see this, let $l \in C(\mathbb{R})$ be such that $Z(l) = \{0\}$ and $\lim_{x \to 0} l^n(x)/x = \infty$, $\forall n \in \mathbb{N}$; see [7], 2G. Now consider $S = \{hl^n \circ f \colon \operatorname{int} Z(h) = \emptyset,\ n = 0, 1, 2, \ldots\}$ and $I = (f)$; note that $l^0 \circ f = 1$. Clearly S is closed under multiplication and $S \cap I = \emptyset$, for otherwise if $S \cap I \neq \emptyset$, then $hl^n \circ f = kf$, for some $k \in C(X)$ and $n \neq 0$. (In the case $n = 0$ we have $\operatorname{int} Z(f) = \emptyset$ and $\partial Z(f) = Z(f)$ which contradicts our hypothesis). By our hypothesis, there exists $x \in Z(f) \setminus \operatorname{int} Z(f)$ such that $x \notin Z(h)$. Now let (x_α) be a net in $X \setminus (Z(f) \cup Z(h))$ such that $x_\alpha \longrightarrow x$. This shows that

$$k(x_\alpha) = h(x_\alpha) \frac{l^n(f(x_\alpha))}{f(x_\alpha)} \longrightarrow \infty$$

which contradicts the continuity of k at x. Hence $S \cap I = \emptyset$ and therefore there exists a prime ideal P such that $P \cap S = \emptyset$ and $I = (f) \subseteq P$. Since S contains all non-zero divisors of $C(X)$, then P is a nonregular prime ideal. On the other hand $l \circ f \notin P$, $Z(l \circ f) = Z(f)$ and $f \in P$ which imply that P is not a z-ideal. Conversely suppose that X is a ∂-space and let P be a nonregular prime ideal in $C(X)$, $\operatorname{int} Z(f) = \operatorname{int} Z(g)$ and $f \in P$. Since X is a ∂-space, then there exists a nonzerodivisor $h \in C(X)$ such that $\partial Z(f) \subseteq Z(h)$ and $\partial Z(fg) \subseteq Z(h)$. Now we define $k(x) = h(x)f(x)$, $\forall x \in \operatorname{Coz}(fg)$ and $k(x) = h(x)$, $\forall x \in Z(fg)$. Obviously k is continuous on $\operatorname{Coz}(fg)$ and on $\operatorname{int} Z(fg)$ and it is not hard to show that k is also continuous on $\partial Z(fg) \subseteq Z(h)$. We show that $fgh = kg$. For $x \notin Z(fg)$, we have $k = fh$ and equality holds. Now suppose that $x \in Z(fg) = Z(f) \cup Z(g)$. If $x \in Z(g)$, then $(fgh)(x) = (kg)(x) = 0$ and if $x \in Z(f)$, then either $x \in \operatorname{int} Z(f) = \operatorname{int} Z(g)$ which again $(fgh)(x) = (kg)(x) = 0$ or $x \in \partial Z(f)$ which implies that $x \in Z(h)$ and hence $(fgh)(x) = (kg)(x) = 0$. Therefore $fgh = kg$ and then $gk \in P$. But $\operatorname{int} Z(k) = \emptyset$ implies that $k \notin P$ and consequently $g \in P$, i.e., P is a z°-ideal. $\square$

Corollary 4.5. *The only nonregular prime ideals of $C(X)$ are minimal prime ideals if and only if X is a ∂-space and an m-space.*

By Proposition 2.4, the following corollary is evident.

Corollary 4.6. *X is a ∂-space if and only if βX is.*

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