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ON A REPRESENTATION OF MONOUNARY ALGEBRAS

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Abstract. In this note we deal with a question concerning monounary algebras which is analogous to an open problem for partially ordered sets proposed by Duffus and Rival.

Keywords: monounary algebra, connectedness, retract, retract irreducibility, representation

MSC 2000: 08A60

1. Introduction

Duffus and Rival [1] studied a certain form of representability of partially ordered sets. The representation under consideration was defined by means of retracts. In [1] it was remarked that "the following important problem remains unsolved:

($*$) Does every partially ordered set have a representation $\{P_i\colon i \in I\}$ such that P_i for each $i \in I$ is irreducible?"

(The detailed definitions of the notions of representation and irreducibility are recalled in Section 2 below.)

We remark that a monounary algebra can be viewed as a particular case of a quasi-ordered set. Namely, a monounary algebra is defined to be an algebraic structure $\mathscr{A} = (A, f)$, where A is a non-empty set and f is a unary operation on A. To each monounary algebra $\mathscr{A}$ there corresponds a quasi-ordered set $Q = (A, \leqslant)$, where the relation $\leqslant$ is defined as follows: if $a, b \in A$, then $a \leqslant b$, whenever $f^n(a) = b$ for some $n \in \mathbb{N} \cup \{0\}$. Conversely, the quasi-ordered set $Q = (A, \leqslant)$ uniquely defines a monounary algebra $\mathscr{A} = (A, f)$.

Retracts and retract irreducibility of monounary algebras were studied in the author's papers [2]–[7]. Let $\mathscr{U}$ be the class of all monounary algebras and let $\mathscr{U}_c$ be

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the class of all connected monounary algebras. For $\mathscr{A} \in \mathscr{U}$ let $R(\mathscr{A})$ be the system of all isomorphic copies of all retracts of $\mathscr{A}$.

In the present paper we deal with a question analogous to $(*)$ concerning a representation of a monounary algebra $\mathscr{A}$ in a class $\mathscr{K}$ for the case when $\mathscr{K} \in \{\mathscr{U}, \mathscr{U}_c, R(\mathscr{A})\}$; let us denote this question by $(**)$. We prove that the answer to $(**)$ is "No".

2. On the question $(**)$

We start by recalling some definition.

First we recall some definitions for partially ordered sets.

Let P be a partially ordered set and let $R(P)$ be the system of all partially ordered sets Q such that Q is isomorphic to some retract of P. We say that P is irreducible, if, whenever $P_i \in R(P)$ for $i \in I$ and $P \in R\Big(\prod\limits_{i\in I} P_i\Big)$, then there is $j \in I$ such that $P \in R(P_j)$. If $P \in R\Big(\prod\limits_{i\in I} P_i\Big)$ and $P_i \in R(P)$ for each $i \in I$, then the system $\{P_i \colon i \in I\}$ is called a representation of P. (Cf. Duffus and Rival [1].)

We will use these definitions of representation and of irreducibility for monounary algebras.

Let $\mathscr{A} = (A, f) \in \mathscr{U}$. A nonempty subset M of A is said to be a retract of $\mathscr{A}$ if there is a mapping h of A onto M such that h is an endomorphism of $\mathscr{A}$ and $h(x) = x$ for each $x \in M$. The mapping h is called a retraction endomorphism corresponding to the retract M. Further, let $R(\mathscr{A})$ be the system of all monounary algebras $\mathscr{B}$ such that $\mathscr{B}$ is isomorphic to (M, f) for some retract M of $\mathscr{A}$.

Let $\mathscr{K}$ be a system of monounary algebras. In [4] there was introduced the following definition: an element $\mathscr{A}$ of $\mathscr{U}$ is said to be retract irreducible in $\mathscr{K}$, if, whenever $\mathscr{B}_i \in \mathscr{K}$ for $i \in I$ and $\mathscr{A} \in R\Big(\prod\limits_{i\in I} \mathscr{B}_i\Big)$, then there is $j \in I$ such that $\mathscr{A} \in R(\mathscr{B}_j)$.

In [2] and [3] there were described all $\mathscr{A} \in \mathscr{U}_c$ which are retract irreducible in $\mathscr{U}_c$ and in [4] all $\mathscr{A} \in \mathscr{U}_c$ which are retract irreducible in $\mathscr{U}$. Further, in [6] and [7] there were found all $\mathscr{A} \in \mathscr{U}_c$ such that $\mathscr{A}$ is retract irreducible in $R(\mathscr{A})$ (they were denoted as irreducible in the sense of Duffus and Rival, or, shortly, DR-irreducible). All $\mathscr{A} \in \mathscr{U}$ which are retract irreducible in $\mathscr{U}$ were described in [5].

Analogously as for partially ordered sets we define the following notion. Let $\mathscr{A} \in \mathscr{U}$, $\mathscr{K} \subseteq \mathscr{U}$. A system $\{B_i \colon i \in I\} \subseteq \mathscr{K}$ will be called a representation of $\mathscr{A}$ in $\mathscr{K}$, if $\mathscr{A} \in R\Big(\prod\limits_{i\in I} \mathscr{B}_i\Big)$.

We will consider the following question for a class $\mathscr{K} \subseteq \mathscr{U}$: $(**)$ Does every monounary algebra $\mathscr{A}$ have a representation $\{\mathscr{B}_i : i \in I\}$ in $\mathscr{K}$ such that $\mathscr{B}_i$ for each $i \in I$ is retract irreducible in $\mathscr{K}$?

The aim of this paper is to prove

Theorem. *There exists a connected monounary algebra $\mathscr{A}$ such that if $\mathscr{K} \in \{\mathscr{U}, \mathscr{U}_c, R(\mathscr{A})\}$, then $\mathscr{A}$ possesses no representation $\{\mathscr{B}_i : i \in I\}$ of $\mathscr{A}$ in $\mathscr{K}$ such that for each $i \in I$, $\mathscr{B}_i$ is retract irreducible in $\mathscr{K}$.*

3. The class $\mathscr{K} = R(\mathscr{A})$

In the following notation suppose that distinct symbols mean distinct elements.

3.1. Notation. *For $n \in \mathbb{N}$ let*

$$A_n = \{j_n : j \in \{1, \dots, n\}\}.$$

Put

$$A = \mathbb{N} \cup \bigcup_{n \in \mathbb{N}} A_n.$$

Further let

$$f(n) = n + 1 \quad \text{for each } n \in \mathbb{N},$$
$$f(j_n) = \begin{cases} (j+1)_n & \text{for each } n \in \mathbb{N},\ j \in \{1, \dots, n-1\}, \\ 1 & \text{for each } n \in \mathbb{N},\ j = n. \end{cases}$$

Denote $\mathscr{A} = (A, f)$ (cf. Fig. 1*).*

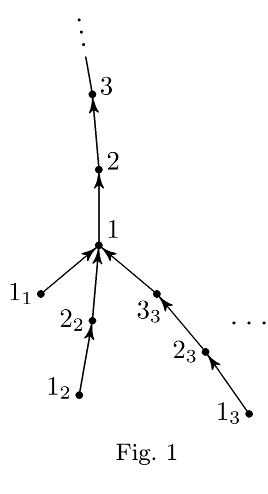

Fig. 1

Then we obviously have

3.2. Lemma. *$\mathscr{A}$ is a connected monounary algebra.*

3.3. Lemma. *Let M be a retract of $\mathscr{A}$. Then $1 \in M$ and* $\operatorname{card}(f^{-1}(1) \cap M) > 2$.

Proof. It is obvious that if M is a retract of $\mathscr{A}$, then $\mathbb{N} \subseteq M$. Suppose that $\operatorname{card}(f^{-1}(1) \cap M) \leqslant 2$. Then there is $n \in \mathbb{N}$ such that $k_k \notin M$ for each $k \in \mathbb{N}$, $k \geqslant n$. Hence $A_k \cap M = \emptyset$ for each $k \in \mathbb{N}$, $k \geqslant n$. Let h be a retraction endomorphism corresponding to M. Denote $z = h(1_n)$. Then

$$z \in M \subseteq \mathbb{N} \cup A_1 \cup \ldots \cup A_{n-1},$$

thus

$$f^n(z) \in \mathbb{N} - \{1\}. \tag{1}$$

Further, $1 \in M$, hence

$$f^n(z) = f^n(h(1_n)) = h(f^n(1_n)) = h(1) = 1,$$

a contradiction to (1). □

3.4. Corollary. *If M is a retract of $\mathscr{A}$, then (M, f) is not retract irreducible in the class $R(\mathscr{A})$.*

Proof. According to [6, 2.9] and [7, 4.1], we obtain that if (M, f) is retract irreducible in $R(\mathscr{A})$, then $\operatorname{card} f^{-1}(x) < 2$ for each $x \in M$. Hence 3.3 yields the required assertion. □

3.5. Proposition. *Let $\{B_i \colon i \in I\}$ be a representation of $\mathscr{A}$ in the class $R(\mathscr{A})$. Then B_i fails to be retract irreducible in $R(\mathscr{A})$ for each $i \in I$.*

Proof. Let $i \in I$. Then there exists a retract M of $\mathscr{A}$ such that

$$\mathscr{B}_i \cong (M, f). \tag{1}$$

By 3.4 and (1), B_i is not retract irreducible in the class $R(\mathscr{A})$. □

4. The class $\mathscr{K} = \mathscr{U}_c$

Let $\mathscr{A}$ be as in Section 3 and suppose that the system $\{\mathscr{B}_i \colon i \in I\} \subseteq \mathscr{U}_c$ is a representation of $\mathscr{A}$ such that if $i \in I$, then $\mathscr{B}_i$ is retract irreducible in $\mathscr{U}_c$. Then [2, (R)] and [3, (R1)] imply

4.1. Lemma. *If $i \in I$, then one of the following conditions is satisfied:*

(i) $\mathscr{B}_i$ *is a cycle with* p^m *elements, where* p *is a prime and* $m \in \mathbb{N}$,

(ii) $\mathscr{B}_i \cong (\mathbb{N}, f)$,

(iii) $\mathscr{B}_i$ *contains a one element cycle* $\{c\}$ *and if* $\{a, b\} \subseteq \mathscr{B}_i$ *with* $f(a) = f(b)$, *then either a=b or* $c \in \{a, b\}$.

The system $\{\mathscr{B}_i : i \in I\} \subseteq \mathscr{U}_c$ is a representation of $\mathscr{A}$, thus there is a retract M of $\prod_{i \in I} \mathscr{B}_i$ such that $\mathscr{A} \cong (M, f)$. Let ν be an isomorphism of $\mathscr{A}$ onto (M, f).

4.2. Lemma. *If $i \in I$, then* (ii) *fails to hold.*

P r o o f. Let $i \in I$ and suppose that (ii) is valid. Then $\mathscr{B}_i = \{c_n : n \in \mathbb{N}\}$ and $f(c_n) = c_{n+1}$ for each $n \in \mathbb{N}$, where $c_k \neq c_l$ for each $k, l \in \mathbb{N}$, $k \neq l$. Denote $t = \nu(1)$ and, if $n \in \mathbb{N}$, $\nu(1_n) = b_n$. Then there is $k \in \mathbb{N}$ such that

$$t(i) = c_k. \tag{1}$$

(The symbol $t(i)$ means the ith coordinate of the element t.)

We have

$$f^k(1_k) = 1,$$

thus

$$f^k(b_k) = f^k(\nu(1_k)) = \nu(f^k(1_k)) = \nu(1) = t,$$

hence (1) implies

$$c_k = t(i) = (f^k(b_k))(i) = f^k(b_k(i)),$$

i.e.,

$$b_k(i) \in f^{-k}(c_k) = \emptyset,$$

a contradiction. □

4.3. Lemma. *If $n \in \mathbb{N}$, then there exist $i_n \in I$, $m_n \in \mathbb{N}$ and a prime p_n such that*

(a) $p_1^{m_1} < p_2^{m_2} < \ldots$,

(b) $\mathscr{B}_{i_n}$ *is a cycle with* $p_n^{m_n}$ *elements.*

P r o o f. By 4.2, if $i \in I$, then $\mathscr{B}_i$ contains a cycle. If the cardinalities of these cycles are bounded, then each connected component of $\prod_{i \in I} \mathscr{B}_i$ contains a cycle, thus each subalgebra of $\prod_{i \in I} \mathscr{B}_i$ contains a cycle, hence (M, f) and $\mathscr{A}$, too, contain a cycle, which is a contradiction. Therefore the assertion is valid according to 4.1 and 4.2. □

4.4. Lemma. *There exist distinct elements* $b_n \in \prod_{i\in I} \mathscr{B}_i$ *for* $n \in \mathbb{N}$ *such that* $f(b_{n+1}) = b_n$ *for each* $n \in \mathbb{N}$.

P r o o f. Assume that, for each $n \in \mathbb{N}$, $\mathscr{B}_{i_n}$ is as in 4.3. By 4.2, if $i \in I$, then $\mathscr{B}_i$ contains a cycle; take an arbitrary element

$$b_1 \in \prod_{i\in I} \mathscr{B}_i$$

such that b₁(i) belongs to the cycle of $\mathscr{B}_i$. By induction, if $k \in \mathbb{N}$, $k > 1$ and b_j is defined for each $j \in \mathbb{N}$, $j < k$, then let b_k be the (unique) element of $\prod_{i\in I} \mathscr{B}_i$ such that

(1) $$b_k(i) \text{ belongs to the cycle of } \mathscr{B}_i,$$

(2) $$f(b_k(i)) = b_{k-1}(i).$$

Then obviously $f(b_{n+1}) = b_n$ for each $n \in \mathbb{N}$.

Suppose that there are $k, l \in \mathbb{N}$, $k < l$ such that $b_k = b_l$. In view of 4.3 (a) there exists $n \in \mathbb{N}$ such that

(3) $$l - k < p_n^{m_n}.$$

We have

$$b_k = f^{l-k}(b_l) = f^{l-k}(b_k),$$

thus

(4) $$b_k(i_n) = (f^{l-k}(b_k))(i_n) = f^{l-k}(b_k(i_n)).$$

The element $b_k(i_n)$ belongs to $\mathscr{B}_{i_n}$, i.e., to a cycle with $p_n^{m_n}$‘ elements, therefore (3) and (4) yield a contradiction. □

4.5. Corollary. *No retract of* $\prod_{i\in I} \mathscr{B}_i$ *is isomorphic to* $\mathscr{A}$.

P r o o f. For $n \in N$ let b_n be as in 4.4. If Q is a retract of $\prod_{i\in I} \mathscr{B}_i$ and φ is a corresponding retraction endomorphism, then either

(a) $\varphi(b_1)$ belongs to a cycle, or

(b) there are distinct elements $q_n \in Q$ for $n \in \mathbb{N}$ such that $\varphi(b_n) = q_n$ for each $n \in \mathbb{N}$.

Let Q be a retract of $\prod_{i\in I} \mathscr{B}_i$, $(Q, f) \cong \mathscr{A}$. Then (a) fails to hold. If (b) is valid, then, for each $n \in \mathbb{N}$,

$$f(q_{n+1}) = f(\varphi(b_{n+1})) = \varphi(f(b_{n+1})) = \varphi(b_n) = q_n,$$

which is again a contradiction to the definition of $\mathscr{A}$. □

As a corollary we obtain

4.6. Proposition. *$\mathscr{A}$ possesses no representation $\{\mathscr{B}_i\colon\ i \in I\}$ of $\mathscr{A}$ in $\mathscr{U}_c$ such that each $\mathscr{B}_i$ for $i \in I$ is retract irreducible in $\mathscr{U}_c$.*

5. The class $\mathscr{K} = \mathscr{U}$

Let $\mathscr{A}$ be as in the previous sections and suppose that the system $\{\mathscr{B}_i\colon\ i \in I\} \subseteq \mathscr{U}$ is a representation of $\mathscr{A}$ such that if $i \in I$, then $\mathscr{B}_i$ is retract irreducible in $\mathscr{U}$. In view of [5, Thm. 4.5] we obtain

5.1. Lemma. *If $i \in I$, then one of the following conditions is satisfied:*
(i) *$\mathscr{B}_i$ contains a cycle with p^m elements, where p is a prime and $m \in \mathbb{N}$,*
(ii) *$\mathscr{B}_i \cong (\mathbb{N}, f)$,*
(iii) *$\mathscr{B}_i$ contains a one element cycle $\{c\}$ and if $\{a, b\} \subseteq \mathscr{B}_i$ with $f(a) = f(b)$, then either a=b or $c \in \{a, b\}$.*

Analogously as above the following assertions can be proved:

5.2. Lemma. *If $i \in I$, then* (ii) *fails to hold.*

5.3. Lemma. *If $n \in \mathbb{N}$, then there exist $i_n \in I$, $m_n \in \mathbb{N}$ and a prime p_n such that*
(a) $p_1^{m_1} < p_2^{m_2} < \dots$,
(b) *$\mathscr{B}_{i_n}$ contains a cycle with $p_n^{m_n}$ elements.*

5.4. Lemma. *There exist distinct elements $b_n \in \prod_{i\in I} \mathscr{B}_i$ for $n \in \mathbb{N}$ such that $f(b_{n+1}) = b_n$ for each $n \in \mathbb{N}$.*

5.5. Lemma. *No retract of $\prod_{i\in I} \mathscr{B}_i$ is isomorphic to $\mathscr{A}$.*

5.6. Proposition. *$\mathscr{A}$ possesses no representation $\{\mathscr{B}_i\colon\ i \in I\}$ of $\mathscr{A}$ in $\mathscr{U}$ such that $\mathscr{B}_i$ for each $i \in I$ is retract irreducible in $\mathscr{U}$.*

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