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EXISTENCE FOR NONOSCILLATORY SOLUTIONS OF HIGHER ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. Consider the forced higher-order nonlinear neutral functional differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+C(t)x(t-\tau)]+\sum_{i=1}^{m}Q_i(t)f_i(x(t-\sigma_i))=g(t), \quad t\geqslant t_0,$$

where $n,m\geqslant 1$ are integers, $\tau,\sigma_i\in\mathbb{R}^+=[0,\infty)$, $C,Q_i,g\in C([t_0,\infty),\mathbb{R})$, $f_i\in C(\mathbb{R},\mathbb{R})$, $(i=1,2,\dots,m)$. Some sufficient conditions for the existence of a nonoscillatory solution of above equation are obtained for general $Q_i(t)$ $(i=1,2,\dots,m)$ and $g(t)$ which means that we allow oscillatory $Q_i(t)$ $(i=1,2,\dots,m)$ and $g(t)$. Our results improve essentially some known results in the references.

Keywords: neutral differential equations, nonoscillatory solutions

MSC 2000: 34K15, 34K11

1. Introduction

Consider the forced higher-order nonlinear neutral functional differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+C(t)x(t-\tau)]+\sum_{i=1}^{m}Q_i(t)f_i(x(t-\sigma_i))=g(t), \quad t\geqslant t_0. \tag{1}$$

With respect to the equation (1), we shall throughout assume the following:

(i) $n,m\geqslant 1$ are integers, $\tau,\sigma_i\in\mathbb{R}^+=[0,\infty)$ $(i=1,2,\dots,m)$;

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(ii) $C, Q_i, g \in C([t_0,\infty),\mathbb{R})$, $f_i \in C(\mathbb{R},\mathbb{R})$ $(i = 1, 2, \ldots, m)$.

Let $r = \max\limits_{1\leqslant i\leqslant m} \{\tau, \sigma_i\}$. By a solution of the equation (1) we mean a function $x \in C([t_1 - r,\infty),\mathbb{R})$, for some $t_1 \geqslant t_0$, such that $x(t) + C(t)x(t-\tau)$ is n-times continuously differentiable on $[t_1,\infty)$ and such that the equation (1) is satisfied for $t \geqslant t_1$.

Oscillation and non-oscillation of neutral functional differential equations has developed very rapidly in recent years. We refer the reader to [1]–[15] and the references cited therein. Oscillatory and nonoscillatory behavior of solutions of the forced first order neutral functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + C(t)x(t-\tau)] + Q_1(t)f_1(x(t-\sigma_1)) = g(t), \quad t \geqslant t_0, \tag{2}$$

and of the second order neutral functional differential equation with positive and negative coefficients

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[x(t) + cx(t-\tau)] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \quad t \geqslant t_0, \tag{3}$$

where $c \neq \pm 1$, $Q_1(t) \geqslant 0$ and $Q_2(t) \geqslant 0$, have been investigated in [8], [12]. Clearly, equations (2) and (3) are special forms of the equation (1). Parhi and Rath [12], Kulenovic and Hadziomerspahic [8] proved the following results by using Banach contraction mapping principle.

Theorem A ([12], Theorems 2.6, 2.8 and 2.10)**.** *Assume that*

H_1) *$C(t)$ is in one of the following ranges:*

$$0 \leqslant C(t) < c_1 < 1, \quad 1 < c_2 \leqslant C(t) \leqslant c_3, \quad c_4 \leqslant C(t) \leqslant c_5 < -1,$$

where c_i $(i = 1, \ldots, 5)$ are positive real numbers.

H_2) *$Q_1(t) \geqslant 0$, $f_1 \in C(\mathbb{R},\mathbb{R})$ is nondecreasing, $xf_1(x) \geqslant 0$ for any $x \neq 0$, and f_1 satisfies the Lipschitz condition on intervals of the type $[a,b]$, $0 < a < b$.*

Further, assume that

$$\int_0^\infty Q_1(t)\,\mathrm{d}t < \infty, \quad \int_0^\infty |g(t)|\,\mathrm{d}t < \infty.$$

Then the equation (2) has a nonoscillatory solution.

Theorem B [8]. *Assume that*

H_3) $c \neq \pm 1$,

H_4) $aQ_1(t) - Q_2(t) \geqslant 0$, *for every* $t \geqslant T$ *and* $a > 0$.

Further, assume that

$$\int_{t_0}^{\infty} Q_1(t)\,\mathrm{d}t < \infty, \quad \int_{t_0}^{\infty} Q_2(t)\,\mathrm{d}t < \infty.$$

Then the equation (3) *has a nonoscillatory solution.*

In this paper, by using Krasnoselskii's and Schauder's fixed point theorems and some new techniques, we obtain some sufficient conditions for the existence of a nonoscillatory solution of (1) for general $Q_i(t)$ $(i = 1, 2, \ldots, m)$ and $g(t)$ which means that we allow oscillatory $Q_i(t)$ $(i = 1, 2, \ldots, m)$ and $g(t)$. In particular, our results improve essentially Theorem A and B by removing the restrictive conditions H_2) and H_4) and relaxing the hypotheses H_1) and H_3).

2. Main results

The following fixed point theorems will be used to prove the main results in this section.

Lemma 1 [5] (Krasnoselskii's Fixed Point Theorem). *Let X be a Banach space, let Ω be a bounded closed convex subset of X and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contractive and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x$$

has a solution in Ω.

Lemma 2 [5], [6] (Schauder's Fixed Point Theorem). *Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S\colon\ \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Sx = x$.*

We will consider the following cases:

$$-1 < c_1 \leqslant C(t) \leqslant 0, \quad -\infty < C(t) \leqslant c_2 < -1, \quad 0 \leqslant C(t) \leqslant c_3 < 1,$$
$$1 < c_4 \leqslant C(t) < \infty, \quad C(t) \equiv 1, \quad C(t) \equiv -1.$$

Our main results are the following six theorems.

Theorem 1. *Assume that* $-1 < c_1 \leqslant C(t) \leqslant 0$ *and that*

$$\int_{t_0}^{\infty} t^{n-1} |Q_i(t)| \, \mathrm{d}t < \infty, \quad i = 1, 2, \ldots, m \tag{4}$$

and

$$\int_{t_0}^{\infty} t^{n-1} |g(t)| \, \mathrm{d}t < \infty. \tag{5}$$

Then (1) *has a nonoscillatory bounded solution.*

P r o o f. By (4) and (5), we choose a $T > t_0$ sufficiently large such that

$$\frac{1}{(n-1)!} \int_T^{\infty} s^{n-1} \left(\sum_{i=1}^m |Q_i(s)| M_1 + |g(s)| \right) \mathrm{d}s \leqslant \frac{1+c_1}{3},$$

where $M_1 = \max\limits_{2(1+c_1)/3 \leqslant x \leqslant 4/3} \{ |f_i(x)| \colon \ 1 \leqslant i \leqslant m \}$.

Let $C([t_0, \infty), \mathbb{R})$ be the set of all continuous functions with the norm $\|x\| = \sup\limits_{t \geqslant t_0} |x(t)| < \infty$. Then $C([t_0, \infty), \mathbb{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbb{R})$ as follows:

$$\Omega = \left\{ x = x(t) \in C([t_0, \infty), \mathbb{R}) \colon \ \frac{2(1+c_1)}{3} \leqslant x(t) \leqslant \frac{4}{3}, \ t \geqslant t_0 \right\}.$$

Define two maps S_1 and $S_2 \colon \ \Omega \to C([t_0, \infty), \mathbb{R})$ as follows:

$$(S_1 x)(t) = \begin{cases} 1 + c_1 - C(t) x(t - \tau), & t \geqslant T, \\ (S_1 x)(T), & t_0 \leqslant t \leqslant T, \end{cases}$$

$$(S_2 x)(t) = \begin{cases} \dfrac{(-1)^{n+1}}{(n-1)!} \displaystyle\int_t^{\infty} (s-t)^{n-1} \left(\sum_{i=1}^m Q_i(s) f_i(x(s - \sigma_i)) - g(s) \right) \mathrm{d}s, & t \geqslant T, \\ (S_2 x)(T), & t_0 \leqslant t \leqslant T. \end{cases}$$

i) We shall show that for any $x, y \in \Omega$, $S_1 x + S_2 y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $t \geqslant T$, we get

$$\begin{aligned}
&(S_1 x)(t) + (S_2 y)(t) \\
&\quad \leqslant 1 + c_1 - C(t) x(t - \tau) \\
&\qquad + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} \left(\sum_{i=1}^m |Q_i(s)| \, |f_i(y(s - \sigma_i))| + |g(s)| \right) \mathrm{d}s \\
&\quad \leqslant 1 + c_1 - \frac{4}{3} c_1 + \frac{1}{(n-1)!} \int_T^{\infty} s^{n-1} \left(\sum_{i=1}^m |Q_i(s)| M_1 + |g(s)| \right) \mathrm{d}s \\
&\quad \leqslant 1 + c_1 - \frac{4}{3} c_1 + \frac{1+c_1}{3} = \frac{4}{3}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&(S_1x)(t)+(S_2y)(t)\\
&\quad\geqslant 1+c_1-C(t)x(t-\tau)-\frac{1}{(n-1)!}\\
&\qquad\times\int_t^\infty (s-t)^{n-1}\biggl(\sum_{i=1}^m |Q_i(s)|\,|f_i(y(s-\sigma_i))|+|g(s)|\biggr)\,\mathrm{d}s\\
&\quad\geqslant 1+c_1-\frac{1}{(n-1)!}\int_T^\infty s^{n-1}\biggl(\sum_{i=1}^m |Q_i(s)|M_1+|g(s)|\biggr)\,\mathrm{d}s\\
&\quad\geqslant 1+c_1-\frac{1+c_1}{3}=\frac{2(1+c_1)}{3}.
\end{aligned}$$

Hence,

$$\frac{2(1+c_1)}{3}\leqslant (S_1x)(t)+(S_2y)(t)\leqslant \frac{4}{3},\quad \text{for } t\geqslant t_0.$$

Thus we have proved that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

ii) We shall show that S_1 is a contractive mapping on Ω.

In fact, for $x,y\in\Omega$ and $t\geqslant T$, we have

$$|(S_1x)(t)-(S_1y)(t)|\leqslant -C(t)|x(t-\tau)-y(t-\tau)|\leqslant -c_1\|x-y\|.$$

This implies that

$$\|S_1x-S_1y\|\leqslant -c_1\|x-y\|.$$

Since $0<-c_1<1$, we conclude that S_1 is a contraction mapping on Ω.

iii) We now show that S_2 is completely continuous.

First, we will show that S_2 is continuous. Let $x_k=x_k(t)\in\Omega$ be such that $x_k(t)\to x(t)$ as $k\to\infty$. Because Ω is closed, $x=x(t)\in\Omega$. For $t\geqslant T$, we have

$$\begin{aligned}
&|(S_2x_k)(t)-(S_2x)(t)|\\
&\quad\leqslant \frac{1}{(n-1)!}\int_t^\infty s^{n-1}\biggl(\sum_{i=1}^m |Q_i(s)|\,|f_i(x_k(s-\sigma_i))-f_i(x(s-\sigma_i))|\biggr)\,\mathrm{d}s\\
&\quad\leqslant \frac{1}{(n-1)!}\int_T^\infty s^{n-1}\biggl(\sum_{i=1}^m |Q_i(s)|\,|f_i(x_k(s-\sigma_i))-f_i(x(s-\sigma_i))|\biggr)\,\mathrm{d}s.
\end{aligned}$$

Since $|f_i(x_k(t-\sigma_i))-f_i(x(t-\sigma_i))|\to 0$ as $k\to\infty$ for $i=1,2,\dots,m$, by applying the Lebesgue dominated convergence theorem, we conclude that $\lim\limits_{k\to\infty}\|(S_2x_k)(t)-(S_2x)(t)\|=0$. This means that S_2 is continuous.

Next, we show that $\mathrm{S}_2\Omega$ is relatively compact. It suffices to show that the family of functions $\{S_2x\colon x\in\Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)$.

The uniform boundedness is obvious. For the equicontinuity, according to Levitan's result, we only need to show that, for any given $\varepsilon > 0$, $[T, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. By (4), for any $\varepsilon > 0$, take $T^* \geqslant T$ large enough so that

$$\frac{1}{(n-1)!}\int_{T^*}^{\infty} s^{n-1}\left(M_1\sum_{i=1}^{m}|Q_i(s)| + |g(s)|\right)\mathrm{d}s < \frac{\varepsilon}{2}.$$

Then for $x \in \Omega$, $t_2 > t_1 \geqslant T^*$

$$\begin{aligned}
&|(S_2x)(t_2) - (S_2x)(t_1)|\\
&\quad\leqslant \frac{1}{(n-1)!}\int_{t_2}^{\infty} s^{n-1}\left(\sum_{i=1}^{m}|Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)|\right)\mathrm{d}s\\
&\qquad + \frac{1}{(n-1)!}\int_{t_1}^{\infty} s^{n-1}\left(\sum_{i=1}^{m}|Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)|\right)\mathrm{d}s\\
&\quad\leqslant \frac{1}{(n-1)!}\int_{t_2}^{\infty} s^{n-1}\left(M_1\sum_{i=1}^{m}|Q_i(s)| + |g(s)|\right)\mathrm{d}s\\
&\qquad + \frac{1}{(n-1)!}\int_{t_1}^{\infty} s^{n-1}\left(M_1\sum_{i=1}^{m}|Q_i(s)| + |g(s)|\right)\mathrm{d}s\\
&\quad< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

For $x \in \Omega$ and $T \leqslant t_1 < t_2 \leqslant T^*$

$$\begin{aligned}
&|(S_2x)(t_2) - (S_2x)(t_1)|\\
&\quad\leqslant \frac{1}{(n-1)!}\int_{t_1}^{t_2} s^{n-1}\left(\sum_{i=1}^{m}|Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)|\right)\mathrm{d}s\\
&\quad\leqslant \frac{1}{(n-1)!}\int_{t_1}^{t_2} s^{n-1}\left(M_1\sum_{i=1}^{m}|Q_i(s)| + |g(s)|\right)\mathrm{d}s\\
&\quad\leqslant \frac{1}{(n-1)!}\max_{T\leqslant s\leqslant T^*}\left\{s^{n-1}\left(M_1\sum_{i=1}^{m}|Q_i(s)| + |g(s)|\right)\right\}(t_2 - t_1).
\end{aligned}$$

Thus there exists a $\delta > 0$ such that

$$|(S_2x)(t_2) - (S_2x)(t_1)| < \varepsilon, \quad \text{if } 0 < t_2 - t_1 < \delta.$$

For any $x \in \Omega$, $t_0 \leqslant t_1 < t_2 \leqslant T$, it is easy to see that

$$|(S_2x)(t_2) - (S_2x)(t_1)| = 0 < \varepsilon.$$

Therefore $\{S_2x\colon\ x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact. By Lemma 1 (Krasnoselskii's fixed point theorem), there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. It is easy to see that $x_0(t)$ is a nonoscillatory solution of the equation (1). The proof is complete. $\square$

Theorem 2. *Assume that* $-\infty < C(t) \equiv c_2 < -1$ *and that* (4) *and* (5) *hold. Then* (1) *has a nonoscillatory bounded solution.*

Proof. By (4) and (5), we choose a $T > t_0$ sufficiently large such that

$$-\frac{1}{c_2(m-1)!}\int_{T+\tau}^{\infty} s^{n-1}\biggl(\sum_{i=1}^{m}|Q_i(s)|M_2 + |g(s)|\biggr)\,\mathrm{d}s \leqslant -\frac{c_2+1}{2},$$

where $M_2 = \max\limits_{-(c_2+1)/2 \leqslant x \leqslant -2c_2}\{|f_i(x)|\colon\ 1 \leqslant i \leqslant m\}$.

Let $C([t_0,\infty),\mathbb{R})$ be the set as in the proof of Theorem 1. We define a closed, bounded and convex subset Ω of $C([t_0,\infty),\mathbb{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0,\infty),\mathbb{R})\colon\ -\frac{c_2+1}{2} \leqslant x(t) \leqslant -2c_2,\ \ t \geqslant t_0\}.$$

Define two maps S_1 and $S_2\colon\ \Omega \to C([t_0,\infty),\mathbb{R})$ as follows:

$$(S_1x)(t) = \begin{cases} -c_2 - 1 - \dfrac{1}{C(t)}x(t+\tau), & t \geqslant T,\\ (S_1x)(T), & t_0 \leqslant t \leqslant T, \end{cases}$$

$$(S_2x)(t) = \begin{cases} \dfrac{(-1)^{n+1}}{C(t)(n-1)!}\displaystyle\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}\biggl(\sum_{i=1}^{m}Q_i(s)f_i(x(s-\sigma_i)) - g(s)\biggr)\,\mathrm{d}s, & t \geqslant T,\\ (S_2x)(T), & t_0 \leqslant t \leqslant T. \end{cases}$$

We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $t \geqslant T$, we get

$$\begin{aligned} &(S_1x)(t) + (S_2y)(t)\\ &\quad\leqslant -c_2 - 1 - \frac{1}{C(t)}x(t+\tau)\\ &\qquad - \frac{1}{C(t)}\frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}\biggl(\sum_{i=1}^{m}|Q_i(s)|\,|f_i(y(s-\sigma_i))| + |g(s)|\biggr)\,\mathrm{d}s\\ &\quad\leqslant -c_2 - 1 + 2 - \frac{1}{c_2}\frac{1}{(n-1)!}\int_{T+\tau}^{\infty}s^{n-1}\biggl(\sum_{i=1}^{m}|Q_i(s)|M_2 + |g(s)|\biggr)\,\mathrm{d}s\\ &\quad\leqslant -c_2 + 1 - \frac{c_2+1}{2} \leqslant -2c_2. \end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
&(S_1x)(t) + (S_2y)(t)\\
&\quad\geqslant -c_2 - 1 - \frac{1}{C(t)}x(t+\tau)\\
&\qquad + \frac{1}{C(t)}\frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t)^{n-1}\Big(\sum_{i=1}^{m}|Q_i(s)|\,|f_i(y(s-\sigma_i))| + |g(s)|\Big)\,\mathrm{d}s\\
&\quad\geqslant -c_2 - 1 + \frac{1}{c_2}\frac{1}{(n-1)!}\int_{T}^{\infty}s^{n-1}\Big(\sum_{i=1}^{m}|Q_i(s)|M_2 + |g(s)|\Big)\,\mathrm{d}s\\
&\quad\geqslant -c_2 - 1 + \frac{c_2+1}{2} = -\frac{c_2+1}{2}.
\end{aligned}
$$

Hence,

$$-\frac{c_2+1}{2} \leqslant (S_1x)(t) + (S_2y)(t) \leqslant -2c_2, \quad \text{for } t \geqslant t_0.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

We shall show that S_1 is a contractive mapping on Ω.

In fact, for $x, y \in \Omega$ and $t \geqslant T$, we have

$$|(S_1x)(t) - (S_1y)(t)| \leqslant -\frac{1}{C(t)}|x(t+\tau) - y(t+\tau)| \leqslant -\frac{1}{c_2}\|x-y\|.$$

This implies that

$$\|S_1x - S_1y\| \leqslant -\frac{1}{c_2}\|x-y\|.$$

Since $0 < -1/c_2 < 1$, we conclude that S_1 is a contractive mapping on Ω.

Proceeding similarly as in the proof of Theorem 1 we obtain that the mapping S_2 is completely continuous. By Lemma 1, there is a $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. Clearly, $x_0 = x_0(t)$ is a bounded nonoscillatory solution of the equation (1). This completes the proof of Theorem 2. □

Theorem 3. *Assume that* $0 \leqslant C(t) \leqslant c_3 < 1$ *and that* (4) *and* (5) *hold. Then* (1) *has a nonoscillatory bounded solution.*

Proof. By (4) and (5), we choose a $T > t_0$ sufficiently large such that

$$\frac{1}{(n-1)!}\int_{T}^{\infty}s^{n-1}\Big(\sum_{i=1}^{m}|Q_i(s)|M_3 + |g(s)|\Big)\,\mathrm{d}s \leqslant 1 - c_3,$$

where $M_3 = \max\limits_{2(1-c_3)\leqslant x\leqslant 4}\{f_i(x)\colon\ 1 \leqslant i \leqslant m\}$.

Let $C([t_0,\infty),\mathbb{R})$ be the set as in the proof of Theorem 1. We define a closed, bounded and convex subset Ω of $C([t_0,\infty),\mathbb{R})$ as follows:

$$\Omega=\{x=x(t)\in C([t_0,\infty),\mathbb{R})\colon\ 2(1-c_3)\leqslant x(t)\leqslant 4,\ \ t\geqslant t_0\}.$$

Define two maps S_1 and $S_2\colon\ \Omega\to C([t_0,\infty),\mathbb{R})$ as follows:

$$(S_1x)(t)=\begin{cases}3+c_3-C(t)x(t-\tau), & t\geqslant T,\\ (S_1x)(T), \quad t_0\leqslant t\leqslant T,\end{cases}$$

$$(S_2x)(t)=\begin{cases}\dfrac{(-1)^{n+1}}{(n-1)!}\displaystyle\int_t^\infty(s-t)^{n-1}\Bigl(\sum_{i=1}^m Q_i(s)f_i(x(s-\sigma_i))-g(s)\Bigr)\,\mathrm{d}s, & t\geqslant T,\\ (S_2x)(T), \quad t_0\leqslant t\leqslant T.\end{cases}$$

We shall show that for any $x,y\in\Omega$, $S_1x+S_2y\in\Omega$.

In fact, for every $x,y\in\Omega$ and $t\geqslant T$, we get

$$\begin{aligned}&(S_1x)(t)+(S_2y)(t)\\ &\quad\leqslant 3+c_3-C(t)x(t-\tau)\\ &\qquad+\frac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}\Bigl(\sum_{i=1}^m|Q_i(s)|\,|f_i(y(s-\sigma_i))|+|g(s)|\Bigr)\,\mathrm{d}s\\ &\quad\leqslant 3+c_3+\frac{1}{(n-1)!}\int_T^\infty s^{n-1}\Bigl(\sum_{i=1}^m|Q_i(s)|M_3+|g(s)|\Bigr)\,\mathrm{d}s\\ &\quad\leqslant 3+c_3+1-c_3=4.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&(S_1x)(t)+(S_2y)(t)\\ &\quad\geqslant 3+c_3-C(t)x(t-\tau)\\ &\qquad-\frac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}\Bigl(\sum_{i=1}^m|Q_i(s)|\,|f_i(y(s-\sigma_i))+|g(s)|\Bigr)\,\mathrm{d}s\\ &\quad\geqslant 3+c_3-4c_3-\frac{1}{(n-1)!}\int_T^\infty s^{n-1}\Bigl(\sum_{i=1}^m|Q_i(s)|M_3+|g(s)|\Bigr)\,\mathrm{d}s\\ &\quad\geqslant 3+c_3-4c_3-(1-c_3)=2(1-c_3).\end{aligned}$$

Hence,

$$2(1-c_3)\leqslant(S_1x)(t)+(S_2y)(t)\leqslant 4,\quad \text{for } t\geqslant t_0.$$

Thus we have proved that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

Proceeding similarly as in the proof of Theorem 1 we obtain that the mapping S_1 is a contractive mapping on Ω and the mapping S_2 is completely continuous. By Lemma 1, there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. Clearly, $x_0 = x_0(t)$ is a bounded nonoscillatory solution of the equation (1). This completes the proof of Theorem 3. □

Theorem 4. *Assume that* $1 < c_4 \equiv C(t) < \infty$ *and that* (4) *and* (5) *hold. Then* (1) *has a nonoscillatory bounded solution.*

Proof. By (4) and (5), we choose a $T > t_0$ sufficiently large such that

$$\frac{1}{c_4(n-1)!}\int_{T+\tau}^{\infty} s^{n-1}\Bigl(\sum_{i=1}^{m}|Q_i(s)|M_4 + |g(s)|\Bigr)\,\mathrm{d}s \leqslant c_4 - 1,$$

where $M_4 = \max\limits_{2(c_4-1)\leqslant x\leqslant 4c_4}\{f_i(x)\colon\ i = 1,2,\ldots,m\}$.

Let $C([t_0,\infty),\mathbb{R})$ be the set as in the proof of Theorem 1. We define a closed, bounded and convex subset Ω of $C([t_0,\infty),\mathbb{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0,\infty),\mathbb{R})\colon\ 2(c_4-1) \leqslant x(t) \leqslant 4c_4,\ \ t \geqslant t_0\}.$$

Define two maps S_1 and $S_2\colon\ \Omega \to C([t_0,\infty),\mathbb{R})$ as follows:

$$(S_1x)(t) = \begin{cases} 3c_4 + 1 - \dfrac{1}{C(t)}x(t+\tau), & t \geqslant T,\\ (S_1x)(T), \quad t_0 \leqslant t \leqslant T, \end{cases}$$

$$(S_2x)(t) = \begin{cases} \dfrac{(-1)^{n+1}}{C(t)(n-1)!}\displaystyle\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1} \\ \qquad \times\Bigl(\displaystyle\sum_{i=1}^{m} Q_i(s)f_i(x(s-\sigma_i)) - g(s)\Bigr)\,\mathrm{d}s, \quad t \geqslant T,\\ (S_2x)(T), \quad t_0 \leqslant t \leqslant T. \end{cases}$$

We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $t \geqslant T$, we get

$$\begin{aligned} &(S_1x)(t) + (S_2y)(t)\\ &\quad\leqslant 3c_4 + 1 - \frac{1}{C(t)}x(t+\tau)\\ &\qquad + \frac{1}{C(t)}\frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}\Bigl(\sum_{i=1}^{m}|Q_i(s)|\,|f_i(y(s-\sigma_i))| + |g(s)|\Bigr)\,\mathrm{d}s\\ &\quad\leqslant 3c_4 + 1 + \frac{1}{c_4}\frac{1}{(n-1)!}\int_{T+\tau}^{\infty}s^{n-1}\Bigl(\sum_{i=1}^{m}(|Q_i(s)|M_4 + |g(s)|\Bigr)\,\mathrm{d}s\\ &\quad\leqslant 3c_4 + 1 + (c_4 - 1) = 4c_4. \end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
&(S_1x)(t)+(S_2y)(t)\\
&\quad\geqslant 3c_4+1-\frac{1}{C(t)}x(t+\tau)\\
&\qquad-\frac{1}{C(t)}\frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t)^{n-1}\Bigl(\sum_{i=1}^{m}|Q_i(s)|\,|f_i(y(s-\sigma_i))|+|g(s)|\Bigr)\,\mathrm{d}s\\
&\quad\geqslant 3c_4+1-4-\frac{1}{c_4}\frac{1}{(n-1)!}\int_{T}^{\infty}s^{n-1}\Bigl(\sum_{i=1}^{m}|Q_i(s)|M_4+|g(s)|\Bigr)\,\mathrm{d}s\\
&\quad\geqslant 3c_4-3-(c_4-1)=2(c_4-1).
\end{aligned}
$$

Hence,

$$2(c_4-1)\leqslant S_1x(t)+S_2y(t)\leqslant 4c_4,\quad \text{for } t\geqslant t_0.$$

Thus we have proved that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

Proceeding similarly as in the proof of Theorem 1 we obtain that the mapping S_1 is a contractive mapping on Ω and the mapping S_2 is completely continuous. By Lemma 1, there is an $x_0\in\Omega$ such that $S_1x_0+S_2x_0=x_0$. Clearly, $x_0=x_0(t)$ is a bounded nonoscillatory solution of the equation (1). This completes the proof of Theorem 4. □

Theorem 5. *Assume that $C(t)\equiv 1$ and that (4) and (5) hold. Then (1) has a nonoscillatory bounded solution.*

Proof. By (4) and (5), we choose a $T>t_0$ sufficiently large such that

$$\frac{1}{(n-1)!}\int_{T+\tau}^{\infty}s^{n-1}\Bigl(\sum_{i=1}^{m}|Q_i(s)|M_5+|g(s)|\Bigr)\,\mathrm{d}s\leqslant 1,$$

where $M_5=\max\limits_{2\leqslant x\leqslant 4}\{f_i(x)\colon\ 1\leqslant i\leqslant m\}$.

We define a closed, bounded and convex subset Ω of $C([t_0,\infty),\mathbb{R})$ as follows:

$$\Omega=\{x=x(t)\in C([t_0,\infty),\mathbb{R})\colon\ 2\leqslant x(t)\leqslant 4,\ \ t\geqslant t_0\}.$$

Define a mapping $S\colon\ \Omega\to C([t_0,\infty),\mathbb{R})$ as follows:

$$
(Sx)(t)=\begin{cases}
3+\dfrac{(-1)^{n+1}}{(n-1)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}(s-t)^{n-1}\\
\qquad\times\Bigl(\displaystyle\sum_{i=1}^{m}Q_i(s)f_i(x(s-\sigma_i))-g(s)\Bigr)\,\mathrm{d}s, & t\geqslant T,\\
(Sx)(T), \quad t_0\leqslant t\leqslant T.
\end{cases}
$$

We shall show that $S\Omega \subset \Omega$.

In fact, for every $x \in \Omega$ and $t \geqslant T$, we get

$$\begin{aligned}(Sx)(t) &\leqslant 3 + \frac{1}{(n-1)!} \\ &\quad\times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} (s-t)^{n-1} \Big(\sum_{i=1}^{m} |Q_i(s)|\, |f_i(x(s-\sigma_i))| + |g(s)| \Big)\, \mathrm{d}s \\ &\leqslant 3 + \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} s^{n-1} \Big(\sum_{i=1}^{m} |Q_i(s)| M_5 + |g(s)| \Big)\, \mathrm{d}s \leqslant 4.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}(Sx)(t) &\geqslant 3 - \frac{1}{(n-1)!} \\ &\quad\times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} (s-t)^{n-1} \Big(\sum_{i=1}^{m} |Q_i(s)|\, |f_i(x(s-\sigma_i))| + |g(s)| \Big)\, \mathrm{d}s \\ &\geqslant 3 - \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} s^{n-1} \Big(\sum_{i=1}^{m} |Q_i(s)| M_5 + |g(s)| \Big)\, \mathrm{d}s \geqslant 2.\end{aligned}$$

Hence, $S\Omega \subset \Omega$.

Proceeding similarly as in the proof of Theorem 1 we obtain that the mapping S is completely continuous. By Lemma 2, there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$, that is

$$x_0(t) = \begin{cases} 3 + \dfrac{(-1)^{n+1}}{(n-1)!} \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} (s-t)^{n-1} \\ \qquad \times \Big(\displaystyle\sum_{i=1}^{m} Q_i(s) f_i(x(t-\sigma_i)) - g(s) \Big)\, \mathrm{d}s, \quad t \geqslant T, \\ x_0(T), \quad t_0 \leqslant t \leqslant T. \end{cases}$$

It follows that

$$\begin{aligned}x(t) + x(t-\tau) &= 6 + \frac{(-1)^{n+1}}{(n-1)!} \\ &\quad\times \int_t^{\infty} (s-t)^{n-1} \Big(\sum_{i=1}^{m} Q_i(t) f_i(x(t-\sigma_i))) - g(t) \Big)\, \mathrm{d}s, \ \ t \geqslant T.\end{aligned}$$

Clearly, $x_0 = x_0(t)$ is a bounded nonoscillatory solution of the equation (1). This completes the proof of Theorem 5. □

Remark 1. For the special case $n = 1$ or $n = 2$, Theorems 1–5 improve essentially Theorem A and B by removing the restrictive conditions H_2) and H_4) and relaxing the hypotheses H_1) and H_3).

Remark 2. For the special case $C(t) \equiv -1$, it is also possible that the equation (1) has no nonoscillatory solution in spite of the fact that (4) and (5) hold. For example, consider the neutral differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(x(t) - x(t-\tau)) + \frac{1}{t^\alpha}x(t-\sigma) = 0, \tag{6}$$

where n is an odd integer, $\tau > 0$, $\sigma \geqslant 0$, $n < \alpha < n+1$. Clearly, (4) and (5) hold. But, by Theorem 3.2 in [13], the equation (6) has no nonoscillatory solution.

Theorem 6. *Assume that* $C(t) \equiv -1$ *and that*

$$\int_{t_0}^{\infty} t^n |Q_i(t)| \,\mathrm{d}t < \infty, \quad i = 1, 2, \ldots, m \tag{7}$$

and

$$\int_{t_0}^{\infty} t^n |g(t)| \,\mathrm{d}t < \infty. \tag{8}$$

Then (1) *has a nonoscillatory bounded solution.*

Proof. By a known result [5, Theorem 3.2.6], (7) and (8) are equivalent to

$$\sum_{j=0}^{\infty} \int_{t_0 + j\tau}^{\infty} t^{n-1} |Q_i(t)| \,\mathrm{d}t < \infty, \quad i = 1, 2, \ldots, m \tag{9}$$

and

$$\sum_{j=0}^{\infty} \int_{t_0 + j\tau}^{\infty} t^{n-1} |g(t)| \,\mathrm{d}t < \infty, \tag{10}$$

respectively. We choose a sufficiently large $T > t_0$ such that

$$\frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} s^{n-1} \left(\sum_{i=1}^{m} |Q_i(s)| M_6 + |g(s)| \right) \mathrm{d}s \leqslant 1,$$

where $M_6 = \max\limits_{0 \leqslant x \leqslant 1} \{f_i(x) \colon\ 1 \leqslant i \leqslant m\}$.

We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbb{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), \mathbb{R}) \colon\ 2 \leqslant x(t) \leqslant 4,\ \ t \geqslant t_0\}.$$

Define a mapping $S\colon \Omega \to C([t_0,\infty),\mathbb{R})$ as follows:

$$(Sx)(t) = \begin{cases} 3 + \dfrac{(-1)^n}{(n-1)!} \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} (s-t)^{n-1} \\ \quad \times \Bigl(\displaystyle\sum_{i=1}^{m} Q_i(s) f_i(x(s-\sigma_i)) - g(s)\Bigr)\,\mathrm{d}s, \qquad t \geqslant T, \\ \\ (Sx)(T), \quad t_0 \leqslant t \leqslant T. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$ and $t \geqslant T$, we get

$$\begin{aligned} (Sx)(t) &\leqslant 3 + \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} (s-t)^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)|\Bigr)\,\mathrm{d}s \\ &\leqslant 3 + \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} s^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)| M_6 + |g(s)|\Bigr)\,\mathrm{d}s \leqslant 4. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} (Sx)(t) &\geqslant 3 - \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} (s-t)^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)|\Bigr)\,\mathrm{d}s \\ &\geqslant 3 - \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} s^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)| M_6 + |g(s)|\Bigr)\,\mathrm{d}s \geqslant 2. \end{aligned}$$

Hence, $S\Omega \subset \Omega$.

We now show that S is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geqslant T$, we have

$$\begin{aligned} &|(Sx_k)(t) - (Sx)(t)| \\ &\quad\leqslant \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} s^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)|\,|f_i(x_k(s-\sigma_i)) - f_i(x(s-\sigma_i))|\Bigr)\,\mathrm{d}s. \end{aligned}$$

Since $|f_i(x_k(t-\sigma_i)) - f_i(x(t-\sigma_i))| \to 0$ as $k \to \infty$ for $i = 1,2,\dots m$, by applying the Lebesgue dominated convergence theorem, we conclude that $\lim\limits_{k\to\infty} \|(Sx_k)(t) - (Sx)(t)\| = 0$. This means that S is continuous.

In the following, we show that $S\Omega$ is relatively compact. By (9) and (10), for any $\varepsilon > 0$, take $T^* \geqslant T$ large enough so that

$$\frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{T^*+j\tau}^{\infty} s^{n-1} \Bigl(M_6 \sum_{i=1}^{m} |Q_i(s)| + |g(s)|\Bigr)\,\mathrm{d}s < \frac{\varepsilon}{2}.$$

Then for $x \in \Omega$, $t_2 > t_1 \geqslant T^*$

$$
\begin{aligned}
&|(Sx)(t_2) - (Sx)(t_1)| \\
&\quad \leqslant \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t_2+j\tau}^{\infty} s^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)| \Bigr)\,\mathrm{d}s \\
&\qquad + \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t_1+j\tau}^{\infty} s^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)| \Bigr)\,\mathrm{d}s \\
&\quad \leqslant \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t_2+j\tau}^{\infty} s^{n-1} \Bigl(M_6 \sum_{i=1}^{m} |Q_i(s)| + |g(s)| \Bigr)\,\mathrm{d}s \\
&\qquad + \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t_1+j\tau}^{\infty} s^{n-1} \Bigl(M_6 \sum_{i=1}^{m} |Q_i(s)| + |g(s)| \Bigr)\,\mathrm{d}s \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

For $T \leqslant t_1 < t_2 \leqslant T^*$, we choose a sufficiently large $J \in \mathbb{N}^+$ such that $T + j\tau \geqslant T^*$ if $j \geqslant J$. For $x \in \Omega$

$$
\begin{aligned}
&|(Sx)(t_2) - (Sx)(t_1)| \\
&\quad \leqslant \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{t_1+j\tau}^{t_2+j\tau} s^{n-1} \Bigl(\sum_{i=1}^{m} |Q_i(s)|\,|f_i(x(s-\sigma_i))| + |g(s)| \Bigr)\,\mathrm{d}s \\
&\quad \leqslant \frac{1}{(n-1)!} \Bigl[\sum_{j=1}^{J} \int_{t_1+j\tau}^{t_2+j\tau} s^{n-1} \Bigl(M_6 \sum_{i=1}^{m} |Q_i(s)| + |g(s)| \Bigr)\,\mathrm{d}s \\
&\qquad + \sum_{j=J+1}^{\infty} \int_{t_1+j\tau}^{t_2+j\tau} s^{n-1} \Bigl(M_6 \sum_{i=1}^{m} |Q_i(s)| + |g(s)| \Bigr)\,\mathrm{d}s \Bigr] \\
&\quad \leqslant \frac{1}{(n-1)!} \Bigl[\max_{T+\tau \leqslant s \leqslant T^*+(J-1)\tau} \Bigl\{ s^{n-1} \Bigl(M_6 \sum_{i=1}^{m} |Q_i(s)| + |g(s)| \Bigr) \Bigr\} J(t_2 - t_1) \\
&\qquad + \sum_{j=1}^{\infty} \int_{T^*+j\tau}^{\infty} s^{n-1} \Bigl(M_6 \sum_{i=1}^{m} |Q_i(s)| + |g(s)| \Bigr)\,\mathrm{d}s \Bigr].
\end{aligned}
$$

Thus there exists a $\delta > 0$ such that

$$|(Sx)(t_2) - (Sx)(t_1)| < \varepsilon, \quad \text{if } 0 < t_2 - t_1 < \delta.$$

For any $x \in \Omega$, $t_0 \leqslant t_1 < t_2 \leqslant T$, it is easy to see that

$$|(Sx)(t_2) - (Sx)(t_1)| = 0 < \varepsilon.$$

Therefore $\{Sx\colon x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S\Omega$ is relatively compact. By Lemma 2 (Schauder's fixed point theorem), there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. That is,

$$x_0(t) = \begin{cases} 3 + \dfrac{(-1)^n}{(n-1)!} \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} (s-t)^{n-1} \\ \quad \times \Bigl(\displaystyle\sum_{i=1}^{m} Q_i(s) f_i(x_0(s-\sigma_i)) - g(s)\Bigr)\, \mathrm{d}s, \quad t \geqslant T, \\ x_0(T), \quad t_0 \leqslant t \leqslant T. \end{cases}$$

It follows that

$$x(t) - x(t-\tau) = \frac{(-1)^{n+1}}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} \Bigl(\sum_{i=1}^{m} Q_i(t) f_i(x(t-\sigma_i))) - g(t)\Bigr)\, \mathrm{d}s, \quad t \geqslant T.$$

Clearly, $x_0 = x_0(t)$ is a bounded nonoscillatory solution of the equation (1). This completes the proof of Theorem 6. □

Remark 3. Only minor adjustments are necessary to discuss the neutral functional differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t) + C(t)x(t-\tau)] + F(t, x(\sigma_1(t)), \ldots, x(\sigma_m(t))) = g(t), \quad t \geqslant t_0$$

where $F\colon [t_0, \infty) \times \mathbb{R} \times \ldots \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded, $\sigma_i(t) \to \infty$ ($i = 1, 2, \ldots, m$) as $t \to \infty$, and $m \geqslant 1$ is an integer. We omit the details.

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