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# A MORITA TYPE THEOREM FOR A SORT OF QUOTIENT CATEGORIES

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*Abstract.* We consider the quotient categories of two categories of modules relative to the Serre classes of modules which are bounded as abelian groups and we prove a Morita type theorem for some equivalences between these quotient categories.

*Keywords*: Morita theorem, quotient category, equivalent categories, adjoint functors

*MSC 2000*: 16D90, 16A50

## 1. Introduction

The notions "quasi-isomorphism", "quasi-direct decomposition" and other similar notions became important in the torsion free Abelian Groups Theory because they allowed B. Jönson to enunciate a Krull-Schmidt type theorem for torsion free groups of finite rank (see [5, Corollary 7.9]). In [13], E. Walker extended these notions to the category of abelian groups, observing that they originate in the quotient category $\mathcal{A}b/\mathcal{B}$, where $\mathcal{A}b$ is the category of abelian groups and $\mathcal{B}$ is the class of all bounded abelian groups. In [3], the authors introduced the notion of almost-flat modules in order to answer the question which of the properties of torsion free abelian groups as modules over their endomorphism rings are preserved by quasi-isomorphisms. The notion of almost (quasi-)projective module was used in [11] and [12] in order to characterize the torsion free abelian groups projective as modules over their endomorphism rings. In the same context Albrecht extended in [1] the classes of $A$-static groups and $A$-adstatic modules to the classes of almost $A$-static groups and almost $A$-adstatic modules requesting the arrows of adjunction, induced by the pair of adjoints functors $\operatorname{Hom}(A,-)\colon \mathcal{A}b \rightleftarrows \text{Mod-}E\colon -\otimes_E A$, to be quasi-isomorphisms. The class of almost $A$-static modules was used in [6] in order to characterize the modules which are almost-flat over their endomorphism rings.

In [7], using the Elbert Walker's techniques which were developed in [13], the authors give natural interpretations for the notions "almost projective" and "almost flat". We recall here some of the constructions and notions used in this paper: Let $\Sigma$ be a multiplicatively closed set of non-zero integers. We will say that an abelian group $A$ is $\Sigma$-bounded if there exists $n \in \Sigma$ such that $nA = 0$. If $R$ is a unital ring and Mod-$R$ is the category of right $R$-modules, then the class $\mathcal{S}$ of all right $R$-modules which are $\Sigma$-bounded as abelian groups is a Serre class. Hence the quotient category Mod-$R/\mathcal{S}$ exists and it is proved (analogously to [13, Theorem 3.1]) that this category is equivalent to the category $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$ which has as objects all the right $R$-modules and if $M, N \in$ Mod-$R$, then

$$\mathrm{Hom}_{\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R}(M, N) = \mathbb{Z}[\Sigma^{-1}] \otimes_{\mathbb{Z}} \mathrm{Hom}_R(M, N).$$

We will denote by $\mathbf{q}\colon$ Mod-$R \to \mathbb{Z}[\Sigma^{-1}]$Mod-$R$ the canonical functor. Note that $\mathbf{q}(M) = M$ for any $M \in$ Mod-$R$ and $\mathbf{q}(f) = 1 \otimes f$ for all $R$-homomorphisms $f$. Observe that every homomorphism in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$ can be written as $\frac{1}{n}f \overset{\text{not}}{=} \frac{1}{n} \otimes f$ where $f$ is an $R$-homomorphism and, as the multiplication by $n \in \Sigma$ represents an automorphism in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$ for all $n \in \Sigma$, $\frac{1}{n}f$ is respectively a monomorphism, an epimorphism, an isomorphism if and only if $\mathbf{q}(f)$ has the same property. We will say: a $R$-homomorphism $f$ is a $\mathbf{q}$-*monomorphism* if $\mathbf{q}(f)$ is a monomorphism (this means $\mathrm{Ker}(f)$ is a $\Sigma$-bounded group), it is an $\mathbf{q}$-*epimorphism* if $\mathbf{q}(f)$ is an epimorphism (equivalently, $\mathrm{Coker}(f)$ is a $\Sigma$-bounded group), and it is a $\mathbf{q}$-isomorphism if $\mathbf{q}(f)$ is an isomorphism. Note that $f\colon M \to N$ is a $\mathbf{q}$-isomorphism if and only if there exists an integer $n \in \Sigma$ and an $R$-homomorphism $g\colon N \to M$ such that $gf = n1_M$ and $fg = n1_N$. We say that $g$ is a $\mathbf{q}$-*inverse* for $f$. A $\mathbf{q}$-epimorphism ($\mathbf{q}$-monomorphism) $f\colon M \to N$ $\mathbf{q}$-*splits* if $\mathbf{q}(f)$ splits in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$. This means that there exists an $R$-homomorphism $g\colon N \to M$ and an integer $n \in \Sigma$ such that $fg = n1_N$ ($gf = n1_M$). Observe that in the case $\Sigma = \mathbb{Z}^\star$ is the set of all non-zero integers we find again Albrecht's "quasi-notions" presented in [2].

The definition of the quotient category modulo a Serre subcategory as a category of additive fractions is given in [9, Section 4.7] (see also [8, Corollaire 3.2]). Let us observe that if $F\colon$ Mod-$R \to$ Mod-$S$ is an additive functor, then it induces a canonical functor $qF\colon \mathbb{Z}[\Sigma^{-1}]$Mod-$R \to \mathbb{Z}[\Sigma^{-1}]$Mod-$S$ such that $\mathbf{q}F = qF\mathbf{q}$, where $\mathbf{q}$ denotes both the canonical functors Mod-$R \to \mathbb{Z}[\Sigma^{-1}]$Mod-$R$ and Mod-$S \to \mathbb{Z}[\Sigma^{-1}]$Mod-$S$. In [7] it is proved that a right $R$-module $P$ is projective in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$ (called $\Sigma$-*almost projective*) if and only if the functor $q\,\mathrm{Hom}_R(P, -)\colon \mathbb{Z}[\Sigma^{-1}]$Mod-$R \to \mathbb{Z}[\Sigma^{-1}]\mathcal{A}b$ is exact. In the case $\Sigma = \mathbb{Z}^\star$ this notion coincides with the notion presented and used in [12] and [11]. It is proved that an $R$-module $P$ is $\Sigma$-almost projective if and only if there exist a projective (free) module $F$ and a $\mathbf{q}$-epimorphism $\alpha\colon F \to P$

such that $\alpha$ **q**-splits. Note that the almost flat left $R$-modules, introduced in [3], are just the left $R$-modules $A$ such that the functor $q(-\otimes_R A)\colon \mathbb{Q}\text{Mod-}R \to \mathbb{Q}\mathcal{A}b$ is exact.

The main result of this paper is Theorem 3.10, which proves a statement analogous to the Morita Theorem, [4, Theorem 22.2] and [14, 46.4], for the special case of equivalences between two quotient categories $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ and $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}S$. As the Morita Theorem justifies the introduction of static and adstatic modules, our theorem justifies, in the case $\Sigma = \mathbb{Z}^\star$, the definitions of almost static and almost adstatic modules.

## 2. **q**-generators

Recall that if $G$ is a right $R$-module with the endomorphism ring $E$, then $G$ becomes a left $E$-module and we have a pair of adjoint functors

$$\mathrm{H}_G = \mathrm{Hom}_R(G,-)\colon \text{Mod-}R \rightleftarrows \text{Mod-}E\colon -\otimes_E G = \mathrm{T}_G$$

with the canonical arrows $\varphi\colon \mathrm{T}_G\mathrm{H}_G \to 1$, $\varphi_M(\alpha\otimes g) = \alpha(g)$ and $\psi\colon 1 \to \mathrm{H}_G\mathrm{T}_G$, $\psi_X(x)(g) = x\otimes g$, for all $M \in \text{Mod-}R$ and $X \in \text{Mod-}E$. They induce the pair of adjoint functors

$$q\mathrm{H}_G\colon \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightleftarrows \mathbb{Z}[\Sigma^{-1}]\text{Mod-}E\colon q\mathrm{T}_G.$$

We mention that $q\mathrm{H}_G(M) = \mathrm{H}_G(M)$ for all $M \in \text{Mod-}R$ and that for every $R$-homomorphism $\gamma\colon M \to N$ we have $q\mathrm{H}_G\left(\frac{1}{n}\gamma\right) = \frac{1}{n}\mathrm{H}_G(\gamma)$.

We will say that $G \in \text{Mod-}R$ is a **q**-*generator* if $q\mathrm{H}_G\colon q\text{Mod-}R \to q\text{Mod-}E$ is a faithful functor.

**Proposition 2.1.** *If $G \in \text{Mod-}R$, then the following assertions are equivalent:*

a) *$G$ is a **q**-generator;*
b) *for each $M \in \text{Mod-}R$, there exists a set $\Lambda$ and a **q**-epimorphism $\varphi\colon G^{(\Lambda)} \to M$;*
c) *for every right $R$-module $M$ the canonical homomorphism $\varphi_M\colon \mathrm{T}_G\mathrm{H}_G(M) \to M$ is a **q**-epimorphism.*

Proof. a) $\Rightarrow$ b) If $\Lambda = \mathrm{Hom}_R(G,M)$, we will prove that the canonical homomorphism $\varphi\colon G^{(\Lambda)} \to M$, induced by the direct sum and the family of homomorphisms $\{\alpha\colon G \to M \mid \alpha \in \Lambda\}$, is an epimorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$. Let $(1/n)\gamma\colon M \to N$ be a homomorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ such that $((1/n)\gamma)\varphi = 0$ (in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$). Then $\gamma\varphi = 0$ in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$, hence $\mathrm{Im}(\gamma\varphi)$ is $\Sigma$-bounded. Therefore, there exists an integer $n \in \Sigma$ such that $n\gamma\varphi i_\alpha = 0$ for all $\alpha \in \Lambda$

($i_\alpha\colon G \to G^{(\Lambda)}$ is the canonical injection). It follows that $n\gamma\alpha = 0$ for all $\alpha \in \Lambda$ and we obtain $q\mathrm{H}_G(\gamma) = 0$. Because $q\mathrm{H}_G$ is faithful we obtain $\gamma = 0$ in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$, hence $(1/n)\gamma = 0$. Then $\varphi$ is an epimorphism in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$.

b) $\Rightarrow$ c) Let $\Lambda$ be a set such that there exists an $R$-homomorphism $\varphi\colon G^{(\Lambda)} \to M$ which is a **q**-epimorphism. Then we can find an integer $n \in \Sigma$ with $n\,\mathrm{Coker}(\varphi) = 0$. We obtain that for every $x \in M$ there exist $\lambda_1, \ldots, \lambda_k \in \Lambda$ and $g_1 \in G_{\lambda_1} = G, \ldots, g_k \in G_{\lambda_k} = G$ such that $nx = \varphi(g_1 \oplus \ldots \oplus g_k)$. It follows that $\varphi_M(\varphi i_{\lambda_1} \otimes g_1 + \ldots + \varphi i_{\lambda_k} \otimes g_k) = nx$ and this shows that $\varphi_M$ is a **q**-epimorphism.

c) $\Rightarrow$ a) If $\alpha\colon M \to N$ is an $R$-homomorphism such that $q\mathrm{H}_G(\alpha) = 0$, we use the commutative diagram

$$\begin{array}{ccc} \mathrm{T}_G\mathrm{H}_G(M) & \xrightarrow{\varphi_M} & M \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathrm{T}_G\mathrm{H}_G(N) & \xrightarrow{\varphi_N} & N \end{array}$$

to obtain $\alpha\varphi_M = 0$ in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$. Then $\alpha = 0$ in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$ because $\varphi_M$ represents an epimorphism in this category. $\square$

**Example 2.2.** If $p$ is a prime, the module $\mathbb{Q}$ is a **q**-generator in the category Mod-$\mathbb{Q} \times \mathbb{Z}(p)$ which is not a generator.

**Lemma 2.3.** *If $G$ is a* **q***-generator in* Mod-$R$ *and $M \in$* Mod-$R$ *is* **q***-isomorphic with a finitely generated right $R$-module, then there exists an integer $k > 0$ and an $R$-homomorphism $G^k \to M$ which is a* **q***-epimorphism.*

P r o o f. Observe that we can suppose without loss of generality that $M$ is a finitely generated right $R$-module. If $\langle x_1, \ldots, x_m\rangle = M$ and $\varphi\colon G^{(I)} \to M$ is an $R$-homomorphism such that $n\,\mathrm{Coker}(\varphi) = 0$ for an integer $n \in \Sigma$, we fix $g_1, \ldots, g_m \in G$ such that $\varphi(g_i) = nx_i$ for all $i = 1, \ldots, m$. Then there exists a submodule $G^{(J)}$ in $G^{(I)}$ which is a finite direct sum of copies of $G$ such that $g_i \in G^{(J)}$ for all $i = 1, \ldots, n$. The restriction $\varphi_{|G^{(J)}}\colon G^{(J)} \to M$ is a **q**-epimorphism. $\square$

We will say that an $R$-module $M$ is an $\Sigma$-*almost finitely generated* module if it is **q**-isomorphic to a finitely generated $R$-module.

**Proposition 2.4.** *If $G$ is a* **q***-generator in* Mod-$R$ *where $E$ is the endomorphism ring of $G$ and $B$ the biendomorphism ring of $G$, then:*

a) *$G$ is $\Sigma$-almost projective and $\Sigma$-almost finitely generated as a left $E$-module;*
b) *if $\vartheta\colon R \to B$, $\vartheta(r)(g) = gr$ is the canonical ring homomorphism, then* $\mathrm{Ker}(\vartheta)$ *and* $B/\mathrm{Im}(\vartheta)$ *are $\Sigma$-bounded as abelian groups.*

P r o o f. Using the previous lemma we find an exact sequence $G^k \xrightarrow{\varphi} R \to H \to 0$ in Mod-$R$ such that $H$ is $\Sigma$-bounded as an abelian group. Because $R$ is $\Sigma$-almost projective, there exist $n \in \Sigma$ and $\psi\colon R \to G^k$ such that $\varphi\psi = n1_R$.

a) We apply the contravariant functor $\mathrm{Hom}_R(-, G)\colon$ Mod-$R \to E$-Mod to the previous exact sequence concluding that the sequence

$$0 \to \mathrm{Hom}_R(H, G) \to \mathrm{Hom}_R(R, G) \overset{\mathrm{Hom}_R(\varphi,G)}{\longrightarrow} \mathrm{Hom}_R(G^k, G)$$

is exact. Therefore $\mathrm{Hom}_R(\varphi, G)$ is a $\mathbf{q}$-monomorphism which splits in the category $\mathbb{Z}[\Sigma^{-1}](E\text{-Mod})$ because we have $\mathrm{Hom}_R(\psi, G)\,\mathrm{Hom}_R(\varphi, G) = n1_{\mathrm{Hom}_R(R,G)}$, hence $G \cong \mathrm{Hom}_R(R, G)$ is $\Sigma$-almost projective as an $E$-module.

b) If $\mu\colon B(G^k) \to B$ is the canonical isomorphism which is presented in [4, 14.2] and $\lambda\colon R \to B(G^k)$ is the canonical ring homomorphism, then $\vartheta = \mu\lambda$. Consider $r \in R$ such that $\lambda(r) = 0$ and $x \in G^k$ with $\varphi(x) = n1$. Then $nr = \varphi(xr) = \varphi(\lambda(r)(x)) = 0$, hence the group $\mathrm{Ker}(\vartheta) = \mathrm{Ker}(\varphi)$ is bounded by $n \in \Sigma$.

To prove that $\mathrm{Coker}(\vartheta)$ is $\Sigma$-bounded, we apply the density theorem, [14, 15.7], to the following hypothesis: $G$ is $\Sigma$-almost finitely generated as a left $E$-module, hence there exist an integer $n \in \Sigma$ and an $E$-homomorphism $\alpha\colon E^k \to G$ with $n\,\mathrm{Coker}(\alpha) = 0$. We consider $x_1, \ldots, x_k \in G$ such that they generate an $E$-submodule $H$ with $nG/H = 0$. It follows that for every $\beta \in B$, the homomorphism $n\beta$ is determined by $\beta(x_1), \ldots, \beta(x_k)$. Then the density theorem shows that $n\beta$ is a multiplication by an element $r \in R$. Therefore, $n(B/\mathrm{Im}(\vartheta)) = 0$. □

## 3. The main theorem

Let $R$ and $S$ be unital rings. Consider a pair of functors $F\colon$ Mod-$R \rightleftarrows$ Mod-$S\colon G$ such that

i) there exists a natural transformation $\varphi\colon GF \to 1_{\text{Mod-}R}$ such that $\varphi_A$ is a $\mathbf{q}$-isomorphism for all $A \in$ Mod-$R$,

ii) there exists a natural transformation $\psi\colon 1_{\text{Mod-}S} \to FG$ such that $\psi_X$ is a $\mathbf{q}$-isomorphism for all $X \in$ Mod-$S$.

We will say that $F$ is a *good* $\mathbf{q}$*-equivalence* and $G$ is a *good* $\mathbf{q}$*-inverse for $F$*. Observe that under these conditions the functors $qF\colon \mathbb{Z}[\Sigma^{-1}]$Mod-$R \rightleftarrows \mathbb{Z}[\Sigma^{-1}]$Mod-$S$: $qG$ are equivalences.

**Remark 3.1.** In this case it follows that

i) for every $A \in$ Mod-$R$ there exist an integer $n_A \in \Sigma$ and $\varphi'_A\colon A \to GF(A)$ such that $\varphi_A\varphi'_A = n_A 1_A$ and $\varphi'_A\varphi_A = n_A 1_{GF(A)}$,

ii) for every $X \in$ Mod-$S$ there exist an integer $m_X \in \Sigma$ and $\psi'_A \colon FG(X) \to X$ such that $\psi_X \psi'_X = m_X 1_{FG(X)}$ and $\psi'_A \psi_A = m_X 1_X$.

Moreover, if $A \cong A'$ and $X \cong X'$, we can suppose that $n_A = n_{A'}$ and $m_X = m_{X'}$.

If $\varphi\colon R \to S$ is a unital ring homomorphism, then we obtain a pair of adjoint functors $(\varphi_\star, \varphi^\star)$, where

$$\varphi_\star\colon S\text{-Mod} \to R\text{-Mod}, \quad \varphi_\star(A) = A, \quad \varphi_\star(f) = f$$

is the restriction of scalars and

$$\varphi^\star\colon R\text{-Mod} \to S\text{-Mod}, \quad \varphi^\star(C) = S \otimes_R C, \quad \varphi^\star(f) = 1_S \otimes f$$

is the extension of scalars [10, IV.9]. They induce the natural transformations

$$\xi\colon \varphi^\star\varphi_\star \to 1_{S\text{-Mod}} \quad \text{with } \xi_A\colon \varphi^\star\varphi_\star(A) \to A, \quad s \otimes a \mapsto sa$$

and

$$\zeta\colon 1_{R\text{-Mod}} \to \varphi_\star\varphi^\star \quad \text{with } \zeta_C\colon C \to \varphi_\star\varphi^\star(C), \quad c \mapsto 1 \otimes c$$

for all $A \in S$-Mod and $C \in R$-Mod.

In [6, Lemma 3.1] the following result is proved which will be useful in the enunciation of our main theorem.

**Proposition 3.2** [6, Lemma 3.1]. *If $R$ and $S$ are rings such that there exists a unital ring homomorphism $\varphi\colon R \to S$ such that the groups $\mathrm{Ker}(\varphi)$ and $S/\mathrm{Im}(\varphi)$ are $\Sigma$-bounded, then $\varphi_\star$ is a good* **q***-equivalence and $\varphi^\star$ is a good* **q***-inverse for $\varphi_\star$.*

We revert to the general case.

**Proposition 3.3.** *Let $F\colon$ Mod-$R \rightleftarrows$ Mod-$S$ be a good* **q***-equivalence and let $G$ be a good* **q***-inverse for $F$. Then*

a) *the group homomorphism*

$$F\colon \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_S(F(A), F(B))$$

*is a* **q***-isomorphism for all $A, B \in$ Mod-$R$;*

a′) *the group homomorphism*

$$G\colon \mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_R(G(X), G(Y))$$

*is a* **q***-isomorphism for all $X, Y \in$ Mod-$S$;*

b) *If $X \in$ Mod-$S$, then there exists $A \in$ Mod-$R$ such that $F(A)$ and $X$ are* **q***-isomorphic.*

P r o o f. a), a′) Let $f\colon A \to B$ be an $R$-homomorphism such that $F(f) = 0$. We obtain that $GF(f) = 0$ and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \varphi_A}\uparrow & & \uparrow{\scriptstyle \varphi'_B} \\ GF(A) & \xrightarrow{GF(f)} & GF(B), \end{array}$$

shows that $\operatorname{Im}(\varphi_A) \subseteq \operatorname{Ker}(f)$. It follows that $\operatorname{Im}(f) \cong A/\operatorname{Ker}(f)$ is a homomorphic image of $A/\operatorname{Im}(\varphi_A)$, hence $n_A \operatorname{Ker}(F) = 0$.

Analogously, it follows that $\operatorname{Ker}(G)$ is bounded by $m_Y$.

If $g \in \operatorname{Hom}_S(F(A), F(B))$ and $h = \varphi_B G(g) \varphi'_A$, we obtain the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & B \\ {\scriptstyle \varphi_A}\uparrow & & \uparrow{\scriptstyle \varphi_B} \\ GF(A) & \xrightarrow{G(n_A g)} & GF(B) \end{array}$$

and it follows that $\varphi_B G(n_A g) = \varphi_B GF(h)$, hence we have $n_B G(n_A g) = n_B G(F(h))$. Because the kernel of $G$ is bounded by $m_{F(B)}$ we find

$$m_{F(B)} n_A n_B g = F(m_{F(B)} n_B h)$$

and this shows that $F$ is **q**-epic.

The proof of the fact that $G$ is a **q**-epimorphism is analogous. We obtain that for every $f\colon G(X) \to G(Y)$ there exists $h = \psi'_Y F(f) \varphi_X$ such that

$$n_{G(X)} m_X m_Y f = G(n_{G(X)} m_X h).$$

The definition of good **q**-equivalences ensures that b) is valid. □

**Proposition 3.4.** *Let $F\colon \text{Mod-}R \rightleftarrows \text{Mod-}S\colon G$ be a pair of functors such that $F$ is a good* **q***-equivalence and $G$ is a good* **q***-inverse for $F$.*

a) *For every family $(X_i)_{i\in I}$ of right $S$-modules, the canonical monomorphism*

$$\gamma\colon \bigoplus_{i\in I} G(X_i) \to G\Big(\bigoplus_{i\in I} X_i\Big)$$

*is a* **q***-isomorphism.*

b) *If $M$ is a right $R$-module and $I$ is a set, then the canonical monomorphism $\gamma\colon F(M)^{(I)} \to F(M^{(I)})$ is a* **q***-isomorphism.*

P r o o f. a) It is enough to prove that every $R$-homomorphism $f\colon G\Big(\bigoplus_{i\in I} X_i\Big) \to M$ such that $f\gamma = 0$ represents the zero homomorphism in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$.

Let $f\colon G\Big(\bigoplus_{i\in I} X_i\Big) \to M$ be an $R$-homomorphism with $f\gamma = 0$. Then $fG(\alpha_i) = 0$ for all $i \in I$ (if $i \in I$, $\alpha_i\colon X_i \to \bigoplus_{i\in I} X_i$ denotes the canonical injection). It follows that $\varphi'_M fG(\alpha_i)\colon G(X_i) \to G(F(M))$ is the zero homomorphism. From the proof of Proposition 3.3 it follows that there exists $\overline{f} \in \mathrm{Hom}_S\Big(\bigoplus_{i\in I} X_i, F(M)\Big)$ such that

$$n_{G(\bigoplus_{i\in I} X_i)} m_{\bigoplus_{i\in I} X_i} m_{F(M)} \varphi'_M f = G\Big(n_{G(\bigoplus_{i\in I} X_i)} m_{\bigoplus_{i\in I} X_i} \overline{f}\Big)$$

and this implies that

$$G\Big(n_{G(\bigoplus_{i\in I} X_i)} m_{\bigoplus_{i\in I} X_i} \overline{f} \alpha_i\Big) = 0$$

for all $i \in I$. Using again Proposition 3.3 we obtain

$$m_{F(M)} n_{G(\bigoplus_{i\in I} X_i)} m_{\bigoplus_{i\in I} X_i} \overline{f} \alpha_i = 0$$

for all $i \in I$. As the family $(\alpha_i)_{i\in I}$ is an epimorphic family, we have

$$m_{F(M)} n_{G(\bigoplus_{i\in I} X_i)} m_{\bigoplus_{i\in I} X_i} \overline{f} = 0$$

and this shows that $\varphi'_M f$ represents the zero homomorphism in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$. As $\varphi'_M$ is a **q**-isomorphism, it follows that $f$ is zero in $\mathbb{Z}[\Sigma^{-1}]$Mod-$R$ and the proof is complete.

In the same way b) follows. Remark that here we use the fact that for every right $R$-module $N$ the kernel of the homomorphism $F\colon \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_S(F(M),F(N))$ is bounded by $n_M$ and we have the direct sum of $I$ copies of $M$. □

**Corollary 3.5.** *If $F\colon$ Mod-$R \rightleftarrows$ Mod-$S\colon G$ is a pair of functors such that $F$ is a good* **q***-equivalence and $G$ is a good* **q***-inverse for $F$, then $F$ and $G$ preserve the* **q***-generators.*

P r o o f. Let $X$ be a **q**-generator in Mod-$S$. If $M$ is a right $R$-module, then there exists a set $I$ and a **q**-epimorphism $\alpha\colon X^{(I)} \to F(M)$. If $\gamma\colon G(X)^{(I)} \to G(X^{(I)})$ is the canonical homomorphism, then $\varphi_M G(\alpha)\gamma\colon G(X)^{(I)} \to M$ is a **q**-epimorphism, hence $G$ preserves the **q**-generators. Similarly, we obtain that $F$ preserves the **q**-generators. □

**Proposition 3.6.** *Let* $F\colon \text{Mod-}R \rightleftarrows \text{Mod-}S\colon G$ *be a pair of functors such that* $F$ *is a good* **q**-*equivalence and* $G$ *is a good* **q**-*inverse for* $F$. *Then*

a) *the natural transformation*

$$\overline{\varphi}\colon \text{Hom}_S(-,F(-)) \to \text{Hom}_R(G(-),-), \quad \overline{\varphi}_{X,A}(\alpha) = \varphi_A G(\alpha),$$

*has the property that* $\overline{\varphi}_{X,A}$ *is a* **q**-*isomorphism for all* $A \in \text{Mod-}R$ *and* $X \in \text{Mod-}S$;

b) *the natural transformation*

$$\overline{\psi}\colon \text{Hom}_S(F(-),-) \to \text{Hom}_R(-,G(-)), \quad \overline{\psi}_{A,X}(\beta) = F(\beta)\psi_X$$

*has the property that* $\overline{\psi}_{A,X}$ *is a* **q**-*isomorphism for all* $A \in \text{Mod-}R$ *and* $X \in \text{Mod-}S$.

Proof. If $A \in \text{Mod-}R$ and $X \in \text{Mod-}S$, then the composition

$$\text{Hom}_S(X,F(A)) \overset{G}{\to} \text{Hom}_R(G(X),G(F(A)) \overset{\text{Hom}_R(G(X),\varphi_A)}{\to} \text{Hom}_R(G(X),A)$$

gives the homomorphism

$$\overline{\varphi}_{X,A}\colon \text{Hom}_S(X,F(A)) \to \text{Hom}_R(G(X),A),$$

hence $\overline{\varphi}_{X,A}$ is a **q**-isomorphism. □

**Lemma 3.7.** *Let* $F_1, F_2\colon \textit{Mod-}R \to \text{Mod-}S$ *be functors and* $\mu\colon F_1 \to F_2$ *a natural transformation. If* $A$ *and* $B$ *are right* $R$-*modules and* $f\colon A \to B\colon g$ *are* $R$-*homomorphisms such that* $\mu_A$ *is an isomorphism and* $fg = n1_B$ *or* $gf = n1_A$, *then* $\mu_B$ *is a* **q**-*isomorphism.*

Proof. Using the fact that $\mu$ is a natural transformation, we obtain

$$F_2(f)\mu_A = \mu_B F_1(f) \quad \text{and} \quad \mu_A F_1(g) = F_2(g)\mu_B.$$

It follows that

$$F_1(f)\mu_A^{-1}F_2(g)\mu_B = n1_{F_1(B)} \quad \text{and} \quad \mu_B F_1(f)\mu_A^{-1}F_2(g) = n1_{F_2(B)}.$$

In the other case, the proof is similar. □

The next lemma is analogous to [4, Proposition 20.11].

**Lemma 3.8.** *Let $P$ be a left $S$-module, $U$ an $S$-$R$-bimodule and $N$ a right $R$-module. If $P$ is $\Sigma$-almost finitely generated and $\Sigma$-almost projective then the canonical homomorphism*

$$\mu\colon \operatorname{Hom}_R(U,N)\otimes_S P \to \operatorname{Hom}_R(\operatorname{Hom}_S(P,U),N), \quad \mu(\alpha\otimes p)(\beta)=\alpha\beta(p)$$

*is a* **q**-*isomorphism.*

P r o o f. Because $P$ is $\Sigma$-almost finitely generated and $\Sigma$-almost projective, there exist an integer $n\in\Sigma$ and $S$-homomorphisms $f\colon S^k \rightleftarrows P\colon g$ such that $fg=n1_P$. We obtain the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_R(U,N)\otimes_S S^k & \xrightarrow{\mu_{S^k}} & \operatorname{Hom}_R(\operatorname{Hom}_S(S^k,U),N) \\
\upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright \\
\operatorname{Hom}_S(U,N)\otimes_S P & \xrightarrow{\mu_P} & \operatorname{Hom}_R(\operatorname{Hom}_S(P,U),N)
\end{array}$$

and, using [4, Proposition 20.11] and the previous lemma, we conclude $\mu_P$ is a **q**-isomorphism. ☐

In the same way we find an analogue of Proposition 20.10 in [4]:

**Lemma 3.9.** *Let $P$ be a right $R$-module, $U$ an $R$-$S$-bimodule and $N$ a right $S$-module. If $P$ is $\Sigma$-almost projective and $\Sigma$-almost finitely generated, then the canonical homomorphism*

$$\nu\colon N\otimes_S \operatorname{Hom}_R(P,U)\to \operatorname{Hom}_R(P,N\otimes_S U), \quad \nu(n\otimes\alpha)(p)=n\otimes\alpha(p)$$

*is a* **q**-*isomorphism.*

**Theorem 3.10.** *If $R$ and $S$ are rings, then the following are equivalent:*

a) *There exists a functor $F\colon$ Mod-$R\to$ Mod-$S$ which is a good* **q**-*equivalence.*

b) *There exists a right $R$-module $P$ which is $\Sigma$-almost finitely generated, $\Sigma$-almost projective,* **q**-*generator and there exists a unital ring homomorphism $\varphi\colon S\to \operatorname{End}_R(P)$ such that the groups $\operatorname{Ker}(\varphi)$ and $S/\operatorname{Im}(\varphi)$ are $\Sigma$-bounded.*

c) *There exist a right $R$-module $P$ and a unital ring homomorphism*

$$\varphi\colon S\to \operatorname{End}_R(P)$$

*with $\operatorname{Ker}(\varphi)$ and $S/\operatorname{Im}(\varphi)$ $\Sigma$-bounded groups such that*

$$\varphi_\star \operatorname{Hom}_R(P,-)\colon \text{Mod-}R\to\text{Mod-}S$$

*is a good* **q**-*equivalence with*

$$(-\otimes_{\mathrm{End}_R(P)} P)\varphi^\star\colon\ \text{Mod-}S \to \text{Mod-}R$$

*a good* **q**-*inverse.*

*Under these conditions there exists a natural transformation*

$$\mu\colon\ F \to \varphi_\star \operatorname{Hom}_R(P,-)$$

*such that* $\mu_M$ *is a* **q**-*isomorphism for all* $M \in \text{Mod-}R$.

P r o o f. a) $\Rightarrow$ b) If $G$ is a good **q**-inverse for $F$, we denote $P = G(S)$. By virtue of [4, Lemma 20.3], $\varphi = G\colon \operatorname{Hom}_S(S,S) \to \operatorname{Hom}_R(P,P)$ is a unital ring homomorphism. We apply Proposition 3.3 and it follows that $\operatorname{Ker}(\varphi)$ and $\operatorname{Coker}(\varphi)$ are $\Sigma$-bounded as abelian groups. Because an equivalence preserves the projective objects, $P$ is $\Sigma$-almost projective. Corollary 3.5 shows that $P$ is a **q**-generator. Moreover, we observe that $F(R)$ is a **q**-generator, and it follows that there exists an $S$-homomorphism $\alpha\colon F(R)^n \to S$ such that $\operatorname{Coker}(\alpha)$ is $\Sigma$-bounded. If $\delta$ is a **q**-inverse for the canonical **q**-isomorphism $\gamma\colon F(R)^n \to F(R^n)$, then $\alpha\delta\colon F(R^n) \to S$ is a **q**-epimorphism. It follows that $G(\alpha\delta)\colon GF(R^n) \to G(S)$ is a **q**-epimorphism ($qG$ is a equivalence, hence it preserves the epimorphisms). $G(\alpha\delta)\varphi'_{R^n}\colon R^n \to G(S)$ is a **q**-epimorphism, hence $P = G(S)$ is $\Sigma$-almost finitely generated.

b) $\Rightarrow$ c) Let $X \in \text{Mod-}\operatorname{End}_R(P)$. Then Lemma 3.9 shows that there exist **q**-isomorphisms

$$X \to X \otimes_{\mathrm{End}_R(P)} \operatorname{Hom}_R(P,P) \to \operatorname{Hom}_R(P, X \otimes_{\mathrm{End}_R(P)} P)$$

which are natural in $X$.

Recall that if $B = \operatorname{Hom}_{\mathrm{End}_R(P)}(P,P)$ is the biendomorphism ring of $P$ and $\vartheta\colon R \to B$ is the canonical ring homomorphism then $\operatorname{Ker}(\vartheta)$ and $\operatorname{Coker}(\vartheta)$ are $\Sigma$-bounded groups (Proposition 2.4). Using Lemma 3.8, we obtain the **q**-isomorphisms

$$\operatorname{Hom}_R(P,A) \otimes_{\mathrm{End}_R(P)} P \to \operatorname{Hom}_R(\operatorname{Hom}_{\mathrm{End}_R(P)}(P,P), A) \to A$$

which are natural in $A$ and it follows that

$$\operatorname{Hom}_R(P,-)\colon\ \text{Mod-}R \to \text{Mod-}\operatorname{End}_R(P)$$

is a good **q**-equivalence with $-\otimes_{\mathrm{End}_R} P$ a good **q**-inverse. Under these conditions

$$\varphi_\star \operatorname{Hom}_R(P,-)\colon\ \text{Mod-}R \to \text{Mod-}S$$

is a good **q**-equivalence and $(-\otimes_{\mathrm{End}_R} P)\varphi^\star$ is a good **q**-inverse.

c) $\Rightarrow$ a) is obvious.

The last statement follows from the fact that the arrows

$$F(M) \to \mathrm{Hom}_S(S, F(M)) \to \mathrm{Hom}_R(G(S), M) = \varphi_\star \mathrm{Hom}_R(G(S), M)$$

are **q**-isomorphisms natural in $M$. $\square$

**Remark 3.11.** In the case $\Sigma = \{1\}$, the theorem is the classical Morita Theorem.

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