Czechoslovak Mathematical Journal

Jelena V. Manojlović

Integral averages and oscillation of second order sublinear differential equations

Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 1, 41–60

Persistent URL: `http://dml.cz/dmlcz/127958`

Czechoslovak Mathematical Journal, 55 (130) (2005), 41–60

INTEGRAL AVERAGES AND OSCILLATION OF SECOND ORDER SUBLINEAR DIFFERENTIAL EQUATIONS

Jelena V. Manojlović, Niš

(Received February 25, 2002)

Abstract. New oscillation criteria are given for the second order sublinear differential equation

$$[a(t)\psi(x(t))x'(t)]' + q(t)f(x(t)) = 0, \quad t \geqslant t_0 > 0,$$

where $a \in C^1([t_0,\infty))$ is a nonnegative function, $\psi, f \in C(\mathbb{R})$ with $\psi(x) \neq 0$, $xf(x)/\psi(x) > 0$ for $x \neq 0$, ψ, f have continuous derivative on $\mathbb{R} \setminus \{0\}$ with $[f(x)/\psi(x)]' \geqslant 0$ for $x \neq 0$ and $q \in C([t_0,\infty))$ has no restriction on its sign. This oscillation criteria involve integral averages of the coefficients q and a and extend known oscillation criteria for the equation $x''(t) + q(t)x(t) = 0$.

Keywords: oscillation, sublinear differential equation, integral averages

MSC 2000: 34C10, 34C15

1. Introduction

We consider the second order nonlinear differential equation

$$[a(t)\psi(x(t))x'(t)]' + q(t)f(x(t)) = 0 \tag{E}$$

where

(i) $a \in C^1([t_0,\infty))$, $a(t) > 0$ for $t \geqslant t_0$,

(ii) $q \in C([t_0,\infty))$ has no restriction on its sign,

(iii) $\psi, f \in C^1(\mathbb{R})$ satisfy

$$\psi(x) \neq 0, \quad x\,\frac{f(x)}{\psi(x)} > 0 \quad \text{for } x \neq 0, \tag{F$_1$}$$

and $f(x)/\psi(x)$ is *strongly sublinear* in the sense that

$$\int_{0+} \frac{\psi(u)}{f(u)} \,\mathrm{d}u < \infty, \quad \text{and} \quad \int_{0-} \frac{\psi(u)}{f(u)} \,\mathrm{d}u < \infty,$$

(iv) ψ and f are continuously differentiable on $\mathbb{R} \setminus \{0\}$ and satisfy

$$(\mathrm{F}_2) \qquad \Big(\frac{f(x)}{\psi(x)}\Big)' \geqslant 0 \quad \text{for } x \neq 0.$$

We assume throughout that every solution $x(t)$ of the differential equation (E) is nontrivial and can be continued to the right, i.e. every solution $x(t)$ is defined on some ray $[T, \infty)$, where $T \geqslant t_0$ may depend on the particular solution, and

$$\sup\{|x(t)|\colon\ t \geqslant T\} > 0 \quad \text{for every } T \geqslant t_0.$$

The oscillatory character of such solutions is considered in the usual sense, i.e. a solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros, otherwise, it is said to be *nonoscillatory.* The equation (E) is called *oscillatory* if all nontrivial continuable solutions are oscillatory.

In the study of the oscillation of second order nonlinear differential equations, many criteria have been found which involve the average behavior of the integral of the coefficients. The differential equation

$$(\mathrm{E}_1) \qquad x''(t) + q(t)|x(t)|^\gamma \operatorname{sgn} x(t) = 0, \quad t \in [t_0, \infty),$$

where q is a continuous real-valued function on $[t_0, \infty)$ without any restriction on its sign, which is known in the literature as the equation of the Emden-Fowler type, is of particular interest in such averaging techniques. In the sublinear case $0 < \gamma < 1$, Chen [2] proved that the equation (E_1) is oscillatory if there exists a positive function ϱ with $\varrho'' \leqslant 0$ such that

$$(\mathrm{A}_1) \qquad \limsup_{t\to\infty} \frac{1}{t^n - 1} \int_{t_0}^{t} (t-s)^{n-1} \varrho^\gamma(s) q(s) \,\mathrm{d}s = \infty,$$

for some integer $n \geqslant 2$.

Wong [20] also proved: if $\varrho\colon\ [0, \infty) \to [0, \infty)$ is a positive nondecreasing concave function, i.e. $\varrho > 0$, $\varrho' \geqslant 0$, $\varrho'' \leqslant 0$, then

$$(\mathrm{A}_2) \qquad \limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha \varrho(s) q(s) \,\mathrm{d}s = \infty, \quad \text{for some } \alpha > 1,$$

suffices for the oscillation of the differential equation (E_1).

Note that from Wong's result, by choosing $\varrho(t) = t^\beta$, $\beta \in [0,\gamma]$, we obtain the result given by Yan in [19]. By applying Chen's result for $n = 2$ and $\varrho(t) = t^{\beta/\gamma}$, $\beta \in [0,\gamma]$, we obtain the result due to Kwong, Wong [9].

Recently, Philos [18] and Wong and Yeh [21] extended those results to the nonlinear differential equation of the form

$$x''(t) + q(t)f(x(t)) = 0, \quad t \in [t_0,\infty), \tag{E$_2$}$$

where f is a continuous function on the real line $\mathbb{R}$, *strongly sublinear* in the sense that

$$\int_{0+} \frac{\mathrm{d}u}{f(u)} < \infty, \quad \text{and} \quad \int_{0-} \frac{\mathrm{d}u}{f(u)} < \infty,$$

and f has continuous derivative on $\mathbb{R} \setminus \{0\}$ and satisfies

$$xf(x) > 0, \quad \text{and} \quad f'(x) \geqslant 0 \quad \text{for all } x \neq 0.$$

Namely, Philos in [18] introduced the nonnegative constant

$$I_f = \min\left\{ \frac{\inf\limits_{x>0} f'(x)F(x)}{1 + \inf\limits_{x>0} f'(x)F(x)}, \frac{\inf\limits_{x<0} f'(x)F(x)}{1 + \inf\limits_{x<0} f'(x)F(x)} \right\} > 0,$$

where

$$F(x) = \int_{0+}^{x} \frac{\mathrm{d}u}{f(u)} \quad \text{for } x > 0, \qquad F(x) = \int_{0-}^{x} \frac{\mathrm{d}u}{f(u)} \quad \text{for } x < 0,$$

and proved that the equation (E$_2$) is oscillatory if there exists a positive and twice continuously differentiable function ϱ on $[t_0,\infty)$ with $\varrho' \geqslant 0$, $\varrho'' \leqslant 0$ on $[t_0,\infty)$, such that the condition (A$_1$) holds for $\gamma = I_f$ and some integer $n \geqslant 2$.

Moreover, Wong and Yeh [21] proved that the equation (E$_2$) is oscillatory if for some $\alpha > 1$, $q(t)$ satisfies

$$\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha \varrho^\beta(s) q(s)\,\mathrm{d}s = \infty, \tag{A$_3$}$$

where $\beta \in [0, I_f]$ and $\varrho \in C^2([t_0,\infty))$ is a positive concave function.

Letting $n = 2$ in Philos's result we obtain Theorem 1 of [15] and by taking $\varrho(t) = 1$, $t \geqslant t_0$ if $I_f = 0$, or $\varrho(t) = t^{\lambda/\beta}$, $t \geqslant t_0$, $0 \leqslant \lambda \leqslant I_f$ if $I_f > 0$, we have the results of Philos in [14].

Furthermore, Philos in [18] also proved the following theorem:

Theorem A. *Let n be an integer with $n \geqslant 2$ and ϱ be a positive and twice continuously differentiable function on $[t_0, \infty)$ such that*

$$\text{(R)} \qquad [\varrho'(t)]^2 \leqslant -c\,\varrho(t)\varrho''(t) \quad \text{for every } t \geqslant t_0,$$

for some positive constant c. The equation (E_2) is oscillatory if there exists a continuous function φ on $[t_0, \infty)$ with

$$(\mathrm{C}_1) \qquad \int_{t_0}^{\infty} \frac{\varphi_+^2(s)}{s}\,\mathrm{d}s = \infty, \quad \varphi_+(s) = \max\{\varphi(s), 0\}, \quad s \geqslant t_0,$$

and such that

$$(\mathrm{A}_4) \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-1} \varrho^{I_f}(s) q(s)\,\mathrm{d}s \geqslant \varphi(T) \quad \text{for every } T \geqslant t_0.$$

For $n = 2$ we obtain a previous result due to same author [16, Theorem 1].

The purpose of this paper is to prove analogous extensions of the above mentioned results to the more general differential equation (E) by using more general conditions than (A_3) and (A_4). Namely, the established oscillation criterion relates on an integral averaging technique introduced by Philos [17] who used kernel function

$$H\colon\ \mathscr{D} = \{(t,s)\colon\ t \geqslant s \geqslant t_0\} \to \mathbb{R}$$

and obtained new oscillation criteria for the linear differential equation (L). Grace [6], Li, Yeh [10], Wong [22] and the author [11], [12], [13] proceeded further in this direction and established oscillation criteria in terms of more general means for the second order nonlinear equation (E).

A great deal of oscillation criteria for the equation (E) has been obtained by using the technique which involves the average behavior of the integral of the coefficients $a(t)$ and $q(t)$. But, all these results have been obtained under the assumption that for $x \neq 0$, $x\,f(x) > 0$, $\psi(x) > 0$ and either

$$(*) \qquad f'(x) \geqslant k > 0$$

(see [10]), or

$$(**) \qquad \frac{f'(x)}{\psi(x)} \geqslant K > 0,$$

(see [4] and [6]). Very recently, Kirane and Rogovchenko [7] derived new oscillation criteria for the nonlinear equation

$$(\mathrm{E}_p) \qquad [a(t)\psi(x(t))x'(t)]' + p(t)x'(t) + q(t)f\bigl(x(t)\bigr) = 0$$

without the assumption that $f(x)$ has to be nondecreasing. They established oscillation criteria under the following conditions for the nonlinearities $f(x)$ and $\psi(x)$

$$(***) \qquad \frac{f(x)}{x} \geqslant K > 0, \quad 0 < c \leqslant \psi(x) \leqslant c_1, \quad \text{for } x \neq 0.$$

Nevertheless, it still remains to establish the oscillation of the equation (E) without the restriction of the positivity of the function $\psi(x)$. So, an essential feature of the proved results is that the assumption of positivity of the function $\psi(x)$ is not required. Consequently, our criteria cover new classes of equations to which known results do not apply.

2. Main results

In this section we will establish three oscillation criteria for the differential equation (E) supposing that the functions $f(x)$ and $\psi(x)$ satisfy the condition

$$(\mathrm{F}_3) \qquad \frac{f(x)\psi'(x)}{\psi^2(x)} \geqslant \frac{1}{k} > 0, \quad \text{for } x \neq 0.$$

For our purpose, we define

$$\Phi(x) = \int_{0+}^{x} \frac{\psi(u)}{f(u)}\,\mathrm{d}u \quad \text{for } x > 0, \qquad \Phi(x) = \int_{0-}^{x} \frac{\psi(u)}{f(u)}\,\mathrm{d}u \quad \text{for } x < 0$$

and introduce the nonnegative constant

$$M_{f,\psi} = \min\left\{ \frac{\inf\limits_{x>0} \Phi(x)\big(f(x)/\psi(x)\big)'}{1 + \inf\limits_{x>0} \Phi(x)\big(f(x)/\psi(x)\big)'}, \frac{\inf\limits_{x<0} \Phi(x)\big(f(x)/\psi(x)\big)'}{1 + \inf\limits_{x<0} \Phi(x)\big(f(x)/\psi(x)\big)'} \right\}.$$

Also, in order to simplify notation we define the function

$$\chi(t) = \frac{q(t)}{a(t)} - \frac{k}{4}\Big(\frac{a'(t)}{a(t)}\Big)^2.$$

Theorem 2.1. *Let $\varrho \in C^2([t_0,\infty))$ be a positive concave function and let the functions $f, \psi \in C^1(\mathbb{R})$ satisfy the conditions* (iii), (iv) *and* (F_3). *Suppose that there exists a continuous function*

$$H\colon \mathscr{D} = \{(t,s) \mid t \geqslant s \geqslant t_0\} \to \mathbb{R}$$

such that

$$(\mathrm{H}_1)\qquad H(t,t)=0 \quad \text{for } t\geqslant t_0, \qquad H(t,s)>0 \quad \text{for } t>s\geqslant t_0,$$

$$(\mathrm{H}_2)\qquad \frac{\partial H(t,t)}{\partial s}=0 \quad \text{for } t\geqslant t_0, \qquad \frac{\partial H(t,s)}{\partial s}\leqslant 0 \quad \text{for } (t,s)\in\mathscr{D},$$

$$(\mathrm{H}_3)\qquad \frac{\partial^2 H(t,s)}{\partial s^2}\geqslant 0 \quad \text{for } (t,s)\in\mathscr{D}$$

$$(\mathrm{H}_4)\qquad \liminf_{t\to\infty} \frac{\partial H(t,s)}{\partial s}\Big/H(t,s)>-\infty \quad \text{for } s\geqslant t_0.$$

The equation (E) *is oscillatory if for some* $\beta\in[0,M_{f,\psi}]$

$$(\mathrm{C}_2)\qquad \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\varrho^{\beta}(s)\,\chi(s)\,\mathrm{d}s=\infty \quad \text{for every } T\geqslant t_0.$$

Proof. Assume to the contrary that there exists a nonoscillatory solution $x(t)$ on $[T,\infty)$. Furthermore, we define $w(t)$ by

$$w(t)=\varrho^{\beta}(t)\Phi(x(t)), \quad t\geqslant T. \tag{1}$$

By differentiation we obtain for $t\geqslant T$

$$w'(t)=\varrho^{\beta}\frac{\psi(x)}{f(x)}x'+\beta\frac{\varrho'}{\varrho}w, \tag{2}$$

which implies

$$\begin{aligned} w''=\varrho^{\beta}\frac{\psi(x)}{f(x)}x''+\beta\varrho^{\beta}\frac{\varrho'}{\varrho}\frac{\psi(x)}{f(x)}x'-\varrho^{\beta}\psi(x)\Big(\frac{x'}{f(x)}\Big)^2 f'(x)\\ +\varrho^{\beta}\frac{\psi'(x)}{f(x)}x'^2+\beta\Big[\frac{\varrho''}{\varrho}-\Big(\frac{\varrho'}{\varrho}\Big)^2\Big]w+\beta\frac{\varrho'}{\varrho}w'. \end{aligned} \tag{3}$$

Since from (2) we have that

$$\frac{x'}{f(x)}=\frac{1}{\varrho^{\beta}\psi(x)}\Big(w'-\beta\frac{\varrho'}{\varrho}w\Big),$$

we obtain

$$\varrho^\beta \frac{\psi'(x)}{f(x)} x'^2 = \Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 \frac{\psi'(x)f(x)}{\varrho^\beta \psi^2(x)} = \Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 \frac{\psi'(x)f(x)\Phi(x)}{w\psi^2(x)} \tag{4}$$

and

$$\varrho^\beta \psi(x)\Big(\frac{x'}{f(x)}\Big)^2 f'(x) = \Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 \frac{f'(x)}{\varrho^\beta \psi(x)} = \Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 \frac{\Phi(x)f'(x)}{w\psi(x)}. \tag{5}$$

Now, by subtracting (4) and (5), we obtain

$$\begin{aligned}
&\varrho^\beta \frac{\psi'(x)}{f(x)} x'^2 - \varrho^\beta \psi(x)\Big(\frac{x'}{f(x)}\Big)^2 f'(x) \\
&\quad = \frac{\Phi(x)}{w}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 \frac{f(x)\psi'(x) - f'(x)\psi(x)}{\psi^2(x)} \\
&\quad = -\frac{1}{w}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 \Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)'.
\end{aligned}$$

According to selection of the number β,

$$\Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)' \geqslant \frac{\beta}{1-\beta} \quad \text{for } x \neq 0,$$

so that the previous equality becomes

$$\varrho^\beta \frac{\psi'(x)}{f(x)} x'^2 - \varrho^\beta \psi(x)\Big(\frac{x'}{f(x)}\Big)^2 f'(x) \leqslant -\frac{\beta}{1-\beta}\frac{1}{w}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2. \tag{6}$$

Besides,

$$\beta\varrho^\beta \frac{\varrho'}{\varrho}\frac{\psi(x)}{f(x)} x' = \beta\frac{\varrho'}{\varrho}\Big(\varrho^\beta \frac{\psi(x)}{f(x)}x' + \beta\frac{\varrho'}{\varrho}w\Big) - \beta^2\Big(\frac{\varrho'}{\varrho}\Big)^2 w = \beta\frac{\varrho'}{\varrho}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big). \tag{7}$$

Using the equation (E) and the conditions (C_1), (C_2) and (C_3), we have

$$\begin{aligned}
\varrho^\beta \frac{\psi(x)}{f(x)} x'' &= -\varrho^\beta \frac{q}{a} - \varrho^\beta \frac{a'}{a}\frac{\psi(x)}{f(x)} x' - \varrho^\beta \frac{\psi'(x)}{f(x)} x'^2 \\
&= -\varrho^\beta \frac{q}{a} - \varrho^\beta \frac{\psi'(x)}{f(x)}\Big(x' + \frac{a'}{2a}\frac{\psi(x)}{\psi'(x)}\Big)^2 + \frac{\varrho^\beta}{4}\Big(\frac{a'}{a}\Big)^2 \frac{\psi^2(x)}{f(x)\psi'(x)} \\
&\leqslant -\varrho^\beta \frac{q}{a} + \frac{k}{4}\varrho^\beta \Big(\frac{a'}{a}\Big)^2.
\end{aligned} \tag{8}$$

Therefore, (3), (6), (7) and (8) imply

$$\begin{aligned}
w'' \leqslant & -\varrho^\beta \frac{q}{a} + \frac{k}{4}\varrho^\beta\Big(\frac{a'}{a}\Big)^2 + \beta\frac{\varrho'}{\varrho}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big) \\
& - \frac{\beta}{1-\beta}\frac{1}{w}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 + \beta\Big[\frac{\varrho''}{\varrho} - \Big(\frac{\varrho'}{\varrho}\Big)^2\Big]w + \beta\frac{\varrho'}{\varrho}w' \\
= & -\varrho^\beta \frac{q}{a} + \frac{k}{4}\varrho^\beta\Big(\frac{a'}{a}\Big)^2 + \beta\frac{\varrho''}{\varrho}w + 2\beta\frac{\varrho'}{\varrho}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big) \\
& + \beta(\beta-1)\Big(\frac{\varrho'}{\varrho}\Big)^2 w - \frac{\beta}{1-\beta}\frac{1}{w}\Big(w' - \beta\frac{\varrho'}{\varrho}w\Big)^2 \\
= & -\varrho^\beta\Big(\frac{q}{a} - \frac{k}{4}\Big(\frac{a'}{a}\Big)^2\Big) + \beta\frac{\varrho''}{\varrho}w \\
& - \frac{\beta}{1-\beta}\frac{1}{w}\Big(w' - \beta\frac{\varrho'}{\varrho}w - (1-\beta)\frac{\varrho'}{\varrho}w\Big)^2,
\end{aligned}$$

which gives for every $t \geqslant T$

$$w''(t) \leqslant -\varrho^\beta(t)\chi(t) + \beta\frac{\varrho''(t)}{\varrho(t)}w(t) - \frac{\beta}{1-\beta}\frac{1}{w(t)}\Big(w'(t) - \frac{\varrho'(t)}{\varrho(t)}w(t)\Big)^2. \tag{9}$$

Consequently, using the fact that ϱ is a positive and concave function, we have

$$w''(s) \leqslant -\varrho^\beta(s)\chi(s) \ \ \forall s \geqslant T. \tag{10}$$

Multiplying the previous inequality through by $H(t,s)$ and integrating from T to t, we find

$$\int_T^t H(t,s)\varrho^\beta(s)\chi(s)\,\mathrm{d}s \leqslant -\int_T^t H(t,s)w''(s)\,\mathrm{d}s. \tag{11}$$

Using integration by parts, by the conditions $(\mathrm{H})_1$–$(\mathrm{H})_3$, we get

$$\begin{aligned}
-\int_T^t H(t,s)w''(s)\,\mathrm{d}s &= H(t,T)w'(T) - \frac{\partial H}{\partial s}(t,T)w(T) \\
-\int_T^t \frac{\partial^2 H}{\partial s^2}(t,s)w(s)\,\mathrm{d}s &\leqslant H(t,T)w'(T) - \frac{\partial H}{\partial s}(t,T)w(T),
\end{aligned} \tag{12}$$

which by (11) and (H_4), leads us to the following contradiction

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\varrho^\beta(s)\chi(s)\,\mathrm{d}s \\
&\qquad \leqslant w'(T) - w(T)\liminf_{t\to\infty}\frac{\frac{\partial H}{\partial s}(t,T)}{H(t,T)} < \infty.
\end{aligned}$$

□

Remark 1. By applying Theorem 2.1 with $a(t) \equiv 1$ and $\psi(x) \equiv 1$, taking $H(t,s) = (t-s)^{\lambda}$ for some constant $\lambda > 1$, which obviously satisfies the conditions $(\mathrm{H})_1$–$(\mathrm{H})_4$, we obtain the oscillation criterion of Wong and Yeh in [21].

Example 1. Consider the differential equation

$$(\mathrm{E}_3) \qquad [t^{\nu} x^3(t) x'(t)]' + q(t)[|x(t)|^{\alpha+3} + x^4(t)] = 0, \quad 0 < \alpha < 1$$

where $q(t) = \lambda t^{\lambda-1}(2 - \cos t) + t^{\lambda} \sin t$, $\nu < \frac{1}{2}\alpha$ and $\lambda - \nu + \frac{1}{2}\alpha > 0$. Then for all $x \neq 0$

$$x \frac{f(x)}{\psi(x)} > 0, \qquad \frac{f(x)\psi'(x)}{\psi^2(x)} \geqslant 3 = k, \qquad \Big(\frac{f(x)}{\psi(x)}\Big)' = \alpha|x|^{\alpha-1} + 1 > 0,$$

so that the conditions (C_1), (C_2) and (C_3) are satisfied.

Further, for every $x \neq 0$

$$\Phi(x) = \int_{0+}^{|x|} \frac{\mathrm{d}u}{u^{\alpha} + u} \leqslant \int_{0+}^{|x|} \frac{\mathrm{d}u}{u^{\alpha}} = \frac{|x|^{1-\alpha}}{1-\alpha}$$

and consequently

$$\inf_{x>0} \Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)' = \inf_{x<0} \Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)' \leqslant \inf_{x>0} \frac{|x|^{1-\alpha}}{1-\alpha}(\alpha x^{\alpha-1} + 1) = \frac{\alpha}{1-\alpha}.$$

On the other hand, for every $x \neq 0$, we have

$$\Phi(x) = \int_{0+}^{|x|} \frac{\mathrm{d}u}{u^{\alpha} + u} \geqslant \int_{0+}^{|x|} \frac{\mathrm{d}u}{2u^{\alpha}} = \frac{|x|^{1-\alpha}}{2(1-\alpha)}, \quad \text{if } |x| \leqslant 1$$

and

$$\Phi(x) = \int_{0+}^{|x|} \frac{\mathrm{d}u}{u^{\alpha} + u} \geqslant \int_{0+}^{1} \frac{\mathrm{d}u}{u^{\alpha} + u} \geqslant \int_{0+}^{1} \frac{\mathrm{d}u}{2u^{\alpha}} = \frac{1}{2(1-\alpha)}, \quad \text{if } |x| \geqslant 1$$

and therefore for $x \neq 0$ we have

$$\Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)' \geqslant \frac{|x|^{1-\alpha}}{2(1-\alpha)}(\alpha x^{\alpha-1} + 1) = \frac{\alpha + |x|^{1-\alpha}}{2(1-\alpha)} > \frac{\alpha}{2(1-\alpha)} \quad \text{for } |x| \leqslant 1$$

and

$$\Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)' \geqslant \frac{\alpha x^{\alpha-1} + 1}{2(1-\alpha)} > \frac{\alpha}{2(1-\alpha)} \quad \text{for } |x| \geqslant 1.$$

Hence,

$$\inf_{x>0} \Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)' = \inf_{x<0} \Phi(x)\Big(\frac{f(x)}{\psi(x)}\Big)' \geqslant \frac{\alpha}{2(1-\alpha)}.$$

Accordingly, $M_{f,\psi} \geqslant \frac{1}{2}\alpha$.

Next, for any $t \geqslant T \geqslant t_0$, we have

$$\begin{aligned}\int_T^t q(s)\,\mathrm{d}s = \int_T^t d[s^\lambda(2-\cos s)] &= t^\lambda(2-\cos t) - T^\lambda(2+\cos T)\\ &= t^\lambda(2-\cos t) - k_0,\end{aligned}$$

so that

$$t^\lambda - k_0 \leqslant \int_T^t q(s)\,\mathrm{d}s \leqslant 3t^\lambda \quad \text{for every } t \geqslant T \geqslant t_0.$$

Consequently, for arbitrary positive number δ such that $\delta+\lambda>0$, we have

$$\begin{aligned}\frac{1}{t^2}\int_T^t (t-s)^2 s^\delta q(s)\,\mathrm{d}s &= \frac{1}{t^2}\int_T^t (t-s)^2 s^\delta\,\mathrm{d}\Big(\int_T^s q(u)\,\mathrm{d}u\Big)\\ &= \frac{1}{t^2}\int_T^t [2(t-s)s^\delta - \delta s^{\delta-1}(t-s)^2]\Big(\int_T^s q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s\\ &= \frac{1}{t^2}\int_T^t [2t(1+\delta)s^\delta - \delta t^2 s^{\delta-1} - (\delta+2)s^{\delta+1}]\Big(\int_T^s q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s\\ &\geqslant \frac{1}{t^2}\int_T^t [2(1+\delta)ts^\delta(s^\lambda-k_0) - 3(\delta+2)s^{\delta+\lambda+1} - 3\delta t^2 s^{\delta+\lambda-1}]\,\mathrm{d}s\\ &= L_1 t^{\delta+\lambda} + L_2 t^\delta + \frac{L_3}{t^2} + \frac{L_4}{t} + L_5\end{aligned}$$

where

$$L_1 = \frac{2(1+\delta)}{\delta+\lambda+1} - \frac{3(\delta+2)}{\delta+\lambda+2} - \frac{3\delta}{\delta+\lambda}, \quad L_2 = -2k_0, \quad L_5 = \frac{3\delta}{\lambda+\delta}T^{\lambda+\delta}$$
$$L_3 = \frac{3(\delta+2)}{\delta+\lambda+2}T^{\delta+\lambda+2}, \quad L_4 = 2k_0T^{\delta+1} - \frac{2(\delta+1)}{\delta+\lambda+1}T^{\delta+\lambda+1}.$$

Moreover, for every $T \geqslant t_0$

$$\lim_{t\to\infty}\frac{1}{t^2}\int_T^t (t-s)^2 s^{\frac{1}{2}\alpha-2}\,\mathrm{d}s = \frac{2}{2-\alpha}T^{\frac{1}{2}\alpha-1}. \tag{13}$$

Taking $\varrho(t)=t$, $\beta=\frac{1}{2}\alpha$, $\delta=\frac{1}{2}\alpha-\nu>0$ ($\delta+\lambda>0$) and $H(t,s)=(t-s)^2$ for $t \geqslant s \geqslant t_0$, we see that the condition (C_2) is satisfied, because

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_T^t \Big[(t-2)^2 s^{\frac{1}{2}\alpha-\nu}q(s) - \frac{3\nu^2}{4}(t-s)^2 s^{\frac{1}{2}\alpha-2}\Big]\,\mathrm{d}s = \infty.$$

Consequently, the equation (E_3) is oscillatory by Theorem 2.1.

We note that since $f(x) = |x|^{\alpha+3} + x^4$, $0 < \alpha < 1$ and $\psi(x) = x^3$, none of the conditions $(*)$, $(**)$ and $(***)$ is satisfied, so that none of the oscillation criteria in [4], [6], [7] and [10] can cover this result. We believe that none of the known oscillation criteria can really cover this result.

Theorem 2.2. *Let*

(i) $\varrho \in C^2([t_0, \infty))$ *be a positive function which satisfies the condition* (R) *for some positive constant* c,

(ii) $H(t,s)$ *be a twice continuously differentiable function on* $\mathscr{D}$ *with respect to the second variable which satisfies the conditions* $(\mathrm{H})_1$–$(\mathrm{H})_4$ *and*

$$(\mathrm{H}_5) \qquad 0 < \inf_{s \geqslant t_0} \Big[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \Big] \leqslant \infty,$$

$$(\mathrm{H}_6) \qquad \int_{t_0}^{\infty} s\Omega^2(s)\,\mathrm{d}s < \infty, \quad \Omega(s) = \limsup_{t\to\infty} \Big(-\frac{\partial H(t,s)/\partial s}{H(t,s)} \Big), \quad s \geqslant t_0.$$

Then the equation (E) *is oscillatory if there exists a function* $\varphi \in C([t_0, \infty))$ *such that* (C_1) *holds and*

$$(\mathrm{C}_3) \qquad \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\varrho^{\beta}(s)\chi(s)\,\mathrm{d}s \geqslant \varphi(T),$$

for every $T \geqslant t_0$ *and some* $\beta \in [0, M_{f,\psi}]$.

P r o o f. Suppose that the equation (E) possesses a nonoscillatory solution $x(t)$. We consider a $T_0 \geqslant t_0$ such that $x(t) \neq 0$ for all $t \geqslant T_0$ and we define the function $w(t)$ by (1) on $[T_0, \infty)$. Then (9) is satisfied for all $t \geqslant T_0$. Thus, using (12), we obtain for all $t > T \geqslant T_0$

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\varrho^{\beta}(s)\chi(s)\,\mathrm{d}s \\
&\quad \leqslant w'(T) - w(T) \liminf_{t\to\infty} \frac{\frac{\partial H}{\partial s}(t,T)}{H(t,T)} \\
&\qquad - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{\partial^2 H}{\partial s^2}(t,s) w(s)\,\mathrm{d}s \\
&\qquad + \beta \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s) \frac{\varrho''(s)}{\varrho(s)} w(s)\,\mathrm{d}s \\
&\qquad - \frac{\beta}{1-\beta} \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{H(t,s)}{w(s)} \Big(w'(s) - \frac{\varrho'(s)}{\varrho(s)} w(s) \Big)^2 \mathrm{d}s.
\end{aligned}$$

Accordingly, by the condition (C_3), we conclude that

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{H(t,s)}{w(s)} \Big(w'(s) - \frac{\varrho'(s)}{\varrho(s)} w(s)\Big)^2 \, \mathrm{d}s < \infty, \quad T \geqslant T_0, \tag{14}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\Big(-\frac{\varrho''(s)}{\varrho(s)}\Big) w(s) \, \mathrm{d}s < \infty, \quad T \geqslant T_0, \tag{15}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{\partial^2 H}{\partial s^2}(t,s) w(s) \, \mathrm{d}s < \infty, \quad T \geqslant T_0, \tag{16}$$

$$\varphi(T) \leqslant w'(T) + \Omega(T) w(T) \quad \text{for every } T \geqslant T_0. \tag{17}$$

Because of (14) and (16) there exists a sequence $\{\tau_n\}_{n\in N}$ in the interval (T_0, ∞) with $\lim\limits_{n\to\infty} \tau_n = \infty$ and such that

$$\lim_{n\to\infty} \frac{1}{H(\tau_n, T_0)} \int_{T_0}^{\tau_n} \frac{(\tau_n, s)}{H} w(s) \Big(w'(s) - \frac{\varrho'(s)}{\varrho(s)} w(s)\Big)^2 \, \mathrm{d}s < \infty, \tag{18}$$

$$\lim_{n\to\infty} \frac{1}{H(\tau_n, T_0)} \int_{T_0}^{\tau_n} \frac{\partial^2 H}{\partial s^2}(\tau_n, s) w(s) \, \mathrm{d}s < \infty. \tag{19}$$

Now, we shall establish that

$$\limsup_{t\to\infty} \frac{w(t)}{t} < \infty. \tag{20}$$

Let us consider an arbitrary positive constant μ. By the condition (H_5), we can consider a constant ϱ with

$$\inf_{s\geqslant t_0} \Big[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\Big] > \varrho > 0.$$

Suppose that (20) fails. Then there exists a $T_1 \geqslant T_0$ such that

$$\frac{w(t)}{t} \geqslant \frac{\mu}{\varrho} \ \ \forall t \geqslant T_1.$$

Thus, we obtain for $t \geqslant T_1$

$$\begin{aligned} \frac{1}{H(t,T_0)} \int_{T_0}^t \frac{\partial^2 H}{\partial s^2}(t,s) w(s) \, \mathrm{d}s &\geqslant \frac{1}{H(t,T_0)} \int_{T_1}^t \frac{\partial^2 H}{\partial s^2}(t,s) w(s) \, \mathrm{d}s \\ &\geqslant \frac{\mu}{\varrho H(t,T_0)} \int_{T_1}^t \frac{\partial^2 H}{\partial s^2}(t,s) s \, \mathrm{d}s \\ &= \frac{\mu}{\varrho H(t,T_0)} \Big(H(t,T_1) - T_1 \frac{\partial H}{\partial s}(t,T_1)\Big) \\ &\geqslant \frac{\mu}{\varrho} \frac{H(t,T_1)}{H(t,t_0)}. \end{aligned}$$

Since

$$\liminf_{t\to\infty} \frac{H(t,T_1)}{H(t,t_0)} > \varrho,$$

we can choose a $T_2 \geqslant T_1$ so that

$$\frac{H(t,T_1)}{H(t,t_0)} \geqslant \varrho \quad \text{for every } t \geqslant T_2. \tag{21}$$

Consequently,

$$\frac{1}{H(t,T_0)} \int_{T_0}^{t} \frac{\partial^2 H}{\partial s^2}(t,s) w(s)\,\mathrm{d}s \geqslant \mu \ \ \forall t \geqslant T_2.$$

Thus,

$$\frac{1}{H(\tau_n,T_0)} \int_{T_0}^{\tau_n} \frac{\partial^2 H}{\partial s^2}(\tau_n,s) w(s)\,\mathrm{d}s \geqslant \mu \quad \text{for sufficiently large } n,$$

which, since $\mu > 0$ is arbitrary, proves that

$$\lim_{n\to\infty} \frac{1}{H(\tau_n,T_0)} \int_{T_0}^{\tau_n} \frac{\partial^2 H}{\partial s^2}(\tau_n,s) w(s)\,\mathrm{d}s = \infty$$

and therefore contradicts (19). So, we have proved (20).

Next, we shall prove that

$$\int_{T_0}^{\infty} \frac{1}{w(s)} \Big(w'(s) - \frac{\varrho'(s)}{\varrho(s)}\, w(s)\Big)^2 \,\mathrm{d}s < \infty. \tag{22}$$

Suppose to the contrary that there exists a $T_1 \geqslant T_0$ such that

$$\int_{T_0}^{t} \frac{1}{w(s)} \Big(w'(s) - \frac{\varrho'(s)}{\varrho(s)} w(s)\Big)^2 \,\mathrm{d}s \geqslant \frac{\mu}{\varrho} \quad \text{for every } t \geqslant T_1,$$

where μ is arbitrary positive constant. Then, for all $t \geqslant T_1$

$$\begin{aligned}
&\frac{1}{H(t,T_0)} \int_{T_0}^{t} \frac{H(t,s)}{w(s)} \Big(w'(s) - \frac{\varrho'(s)}{\varrho(s)} w(s)\Big)^2 \,\mathrm{d}s \\
&\quad = \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\,\mathrm{d}\Big(\int_{T_0}^{s} \frac{1}{w(\tau)} \Big(w'(\tau) - \frac{\varrho'(\tau)}{\varrho(\tau)} w(\tau)\Big)^2 \,\mathrm{d}\tau\Big) \\
&\quad = \frac{1}{H(t,T_0)} \int_{T_0}^{t} \Big(-\frac{\partial H}{\partial s}(t,s)\Big) \Big(\int_{T_0}^{s} \frac{1}{w(\tau)} \Big(w'(\tau) - \frac{\varrho'(\tau)}{\varrho(\tau)}\, w(\tau)\Big)^2 \,\mathrm{d}\tau\Big) \,\mathrm{d}s \\
&\quad \geqslant \frac{1}{H(t,T_0)} \int_{T_1}^{t} \Big(-\frac{\partial H}{\partial s}(t,s)\Big) \Big(\int_{T_0}^{s} \frac{1}{w(\tau)} \Big(w'(\tau) - \frac{\varrho'(\tau)}{\varrho(\tau)} w(\tau)\Big)^2 \,\mathrm{d}\tau\Big) \,\mathrm{d}s \\
&\quad \geqslant \frac{\mu}{\varrho H(t,T_0)} \int_{T_1}^{t} \Big(-\frac{\partial H}{\partial s}(t,s)\Big) \,\mathrm{d}s = \frac{\mu}{\varrho}\, \frac{H(t,T_1)}{H(t,t_0)}.
\end{aligned}$$

By (21) we get

$$\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{H(t,s)}{w(s)}\Big(w'(s)-\frac{\varrho'(s)}{\varrho(s)}w(s)\Big)^2\,\mathrm{d}s \geqslant \mu \ \ \forall t\geqslant T_2,$$

and consequently for sufficiently large n

$$\frac{1}{H(\tau_n,T_0)}\int_{T_0}^{\tau_n}\frac{H(\tau_n,s)}{w(s)}\Big(w'(s)-\frac{\varrho'(s)}{\varrho(s)}w(s)\Big)^2\,\mathrm{d}s \geqslant \mu$$

which contradicts (18) and therefore proves (22).

By similar arguments, using (15), we can prove that

$$\int_{T_0}^{\infty} w(s)\Big(-\frac{\varrho''(s)}{\varrho(s)}\Big)\,\mathrm{d}s < \infty. \tag{23}$$

Next, using the fact that ϱ is concave function, we obtain for $t\geqslant T_0$

$$\varrho(t)-\varrho(T_0)=\int_{T_0}^{t}\varrho'(s)\,\mathrm{d}s > (t-T_0)\varrho'(t),$$

which ensures that

$$\limsup_{t\to\infty}\frac{t\varrho'(t)}{\varrho(t)} < \infty. \tag{24}$$

Using (R), we derive, for every $t\geqslant T_0$

$$\begin{aligned}
&\int_{T_0}^{t}\frac{1}{w(s)}\Big(w'(s)-\frac{\varrho'(s)}{\varrho(s)}w(s)\Big)^2\,\mathrm{d}s,\\
&\quad=\int_{T_0}^{t}\frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s-2\int_{T_0}^{t}\frac{\varrho'(s)}{\varrho(s)}w'(s)\,\mathrm{d}s+\int_{T_0}^{t}\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 w(s)\,\mathrm{d}s\\
&\quad=\int_{T_0}^{t}\frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s-2\frac{\varrho'(t)}{\varrho(t)}w(t)+2\frac{\varrho'(T_0)}{\varrho(T_0)}w(T_0)\\
&\qquad+2\int_{T_0}^{t}\frac{\varrho''(s)}{\varrho(s)}w(s)\,\mathrm{d}s-\int_{T_0}^{t}\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 w(s)\,\mathrm{d}s\\
&\quad\geqslant\int_{T_0}^{t}\frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s-2\frac{\varrho'(t)}{\varrho(t)}w(t)+2\frac{\varrho'(T_0)}{\varrho(T_0)}w(T_0)\\
&\qquad+(c+2)\int_{T_0}^{t}\frac{\varrho''(s)}{\varrho(s)}w(s)\,\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\int_{T_0}^{\infty} \frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s \leqslant 2\Big[\limsup_{t\to\infty} \frac{t\,\varrho'(t)}{\varrho(t)}\Big]\Big[\limsup_{t\to\infty} \frac{w(t)}{t}\Big] - 2\frac{\varrho'(T_0)}{\varrho(T_0)}w(T_0)$$
$$+(c+2)\int_{T_0}^{\infty}\Big(-\frac{\varrho''(s)}{\varrho(s)}\Big)w(s)\,\mathrm{d}s$$
$$+\int_{T_0}^{\infty}\frac{1}{w(s)}\Big(w'(s)-\frac{\varrho'(s)}{\varrho(s)}w(s)\Big)^2\,\mathrm{d}s,$$

which, because of (20), (22), (23) and (24), implies

$$\int_{T_0}^{\infty}\frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s<\infty. \tag{25}$$

Finally, by using (17)

$$\begin{aligned}\int_{T_0}^{\infty}\frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s &\leqslant \int_{T_0}^{\infty}\frac{[w'(s)+\Omega(s)w(s)]^2}{s}\,\mathrm{d}s\\ &\leqslant M\int_{T_0}^{\infty}\frac{[w'(s)+\Omega(s)w(s)]^2}{w(s)}\,\mathrm{d}s\\ &\leqslant 2M\int_{T_0}^{\infty}\frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s+2M\int_{T_0}^{\infty}\Omega^2(s)w(s)\,\mathrm{d}s\\ &\leqslant 2M\int_{T_0}^{\infty}\frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s+2M^2\int_{t_0}^{\infty}s\Omega^2(s)\,\mathrm{d}s,\end{aligned}$$

where $M=\sup\limits_{t\geqslant T_0} w(t)/t$ and by (20), M is finite. Thus, because of (25), (C_1) and (H_6), we come to a contradiction. $\square$

Remark 2. Taking $H(t,s)=(t-s)^{n-1}$ for some integer $n\geqslant 2$, for the particular case of the equation (E_2), we obtain Theorem A.

We observe that Theorem 2.2 can be applied in some cases in which Theorem 2.1 is not applicable. Such a case is described in the following example:

Example 2. Consider the differential equation (E_3), where $q(t)=t^{\lambda}\cos t$ and $\lambda-\nu+\frac{1}{2}\alpha<0$. Then as in Example 1, the conditions (C_1), (C_2) and (C_3) are satisfied and $M_{f,\psi}\geqslant\frac{1}{2}\alpha$.

We can take $\beta=\frac{1}{2}\alpha$, $\varrho(t)=t^{2\mu/\alpha}$ for some $\mu\in\big[0,\frac{1}{2}\alpha\big)$ and $H(t,s)=(t-s)^2$ for $t\geqslant s\geqslant t_0$. Then, the condition (R) is satisfied for arbitrary constant c such that $c\geqslant 2\mu/(\alpha-2\mu)$ and the conditions $(H)_1$–$(H)_6$ are satisfied.

So, using (13), for $\delta = \lambda - \nu + \mu < 0$ and for every $T \geqslant t_0$, we obtain

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t^2} \int_T^t & \Big[(t-s)^2 s^\delta \cos s - \frac{3\nu^2}{4}(t-s)^2 s^{\frac{1}{2}\alpha - 2}\Big]\,\mathrm{d}s \\ & \geqslant -T^\delta \sin T + T^\delta - \frac{3\nu^2}{4-2\alpha} T^{\frac{1}{2}\alpha-1}.\end{aligned}$$

Since $\delta < 0$ and $\frac{1}{2}\alpha - 1 < 0$, for arbitrary small constant $\varepsilon > 0$, there exists a $t_1 \geqslant t_0$ such that for $T \geqslant t_1$

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_T^t \Big[(t-s)^2 s^\delta \cos s - \frac{3\nu^2}{4}(t-s)^2 s^{\frac{1}{2}\alpha-2}\Big]\,\mathrm{d}s \geqslant -T^\delta \sin T - \varepsilon.$$

Now, set $\varphi(T) = -T^\delta \sin T - \varepsilon$ and consider an integer N such that $2N\pi + \frac{5}{4}\pi \geqslant \max\big\{t_1, \big(1+\sqrt{2}\varepsilon\big)^{1/\delta}\big\}$. Then, for all integers $n \geqslant N$, we have

$$\varphi(T) \geqslant \frac{1}{\sqrt{2}} \quad \text{for every } T \in \big[2n\pi + \tfrac{5}{4}\pi, 2n\pi + \tfrac{7}{4}\pi\big],$$

which implies

$$\begin{aligned}\int_{t_0}^\infty \frac{\varphi^2+(s)}{s}\,\mathrm{d}s & \geqslant \sum_{n=N}^\infty \frac{1}{2}\int_{2n\pi+5\pi/4}^{2n\pi+7\pi/4} \frac{1}{s}\,\mathrm{d}s \\ & = \frac{1}{2}\sum_{n=N}^\infty \ln\bigg(1 + \frac{\frac{1}{2}\pi}{2n\pi + \frac{5}{4}\pi}\bigg) = \infty.\end{aligned}$$

Accordingly, all conditions of Theorem 2.2 are satisfied and hence the equation (E_3) is oscillatory.

Notice that Theorem 2.1 is not applicable to the equation (E_3) in this case, since the condition (C_2) is not satisfied.

Theorem 2.3. *Let*

(i) $\varrho \in C^2([t_0,\infty))$ *be a positive function such that*

$$(\mathrm{R}_1) \qquad \varrho'(t) > 0 \quad \text{and} \quad \varrho''(t) \leqslant 0 \quad \text{for every } t \geqslant t_0,$$

(ii) $H(t,s)$ *be a twice continuously differentiable function on* $\mathscr{D}$ *with respect to the second variable which satisfies the conditions* $(\mathrm{H})_1$–$(\mathrm{H})_6$. *Then the equation* (E) *is oscillatory if there exists a function* $\varphi \in C([t_0,\infty))$ *such that* (C_3) *holds for every* $T \geqslant t_0$ *and some* $\beta \in [0, M_{f,\psi}]$ *and*

$$(\mathrm{C}_4) \qquad \limsup_{t\to\infty} \bigg[\int_{t_0}^t \Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 s\,\mathrm{d}s\bigg]^{-1} \int_{t_0}^t \frac{\varphi_+^2(s)}{s}\,\mathrm{d}s = \infty.$$

P r o o f. Let $x(t)$ be a solution of the differential equation (E) on an interval $[T_0,\infty)$, $T_0\geqslant t_0$, with $x(t)\neq 0$ for all $t\geqslant T_0$. Let $w(t)$ be defined by (1). Then, as in the proof of Theorem 2.2, we derive (17), (20), (22), (23) and (24). Furthermore, for every $t\geqslant T_0$, we obtain

$$\begin{aligned}
\int_{T_0}^t \frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s &= \int_{T_0}^t \frac{1}{w(s)}\Big(w'(s)-\frac{\varrho'(s)}{\varrho(s)}w(s)\Big)^2\,\mathrm{d}s = +2\frac{\varrho'(t)}{\varrho(t)}w(t)-2\frac{\varrho'(T_0)}{\varrho(T_0)}w(T_0)\\
&= +2\int_{T_0}^t\Big(-\frac{\varrho''(s)}{\varrho(s)}\Big)w(s)\,\mathrm{d}s+\int_{T_0}^t\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 w(s)\,\mathrm{d}s\\
&\leqslant \int_{T_0}^t \frac{1}{w(s)}\Big(w'(s)-\frac{\varrho'(s)}{\varrho(s)}w(s)\Big)^2\,\mathrm{d}s+2\frac{\varrho'(t)}{\varrho(t)}w(t)\\
&= +2\int_{T_0}^t\Big(-\frac{\varrho''(s)}{\varrho(s)}\Big)w(s)\,\mathrm{d}s+M\int_{T_0}^t\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 s\,\mathrm{d}s,
\end{aligned}$$

where $M=\sup\limits_{t\geqslant T_0} w(t)/t$. Accordingly, by taking into account (20), (22), (23) and (24), we conclude that there exists a positive constant K such that

$$\int_{T_0}^t \frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s\leqslant K\int_{t_0}^t\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 s\,\mathrm{d}s,\quad t\geqslant T_0. \tag{26}$$

Finally, by (17) and (26), for $t\geqslant T_0$ we have

$$\begin{aligned}
\int_{t_0}^t \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s &= \int_{t_0}^{T_0} \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s+\int_{T_0}^t \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s\\
&\leqslant \int_{t_0}^{T_0} \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s+M\int_{T_0}^t \frac{[w'(s)+\Omega(s)w(s)]^2}{w(s)}\,\mathrm{d}s\\
&\leqslant \int_{t_0}^{T_0} \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s+2M\int_{T_0}^t \frac{[w'(s)]^2}{w(s)}\,\mathrm{d}s+2M\int_{T_0}^t \Omega^2(s)w(s)\,\mathrm{d}s\\
&\leqslant \int_{t_0}^{T_0} \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s+2MK\int_{t_0}^t\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 s\,\mathrm{d}s+2M^2\int_{T_0}^t s\Omega^2(s)\,\mathrm{d}s,
\end{aligned}$$

which for all $t\geqslant T_0$ implies

$$\begin{aligned}
\Big\{\int_{t_0}^t\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 s\,\mathrm{d}s\Big\}^{-1}\int_{t_0}^t \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s &\leqslant \Big\{\int_{t_0}^{T_0}\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 s\,\mathrm{d}s\Big\}^{-1}\int_{t_0}^{T_0} \frac{[\varphi_+(s)]^2}{s}\,\mathrm{d}s\\
&\quad+2MK+2M^2\Big\{\int_{t_0}^{T_0}\Big(\frac{\varrho'(s)}{\varrho(s)}\Big)^2 s\,\mathrm{d}s\Big\}^{-1}\int_{T_0}^t s\Omega^2(s)\,\mathrm{d}s
\end{aligned}$$

and therefore by (H_6) and (C_4) gives the desired contradiction. $\square$

Remark 3. The condition (R) of Theorem 2.2 is stronger than the condition (R_1) required in Theorem 2.3, but on the other hand the assumption (C_1) of Theorem 2.2 is weaker than (C_4) required in the previous theorem.

Remark 4. Letting $H(t,s) = t - s$ in Theorem 2.3, for the special case of the differential equation (E_2), we have the oscillation criteria of Philos [16, Theorem 2].

Example 3. Consider the differential equation

$$(\mathrm{E}_4)\quad [t^2x^3(t)x'(t)]' + [t^3(t+\log t)^{-\frac{1}{2}\alpha}\sin t + 3][|x(t)|^{\alpha+3} + x^4(t)] = 0, \quad t \geqslant t_0 > 1$$

where $0 < \alpha < 1$, $\lambda - \nu + \frac{1}{2}\alpha < 0$. We can take $H(t,s)$ as in Example 2 and $\beta = \frac{1}{2}\alpha$. We define $\varrho(t) = t + \log t$, and observe that (R_1) is fulfilled. Furthermore, for every $t \geqslant T \geqslant t_0$, we have

$$\begin{aligned}\int_T^t H(t,s)\varrho^\beta(s)\chi(s)\,\mathrm{d}s &= \int_T^t (t-s)^2 s\sin s\,\mathrm{d}s \\ &= (t-T)^2T\cos T - t^2\sin T + 2t(2T\sin T + \cos t + 2\cos T) \\ &\quad - 6T\cos T - 6\sin t + 6\sin T - 3T^2\sin T\end{aligned}$$

and consequently

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\varrho^\beta(s)\chi(s)\,\mathrm{d}s = T\cos T - \sin T \geqslant T\cos T - 2.$$

Thus, (C_3) holds with $\varphi(T) = T\cos T - 2$, $T \geqslant t_0$. We consider a number t_1 such that $t_1 \geqslant \max\{t_0, 4\sqrt{2}\}$. Next, we choose an integer N such that $2N\pi - \frac{1}{4}\pi \geqslant t_1$, so that for every integer $n \geqslant N$, we obtain

$$\varphi(T) \geqslant \frac{T}{2\sqrt{2}} \quad \text{for every } T \in \left[2n\pi - \tfrac{1}{4}\pi, 2n\pi + \tfrac{1}{4}\pi\right].$$

Then, for $n \geqslant N$, we get

$$\int_{t_0}^{2n\pi+\pi/4}\frac{\varphi_+^2(s)}{s}\,\mathrm{d}s \geqslant \int_{2n\pi-\pi/4}^{2n\pi+\pi/4}\frac{\varphi_+^2(s)}{s}\,\mathrm{d}s \geqslant \frac{1}{8}\int_{2n\pi-\pi/4}^{2n\pi+\pi/4}s\,\mathrm{d}s = \frac{\pi^2 n}{8},$$

and therefore,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{\log t}\int_{t_0}^t\frac{\varphi_+^2(s)}{s}\,\mathrm{d}s &\geqslant \limsup_{n\to\infty}\frac{1}{\log(2n\pi+\pi/4)}\int_{t_0}^{2n\pi+\pi/4}\frac{\varphi_+^2(s)}{s}\,\mathrm{d}s \\ &\geqslant \lim_{n\to\infty}\frac{\pi^2 n}{8\log(2n\pi+\pi/4)} = \infty.\end{aligned}$$

It is proved in [16, Remark 3] that the condition (C_4) is satisfied if the following condition holds

$$\limsup_{t\to\infty} \frac{1}{\log t} \int_{t_0}^{t} \frac{\varphi_+^2(s)}{s}\, \mathrm{d}s = \infty.$$

Consequently, all conditions of Theorem 2.3 are satisfied and the differential equation (E_4) is oscillatory.

References

[1] *B. Ayanlar and A. Tiryaki*: Oscillation theorems for nonlinear second-order differential equations. Comput. Math. Appl. *44* (2002), 529–538.

[2] *Y. Chen*: On the oscillation of nonlinear second order equations. J. South China Normal Univ. Natur. Sci. Ed. *2* (1986), 99–103.

[3] *S. R. Grace and B. S. Lalli*: On the second order nonlinear oscillations. Bull. Inst. Math. Acad. Sinica *15* (1987), 297–309.

[4] *S. R. Grace*: Oscillation theorems for second order nonlinear differential equations with damping. Math. Nachr. *141* (1989), 117–127.

[5] *S. R. Grace and B. S. Lalli*: Integral averaging techniques for the oscillation of second order nonlinear differential equations. J. Math. Anal. and Appl. *149* (1990), 277–311.

[6] *S R. Grace*: Oscillation theorems for nonlinear differential equations of second order. J. Math. Anal. and Appl. *171* (1992), 220–241.

[7] *M. Kirane and Y. V. Rogovchenko*: Oscillation results for a second order damped differential equation with nonmonotonous nonlinearity. J. Math. Anal. Appl. *250* (2000), 118–138.

[8] *T. Kura*: Oscillation theorems for second order nonlinear differential equations. Proc. Amer. Math. Soc. *84* (1982), 535–538.

[9] *M. K. Kwong and J. S. W. Wong*: On an oscillation theorem of Belohorec. SIAM J. Math. Anal. *14* (1983), 474–476.

[10] *H. J. Li and C. C. Yeh*: Oscillation of second order sublinear differential equations. Dynamic Systems Appl. *6* (1997), 529–534.

[11] *J. V. Manojlović*: Oscillation criteria for second order sublinear differential equation. Math. Comp. Modelling *30* (1999), 109–119.

[12] *J. V. Manojlović*: Oscillation criteria for second order sublinear differential equation. Computers and Mathematics with Applications *39* (2000), 161–172.

[13] *J. V. Manojlović*: Integral averages and oscillation of second order nonlinear differential equations. Computers and Mathematics with Applications *41* (2001), 1521–1534.

[14] *Ch. G. Philos*: Oscillation of sublinear differential equations of second order. Nonlinear Anal. *7* (1983), 1071–1080.

[15] *Ch. G. Philos*: On second order sublinear oscillation. Aequationes Math. *27* (1984), 242–254.

[16] *Ch. G. Philos*: Integral averaging techniques for the oscillation of second order sublinear ordinary differential equations. J. Austral. Math. Soc. (Series A) *40* (1986), 111–130.

[17] *Ch. G. Philos*: Oscillation theorems for linear differential equations of second order. Arch. Math. (Basel) *53* (1989), 482–492.

[18] *Ch. G. Philos*: Integral averages and oscillation of second order sublinear differential equations. Diff. Integ. Equat. *4* (1991), 205–213.

[19] *J. Yan*: A note on second order sublinear oscillation theorems. J. Math. Anal. and Appl. *104* (1984), 103–106.

[20] *J. S. W. Wong*: An oscillation criterion for second order sublinear differential equations. Conf. Proc. Canad. Math. Soc. *8* (1987), 299–302.

[21] *J. S. W. Wong and C. C. Yeh*: An oscillation criterion for second order sublinear differential equations. J. Math. Anal. Appl. *171* (1992), 346–351.

[22] *J. S. W. Wong*: Oscillation criteria for second order nonlinear differential equations involving general means. J. Math. Anal. Appl. *247* (2000), 489–505.

Author's address: University of Niš, Faculty of Science and Mathematics, Višegradska 33, 180 00 Niš, Yugoslavia, e-mail: `jelenam@archimed.filfak.ni.ac.yu`.