Ice B. Risteski

Some higher order complex vector functional equations

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 4, 1015–1034

Persistent URL: http://dml.cz/dmlcz/127948

Czechoslovak Mathematical Journal, 54 (129) (2004), 1015–1034

SOME HIGHER ORDER COMPLEX VECTOR FUNCTIONAL EQUATIONS

ICE B. RISTESKI, Toronto

(Received February 7, 2002)

Abstract. In the present paper some complex vector functional equations of higher order without parameters and with complex parameters are solved.

Keywords: complex vector functional equations

MSC 2000: 39B32, 39B52

0. INTRODUCTION

Functional equations find applications in biology, social sciences, engineering, etc. as well as in many other branches of mathematics. A great number of such applications can be found in [1]. This has led to considerable interest in the study of functional equations and has given rise to numerous articles and monographs on the subject.

The present paper is devoted to the study of some complex vector functional equations of higher order. To the best of our knowledge, up to now this kind of complex vector functional equations has not been considered in literature, and we think that their study will be of interest. For this reason we carried out our research with the goal to shed light on this not sufficiently studied field of complex vector functional equations. The results presented here supplement and generalize some of our previous results [2], [3], [4].

Throughout this paper, $\mathscr{V}$ is an n-dimensional complex vector space. Vectors from $\mathscr{V}$ will be denoted by $\mathbf{Z}_i, \mathbf{U}_i$, and so on, and we also denote $\mathbf{O} = (0, 0, \ldots, 0)^T$ and $\mathbf{I} = (1, 1, \ldots, 1)^T$. If $\mathbf{U}, \mathbf{V} \in \mathscr{V}$, with $\mathbf{U} = (u_1, \ldots, u_n)^T$ and $\mathbf{V} = (v_1, \ldots, v_n)^T$ with respect to some basis, we define $\mathbf{UV} = (u_1v_1, \ldots, u_nv_n)^T$.

1. Higher order functional equation without parameters

In this section the following result will be proved.

Theorem 1.1. *The general solution of the functional equation*

$$\sum_{i=1}^{2n-1} F(\mathbf{Z}_1, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2} \dots, \mathbf{Z}_{2n+i-1}) = \mathbf{O} \tag{1.1}$$
$$(\mathbf{Z}_{2n+i-1} \equiv \mathbf{Z}_i,\ 2 \leqslant i \leqslant 2n-1),$$

where

$$F(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n}) = \prod_{k=1}^{n} f(\mathbf{Z}_{2k-1}, \mathbf{Z}_{2k}) \quad (f\colon\ \mathscr{V}^2 \to \mathscr{V};\ n > 1), \tag{1.2}$$

is given by

$$f(\mathbf{U}, \mathbf{V}) = g(\mathbf{U})h(\mathbf{V}) - g(\mathbf{V})h(\mathbf{U}), \quad n = 2, \tag{1.3}$$

$$f(\mathbf{U}, \mathbf{V}) = \mathbf{O}, \quad n > 2, \tag{1.4}$$

where $g, h\colon\ \mathscr{V} \to \mathscr{V}$ are arbitrary functions.

P r o o f. If we put $\mathbf{Z}_k = \mathbf{U}$ $(1 \leqslant k \leqslant 2n)$, the equation (1.1) takes the form $f(\mathbf{U}, \mathbf{U}) \equiv \mathbf{O}$.

Now, we will distinguish two possibilities:

1°. Let $n = 2$. In this case the equation (1.1) becomes

$$f(\mathbf{Z}_1, \mathbf{Z}_2)f(\mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_1, \mathbf{Z}_3)f(\mathbf{Z}_4, \mathbf{Z}_2) + f(\mathbf{Z}_1, \mathbf{Z}_4)f(\mathbf{Z}_2, \mathbf{Z}_3) = \mathbf{O}. \tag{1.5}$$

Indeed, a straightforward calculation shows that the function (1.3) satisfies the functional equation (1.1) for arbitrary functions g and h.

Conversely, we will prove that every solution of the functional equation (1.1) has the form (1.3).

We denote any nonzero component of a nontrivial solution again by $f\colon\ \mathscr{V}^2 \to \mathbb{C}$. For such a component there exists at least one pair of constant complex vectors $(\mathbf{A}, \mathbf{B})$ $(\mathbf{A}, \mathbf{B} \in \mathscr{V})$ such that $f(\mathbf{A}, \mathbf{B}) \neq 0$. By putting $\mathbf{Z}_1 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{B}$, $\mathbf{Z}_3 = \mathbf{U}$, $\mathbf{Z}_4 = \mathbf{V}$, the *scalar* equation (1.5) takes the form

$$f(\mathbf{U}, \mathbf{V}) = -\frac{f(\mathbf{A}, \mathbf{U})}{f(\mathbf{A}, \mathbf{B})} f(\mathbf{V}, \mathbf{B}) - \frac{f(\mathbf{A}, \mathbf{V})}{f(\mathbf{A}, \mathbf{B})} f(\mathbf{B}, \mathbf{U}). \tag{1.6}$$

If we substitute $\mathbf{U}=\mathbf{B}$ and take into consideration the condition $f(\mathbf{U},\mathbf{U})\equiv 0$, the equation (1.6) implies that

$$f(\mathbf{B},\mathbf{V})=-f(\mathbf{V},\mathbf{B}). \tag{1.7}$$

According to (1.7), the equation (1.6) takes the form

$$f(\mathbf{U},\mathbf{V})=\frac{f(\mathbf{A},\mathbf{U})}{f(\mathbf{A},\mathbf{B})}\,f(\mathbf{B},\mathbf{V})-\frac{f(\mathbf{A},\mathbf{V})}{f(\mathbf{A},\mathbf{B})}\,f(\mathbf{B},\mathbf{U}).$$

If we denote

$$g(\mathbf{U})=\frac{f(\mathbf{A},\mathbf{U})}{f(\mathbf{A},\mathbf{B})},\quad h(\mathbf{U})=f(\mathbf{B},\mathbf{U}),$$

then we have

$$f(\mathbf{U},\mathbf{V})=g(\mathbf{U})h(\mathbf{V})-g(\mathbf{V})h(\mathbf{U}).$$

This is the general solution of the equation (1.1) for $n=2$.

2°. Let $n>2$. If we put $\mathbf{Z}_k=\mathbf{U}$ (k odd) and $\mathbf{Z}_k=\mathbf{V}$ (k even) and take into consideration the property $f(\mathbf{U},\mathbf{U})\equiv\mathbf{O}$, then (1.1) implies that

$$\sum_{i=0}^{n-1} f^{n-i}(\mathbf{U},\mathbf{V})f^{i}(\mathbf{V},\mathbf{U})=\mathbf{O}. \tag{1.8}$$

By the substitutions

$$\mathbf{Z}_1=\mathbf{Z}_4=\mathbf{U},\quad \mathbf{Z}_{2k-1}=\mathbf{U},\quad \mathbf{Z}_2=\mathbf{Z}_3=\mathbf{V},\quad \mathbf{Z}_{2k}=\mathbf{V}\qquad (3\leqslant k\leqslant n)$$

the functional equation (1.1) reduces to

$$f^{n-1}(\mathbf{U},\mathbf{V})f(\mathbf{V},\mathbf{U})=\mathbf{O}. \tag{1.9}$$

Also, the following equality holds:

$$f^{n-1}(\mathbf{V},\mathbf{U})f(\mathbf{U},\mathbf{V})=\mathbf{O}. \tag{1.10}$$

Now, if we put

$$\mathbf{Z}_1=\mathbf{Z}_4=\mathbf{Z}_6=\ldots=\mathbf{Z}_{2r}=\mathbf{Z}_{2r+2}=\mathbf{U},\quad \mathbf{Z}_{2k-1}=\mathbf{U},$$
$$\mathbf{Z}_2=\mathbf{Z}_3=\mathbf{Z}_5=\ldots=\mathbf{Z}_{2r-1}=\mathbf{Z}_{2r+1}=\mathbf{V},\quad \mathbf{Z}_{2k}=\mathbf{V}$$
$$(r+2\leqslant k\leqslant n;\quad 1\leqslant r<n)$$

the equation (1.1) becomes

$$f^{n-r}(\mathbf{U},\mathbf{V})f^{r}(\mathbf{V},\mathbf{U})=\mathbf{O}\quad(1\leqslant r<n). \tag{1.11}$$

According to the equality (1.11), the equation (1.8) yields

$$f(\mathbf{U},\mathbf{V})=\mathbf{O}\quad(n>2).$$

This completes the proof of Theorem 1.1.

2. Higher order functional equation with parameters

In this section we will generalize the results given in the previous section.

Theorem 2.1. *The general solution of the functional equation*

$$\sum_{i=1}^{2n-1}a_iF(\mathbf{Z}_1,\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\ldots,\mathbf{Z}_{2n+i-1})=\mathbf{O} \tag{2.1}$$
$$(\mathbf{Z}_{2n+i-1}\equiv\mathbf{Z}_i;\ 2\leqslant i\leqslant 2n-1),$$

where a_i $(1\leqslant i\leqslant 2n-1)$ *are complex parameters not all of which are equal to zero,*

$$F(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{2n})=\prod_{k=1}^{n}f(\mathbf{Z}_{2k-1},\mathbf{Z}_{2k})\quad(f\colon\mathscr{V}^2\to\mathscr{V};\ n>1), \tag{2.2}$$

is given by

$$f(\mathbf{U},\mathbf{V})=g(\mathbf{U})g(\mathbf{V})\quad(n\geqslant2)\quad\textit{if}\ \sum_{i=1}^{2n-1}a_i=0, \tag{2.3}$$

$$f(\mathbf{U},\mathbf{V})=g(\mathbf{U})h(\mathbf{V})-g(\mathbf{V})h(\mathbf{U})\quad(n=2)\quad\textit{if}\ a_1=a_2=a_3\ (\neq0), \tag{2.4}$$

$$f(\mathbf{U},\mathbf{V})=\mathbf{O}\quad\textit{in all other cases,} \tag{2.5}$$

where $g,h\colon\mathscr{V}\to\mathscr{V}$ *are arbitrary functions.*

P r o o f. First, we will prove the theorem for the case $n=2$. Then the functional equation (2.1) becomes

$$a_1f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_3,\mathbf{Z}_4)+a_2f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_2)+a_3f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_2,\mathbf{Z}_3)=\mathbf{O}. \tag{2.6}$$

By a cyclic permutation of the vectors $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$ in equation (2.6) we obtain

$$(2.7)\quad a_1 f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_2) + a_2 f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_2,\mathbf{Z}_3) + a_3 f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_3,\mathbf{Z}_4) = \mathbf{O},$$

$$(2.8)\quad a_1 f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_2,\mathbf{Z}_3) + a_2 f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_3,\mathbf{Z}_4) + a_3 f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_2) = \mathbf{O}.$$

The system of equations (2.6), (2.7) and (2.8) has a nontrivial solution if and only if the following condition is satisfied:

$$(2.9)\qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} = 0.$$

In all other cases the general solution of the functional equation (2.6) is

$$f(\mathbf{U},\mathbf{V}) \equiv \mathbf{O}.$$

From (2.9) it follows that

$$(2.10)\qquad (a_1+a_2+a_3)[(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2] = 0.$$

We will investigate the following cases:

1°. Let $a_1+a_2+a_3=0$ and $a_1=a_2$ $(\neq 0)$. Then the condition (2.9) is satisfied. The equation (2.6) has the form

$$(2.11)\qquad f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_3,\mathbf{Z}_4) + f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_2) = 2f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_2,\mathbf{Z}_3).$$

By a cyclic permutation of the vectors $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$ from this equation we find

$$(2.12)\qquad f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_2) + f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_2,\mathbf{Z}_3) = 2f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_3,\mathbf{Z}_4).$$

If we eliminate the term $f(\mathbf{Z}_1,\mathbf{Z}_2)f(\mathbf{Z}_3,\mathbf{Z}_4)$ from the equations (2.11) and (2.12), we obtain the equation

$$(2.13)\qquad f(\mathbf{Z}_1,\mathbf{Z}_3)f(\mathbf{Z}_4,\mathbf{Z}_2) = f(\mathbf{Z}_1,\mathbf{Z}_4)f(\mathbf{Z}_2,\mathbf{Z}_3).$$

We denote any nonzero component of a nontrivial solution of the equation (2.11) again by $f\colon \mathscr{V}^2 \to \mathbb{C}$. For such a component there exists at least one pair of constant complex vectors $(\mathbf{A},\mathbf{B})$ $(\mathbf{A},\mathbf{B} \in \mathscr{V})$ such that $f(\mathbf{A},\mathbf{B}) \neq 0$.

By putting $\mathbf{Z}_1=\mathbf{A}$, $\mathbf{Z}_2=\mathbf{U}$, $\mathbf{Z}_3=\mathbf{V}$ and $\mathbf{Z}_4=\mathbf{B}$, the equation (2.13) becomes

$$(2.14)\qquad f(\mathbf{A},\mathbf{B})f(\mathbf{U},\mathbf{V}) = f(\mathbf{A},\mathbf{V})f(\mathbf{B},\mathbf{U}).$$

If we put $\mathbf{U} = \mathbf{B}$ in the last equation, we have

$$f(\mathbf{B}, \mathbf{V}) = \frac{f(\mathbf{B}, \mathbf{B})}{f(\mathbf{A}, \mathbf{B})} f(\mathbf{A}, \mathbf{V}).$$

On the basis of this equality, the equation (2.14) becomes

$$f(\mathbf{U}, \mathbf{V}) = g(\mathbf{U}) g(\mathbf{V}), \tag{2.15}$$

where we put

$$\frac{\sqrt{f(\mathbf{B}, \mathbf{B})}}{f(\mathbf{A}, \mathbf{B})} f(\mathbf{A}, \mathbf{U}) = g(\mathbf{U}).$$

Really, the function (2.15) is a solution of the equation (2.11).

2°. Let $a_1 + a_2 + a_3 = 0$ and $a_1 \neq a_2$. Then the condition (2.9) is satisfied.

For $a_1 + a_2 + a_3 = 0$, the equation (2.6) becomes

$$\begin{aligned} a_1 f(\mathbf{Z}_1, \mathbf{Z}_2) f(\mathbf{Z}_3, \mathbf{Z}_4) + a_2 f(\mathbf{Z}_1, \mathbf{Z}_3) f(\mathbf{Z}_4, \mathbf{Z}_2) \\ = (a_1 + a_2) f(\mathbf{Z}_1, \mathbf{Z}_4) f(\mathbf{Z}_2, \mathbf{Z}_3). \end{aligned} \tag{2.16}$$

If we suppose that $a_1 = 0$, this allows us to divide by $a_2 \neq 0$ and then the equation (2.16) reduces to the equation (2.13), whose general solution is the function (2.15).

Now, we will assume that $a_1 \neq 0$.

Let $f(\mathbf{U}, \mathbf{U}) \equiv \mathbf{O}$. In this case, for any nonzero component of a nontrivial solution of the equation (2.16) (denoted again by f) there exists at least one pair of constant complex vectors $(\mathbf{A}, \mathbf{B})$ such that $f(\mathbf{A}, \mathbf{B}) \neq 0$. By putting $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{A}$, $\mathbf{Z}_2 = \mathbf{Z}_4 = \mathbf{B}$, from the equation (2.16) we obtain

$$(a_1 + a_2) f(\mathbf{B}, \mathbf{A}) = a_1 f(\mathbf{A}, \mathbf{B}). \tag{2.17}$$

For $\mathbf{Z}_1 = \mathbf{U}$, $\mathbf{Z}_2 = \mathbf{B}$, $\mathbf{Z}_3 = \mathbf{Z}_4 = \mathbf{A}$, by virtue of the above relation (2.17), the equation (2.16) reduces to the equation

$$(a_1 - a_2) f(\mathbf{A}, \mathbf{B}) f(\mathbf{U}, \mathbf{A}) = 0,$$

from which it follows that

$$f(\mathbf{U}, \mathbf{A}) = 0. \tag{2.18}$$

By the substitutions $\mathbf{Z}_1 = \mathbf{U}$, $\mathbf{Z}_2 = \mathbf{V}$, $\mathbf{Z}_3 = \mathbf{A}$, $\mathbf{Z}_4 = \mathbf{B}$, the equation (2.16), on the basis of the equality (2.18), yields

$$f(\mathbf{U}, \mathbf{V}) \equiv 0. \tag{2.19}$$

Let $f(\mathbf{U},\mathbf{U}) \not\equiv \mathbf{O}$. In this case, for any component of the solution of the equation (2.16) (denoted again by f) for which $f(\mathbf{U},\mathbf{U}) \not\equiv 0$ there exists at least one constant complex vector $\mathcal{C}$ such that $f(\mathcal{C},\mathcal{C}) \neq 0$. If we put $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_4 = \mathcal{C}$, $\mathbf{Z}_3 = \mathbf{U}$, from (2.16) we obtain

$$f(\mathcal{C},\mathbf{U}) = f(\mathbf{U},\mathcal{C}). \tag{2.20}$$

For $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathcal{C}$, $\mathbf{Z}_3 = \mathbf{U}$, $\mathbf{Z}_4 = \mathbf{V}$, in view of (2.20) we can write the equation (2.16) in the form

$$f(\mathbf{U},\mathbf{V}) = \frac{f(\mathcal{C},\mathbf{U})f(\mathcal{C},\mathbf{V})}{f(\mathcal{C},\mathcal{C})}$$

or

$$f(\mathbf{U},\mathbf{V}) = g(\mathbf{U})g(\mathbf{V}), \tag{2.21}$$

with the notation $f(\mathcal{C},\mathbf{U}) = \sqrt{f(\mathcal{C},\mathcal{C})}g(\mathbf{U})$.

It is not hard to check that the function (2.21) is really a solution of the equation (2.16).

Since (2.21) includes the trivial solution (2.19) as well as solutions with zero components, the general solution of the equation (2.16) is given by the formula (2.21).

3°. Let $a_1 + a_2 + a_3 \neq 0$. In view of (2.10), either $f(\mathbf{U},\mathbf{V}) \equiv \mathbf{O}$ or

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 = 0. \tag{2.22}$$

Let us suppose that $f(\mathbf{U},\mathbf{V}) \not\equiv \mathbf{O}$. Since a_1, a_2, a_3 are complex numbers, the equality (2.22) does not directly imply

$$a_1 = a_2 = a_3. \tag{2.23}$$

Adding together the equations (2.6), (2.7) and (2.8), by virtue of the condition $a_1 + a_2 + a_3 \neq 0$ we obtain the equation (1.5), which was solved in the previous section. Its solution is given by formula (1.3). It remains to show that (2.23) holds.

According to (1.3), the solution of equation (1.5) satisfies

$$f(\mathbf{U},\mathbf{V}) = -f(\mathbf{V},\mathbf{U}), \quad f(\mathbf{U},\mathbf{U}) \equiv \mathbf{O}. \tag{2.24}$$

If we put $\mathbf{Z}_1 = \mathbf{Z}_4 = \mathbf{U}$, $\mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{V}$, from (2.6) we find

$$(a_2 - a_1)f^2(\mathbf{U},\mathbf{V}) = \mathbf{O}. \tag{2.25}$$

Since $f(\mathbf{U},\mathbf{V}) \not\equiv \mathbf{O}$, the equality (2.25) implies $a_1 = a_2$. Now, from the condition (2.22) we find (2.23), and we have (2.4). Thus the theorem is proved for $n = 2$.

Now we pass to the proof of the theorem for $n > 2$.

First we will investigate the case

$$\sum_{i=1}^{2n-1} a_i \neq 0.$$

By putting $\mathbf{Z}_i = \mathbf{U}$ $(1 \leqslant i \leqslant 2n)$, from the equation (2.1) we obtain the identity

$$f(\mathbf{U}, \mathbf{U}) \equiv \mathbf{O}. \tag{2.26}$$

Next, we assume $a_1 \neq 0$. We may assume this without loss of generality since if $a_1 = 0$, then there must be at least one $a_i \neq 0$ $(2 \leqslant i \leqslant 2n-2)$, and by a cyclic permutation of the vectors $\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_{2n}$ we may achieve that the coefficient at the term $f(\mathbf{Z}_1, \mathbf{Z}_2) f(\mathbf{Z}_3, \mathbf{Z}_4) \ldots f(\mathbf{Z}_{2n-1}, \mathbf{Z}_{2n})$ be different from zero.

By introducing the substitutions

$$\mathbf{Z}_1 = \mathbf{Z}_3 = \ldots = \mathbf{Z}_{2n-1} = \mathbf{U} \qquad \text{and} \qquad \mathbf{Z}_2 = \mathbf{Z}_4 = \ldots = \mathbf{Z}_{2n} = \mathbf{V}$$

and taking into consideration the identity (2.26), the equation (2.1) takes the form

$$\sum_{i=0}^{n-1} a_{2i+1} f^{n-i}(\mathbf{U}, \mathbf{V}) f^i(\mathbf{V}, \mathbf{U}) = \mathbf{O}. \tag{2.27}$$

For

$$\mathbf{Z}_3 = \mathbf{Z}_5 = \ldots = \mathbf{Z}_{2r+1} = \mathbf{V}, \quad \mathbf{Z}_4 = \mathbf{Z}_6 = \ldots = \mathbf{Z}_{2r+2} = \mathbf{U},$$
$$\mathbf{Z}_1 = \mathbf{Z}_{2i-1} = \mathbf{U}, \quad \mathbf{Z}_2 = \mathbf{Z}_{2i} = \mathbf{V}$$
$$(r+2 \leqslant i \leqslant n;\ 1 \leqslant r < n)$$

the equation (2.1) yields

$$a_1 f^{n-r}(\mathbf{U}, \mathbf{V}) f^r(\mathbf{V}, \mathbf{U}) = \mathbf{O} \quad (1 \leqslant r < n)$$

so that from (2.27) we deduce $f(\mathbf{U}, \mathbf{V}) \equiv \mathbf{O}$.

Now, we pass to the investigation of the case

$$\sum_{i=1}^{2n-1} a_i = 0.$$

If we assume $f(\mathbf{U}, \mathbf{U}) \equiv \mathbf{O}$, then the general solution of the equation (2.1) is (2.5), which may be proved as in the case $\sum\limits_{i=1}^{2n-1} a_i \neq 0$.

If we assume that $f(\mathbf{U},\mathbf{U}) \not\equiv \mathbf{O}$, then for each component of f (denoted again by f) such that $f(\mathbf{U},\mathbf{U}) \not\equiv 0$ there exists at least one complex constant vector $\mathcal{C}$ such that $f(\mathcal{C},\mathcal{C}) \neq 0$.

There is at least one index $r \in \{1, 2, \ldots, 2n-1\}$ such that

$$\sum_{i=0}^{n-2} a_{2i+r} \neq 0 \quad \text{where } a_j = a_{j-2n+1} \ (j > 2n-1).$$

In fact, if such an index did not exist, then we would have the system of $2n-1$ linear homogeneous equations

$$\sum_{i=0}^{n-2} a_{2i+r} = 0 \quad (1 \leqslant r \leqslant 2n-1),$$

which has only the trivial solution

$$a_1 = a_2 = \ldots = a_{2n-1} = 0,$$

but this contradicts the assumption that at least one of these parameters is distinct from zero.

Now we assume that

$$\sum_{i=0}^{n-2} a_{2i+1} \neq 0. \tag{2.28}$$

We may assume this because the case when

$$\sum_{i=0}^{n-2} a_{2i+1} = 0 \quad \text{and} \quad \sum_{i=0}^{n-2} a_{2i+r} \neq 0 \quad (r \in \{2, 3, \ldots, 2n-1\})$$

can be reduced to the case (2.28) by a cyclic permutation of the variables $\mathbf{Z}_2, \mathbf{Z}_3, \ldots,$ $\mathbf{Z}_{2n}$ and a simple renumeration of the variables.

By putting

$$\mathbf{Z}_{2n-1} = \mathbf{U}, \quad \mathbf{Z}_{2n} = \mathbf{V}, \quad \mathbf{Z}_i = \mathcal{C} \quad (1 \leqslant i \leqslant 2n-2)$$

in (2.1), we obtain

$$\begin{aligned}\sum_{i=0}^{n-2} a_{2i+1} f^{n-1}(\mathcal{C},\mathcal{C}) f(\mathbf{U},\mathbf{V}) + \sum_{i=0}^{n-2} a_{2i+1} f^{n-2}(\mathcal{C},\mathcal{C}) f(\mathcal{C},\mathbf{U}) f(\mathbf{V},\mathcal{C}) \\ + a_{2n-1} f^{n-2}(\mathcal{C},\mathcal{C}) f(\mathcal{C},\mathbf{U}) f(\mathcal{C},\mathbf{V}) = 0.\end{aligned} \tag{2.29}$$

Since

$$a_{2n-1} = -\sum_{i=0}^{n-2} a_{2i+1} - \sum_{i=0}^{n-2} a_{2i+2},$$

according to the assumption (2.28), for $\mathbf{V} = \mathcal{C}$, the equation (2.29) implies $f(\mathcal{C}, \mathbf{U}) = f(\mathbf{U}, \mathcal{C})$.

By introducing the notation $f(\mathcal{C}, \mathbf{U}) = g(\mathbf{U})\sqrt{f(\mathcal{C}, \mathcal{C})}$, from (2.29) we conclude

$$f(\mathbf{U}, \mathbf{V}) = g(\mathbf{U})g(\mathbf{V}).$$

Since in the case $\sum_{i=1}^{2n-1} a_i = 0$ this function is really a solution of the equation (2.1), this means that Theorem 2.1 is completely proved. □

3. Nonlinear operator functional equation

In this section a nonlinear operator functional equation of kth order will be solved.

Definition 3.1. Let Ψ_{ij} be the operator which transposes (changes the places of) the ith and jth argument of the function F, i.e.,

$$\begin{aligned} \Psi_{ij} F(\mathbf{Z}_1, \ldots, \mathbf{Z}_{i-1}, \mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{j-1}, \mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_n) \\ = F(\mathbf{Z}_1, \ldots, \mathbf{Z}_{i-1}, \mathbf{Z}_j, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{j-1}, \mathbf{Z}_i, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_n). \end{aligned} \tag{3.1}$$

Theorem 3.1. *The general solution of the functional equation*

$$aF(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{kn}) \sum_{r=n+1}^{kn} \Psi_{nr} F(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{n-1}, \mathbf{Z}_n, \mathbf{Z}_{n+1}, \ldots, \mathbf{Z}_{kn}), \tag{3.2}$$

where a is a complex parameter and

$$F(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{kn}) = \prod_{i=0}^{k-1} f(\mathbf{Z}_{ni+1}, \mathbf{Z}_{ni+2}, \ldots, \mathbf{Z}_{ni+n}) \tag{3.3}$$

($f\colon \mathscr{V}^n \to \mathscr{V};\ n \geqslant 2$), *has components given by*

$$f(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n) = \begin{vmatrix} H_1(\mathbf{U}_1) & H_1(\mathbf{U}_2) & \ldots & H_1(\mathbf{U}_n) \\ H_2(\mathbf{U}_1) & H_2(\mathbf{U}_2) & \ldots & H_2(\mathbf{U}_n) \\ \vdots & & & \\ H_n(\mathbf{U}_1) & H_n(\mathbf{U}_2) & \ldots & H_n(\mathbf{U}_n) \end{vmatrix} \quad \textit{if } a = k-1, \tag{3.4}$$

$$(3.5)\quad f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n)=\begin{cases}\prod_{i=1}^{n}K(\mathbf{U}_i) & \textit{if}\quad a=n(k-1),\\ \textit{or } 0\end{cases}$$

$$(3.6)\quad f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n)\equiv 0\quad \textit{if}\quad a\neq r(k-1)\quad (1\leqslant r\leqslant n),$$

where H_i $(1\leqslant i\leqslant n)$, K *are arbitrary functions in* $\mathscr{V}$.

For the proof of this theorem in the case $a=k-1$ we need the following result.

Lemma 3.2. *If* $f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_n)$ *is a nonzero solution of the equation* (3.2) *for* $a=k-1$ *and* $\mathbf{A}_i$ $(1\leqslant i\leqslant n)$ *are constant complex vectors such that*

$$(3.7)\qquad f(\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_n)\neq 0,$$

then the equality

$$(3.8)\qquad f(\mathbf{U}_1,\dots,\mathbf{U}_{i-1},\mathbf{A}_n,\mathbf{U}_i,\dots,\mathbf{U}_{n-2},\mathbf{A}_n)\equiv 0$$

holds for any $1\leqslant i\leqslant n-1$.

Proof. We will suppose that this is not true, i.e., there exist vectors $\mathbf{U}_\nu\in\mathscr{V}$, $\nu=1,2,\dots,n-2$, such that

$$(3.9)\qquad f(\mathbf{U}_1,\dots,\mathbf{U}_{i-1},\mathbf{A}_n,\mathbf{U}_i,\dots,\mathbf{U}_{n-2},\mathbf{A}_n)\neq 0.$$

By putting

$$\mathbf{Z}_\nu=\mathbf{A}_\nu\quad(1\leqslant\nu\leqslant n),$$
$$\mathbf{Z}_{rn+\nu}=\begin{cases}\mathbf{U}_\nu & (1\leqslant\nu\leqslant i-1),\\ \mathbf{A}_n & (\nu=i\ \text{ or }\ \nu=n),\\ \mathbf{U}_{\nu-1} & (i+1\leqslant\nu\leqslant n-1)\end{cases}$$

for $1\leqslant r\leqslant k-1$, the equation (3.2) becomes

$$\begin{aligned}(3.10)\quad &(k-1)[f(\mathbf{U}_1,\dots,\mathbf{U}_{i-1},\mathbf{A}_n,\mathbf{U}_i,\dots,\mathbf{U}_{n-2},\mathbf{A}_n)]^{k-2}\\ &\times[f(\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_{n-1},\mathbf{A}_n)f(\mathbf{U}_1,\dots,\mathbf{U}_{i-1},\mathbf{A}_n,\mathbf{U}_i,\dots,\mathbf{U}_{n-2},\mathbf{A}_n)\\ &+f(\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_{n-1},\mathbf{U}_1)f(\mathbf{A}_n,\mathbf{U}_2,\dots,\mathbf{U}_{i-1},\mathbf{A}_n,\mathbf{U}_i,\dots,\mathbf{U}_{n-2},\mathbf{A}_n)\\ &+f(\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_{n-1},\mathbf{U}_2)f(\mathbf{U}_1,\mathbf{A}_n,\dots,\mathbf{U}_{i-1},\mathbf{A}_n,\mathbf{U}_i,\dots,\mathbf{U}_{n-2},\mathbf{A}_n)\\ &+f(\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_{n-1},\mathbf{U}_{n-2})f(\mathbf{U}_1,\dots,\mathbf{U}_{i-1},\mathbf{A}_n,\mathbf{U}_i,\dots,\\ &\qquad\mathbf{U}_{n-3},\mathbf{A}_n,\mathbf{A}_n)]=0.\end{aligned}$$

According to the hypothesis (3.9), it follows from (3.10) that

$$
\begin{aligned}
(3.11)\quad & f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{A}_n) f(\mathbf{U}_1, \dots, \mathbf{U}_{i-1}, \mathbf{A}_n, \mathbf{U}_i, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) \\
& + f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_1) f(\mathbf{A}_n, \mathbf{U}_2, \dots, \mathbf{U}_{i-1}, \mathbf{A}_n, \mathbf{U}_i, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) \\
& + f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_2) f(\mathbf{U}_1, \mathbf{A}_n, \dots, \mathbf{U}_{i-1}, \mathbf{A}_n, \mathbf{U}_i, \dots, \mathbf{U}_{n-2}, \mathbf{A}_n) \\
& + \dots + f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{U}_{n-2}) f(\mathbf{U}_1, \dots, \mathbf{U}_{i-1}, \mathbf{A}_n, \mathbf{U}_i, \dots, \\
& \qquad \mathbf{U}_{n-3}, \mathbf{A}_n, \mathbf{A}_n) = 0.
\end{aligned}
$$

Let $E_{n-2} = \{1, 2, 3, \dots, n-2\}$, and let S_r $(0 < r \leqslant n-2)$ be a subset of the set E_{n-2} which contains r elements. For $r = n-2$ we have $S_{n-2} = E_{n-2}$. Putting in (3.2) $\mathbf{Z}_\nu = \mathbf{A}_n$ $(1 \leqslant \nu \leqslant kn)$, we obtain

$$(3.12)\qquad f(\mathbf{A}_n, \mathbf{A}_n, \dots, \mathbf{A}_n) = 0.$$

Now, we suppose that

$$(3.13)\qquad f(\mathbf{V}_1, \dots, \mathbf{V}_{i-1}, \mathbf{A}_n, \mathbf{V}_i, \dots, \mathbf{V}_{n-2}, \mathbf{A}_n) = 0$$

holds for each of the $\binom{n-2}{r}$ sets S_r, where

$$(3.14)\qquad \mathbf{V}_\nu = \begin{cases} \mathbf{A}_n, & \nu \in S_r, \\ \mathbf{Y}_\nu, & \nu \in E_{n-2} \setminus S_r. \end{cases}$$

Under this assumption we will show that

$$(3.15)\qquad f(\mathbf{W}_1, \dots, \mathbf{W}_{i-1}, \mathbf{A}_n, \mathbf{W}_i, \dots, \mathbf{W}_{n-2}, \mathbf{A}_n) = 0$$

holds for each of the $\binom{n-2}{r-1}$ sets S_{r-1}, where

$$(3.16)\qquad \mathbf{W}_\nu = \begin{cases} \mathbf{A}_n, & \nu \in S_{r-1}, \\ \mathbf{Y}_\nu, & \nu \in E_{n-2} \setminus S_{r-1}. \end{cases}$$

Putting $\mathbf{U}_\nu = \mathbf{W}_\nu$ $(1 \leqslant \nu \leqslant n-2)$ in (3.11), on the basis of the hypothesis (3.13) we obtain

$$r f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{n-1}, \mathbf{A}_n) f(\mathbf{W}_1, \dots, \mathbf{W}_{i-1}, \mathbf{A}_n, \mathbf{W}_i, \dots, \mathbf{W}_{n-2}, \mathbf{A}_n) = 0.$$

From this relation $(r \geqslant 1)$ we conclude

$$f(\mathbf{W}_1, \dots, \mathbf{W}_{i-1}, \mathbf{A}_n, \mathbf{W}_i, \dots, \mathbf{W}_{n-2}, \mathbf{A}_n) = 0.$$

Consequently, by induction we have proved that

$$f(\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{A}_n, \mathbf{U}_i, \ldots, \mathbf{U}_{n-2}, \mathbf{A}_n) = 0$$

if exactly r $(0 \leqslant r \leqslant n-2)$ elements among $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{n-2}$ are equal to $\mathbf{A}_n$.

For $r = 0$ we obtain

$$f(\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{A}_n, \mathbf{U}_i, \ldots, \mathbf{U}_{n-2}, \mathbf{A}_n) \equiv 0,$$

which contradicts the hypothesis (3.9). This completes the proof of the lemma. □

Remark 3.1. In particular, for $i = 1$ the identity (3.8) takes the form

$$f(\mathbf{A}_n, \mathbf{U}_1, \ldots, \mathbf{U}_{n-2}, \mathbf{A}_n) \equiv 0.$$

We shall use this identity in the proof of Theorem 3.1. A slight generalization with the ith (instead of the first) argument of f equal to $\mathbf{A}_n$ was kindly suggested by the referee. However, the fact that exactly the nth argument of f is equal to $\mathbf{A}_n$ is intrinsically related to the equation (3.2). It cannot be generalized without generalizing the equation as well.

Proof of Theorem 3.1. We will prove Theorem 3.1 by induction.

For $n = 2$, the functional equation (3.2) takes the form

$$\begin{aligned}(k-1)&f(\mathbf{Z}_1, \mathbf{Z}_2)f(\mathbf{Z}_3, \mathbf{Z}_4)\ldots f(\mathbf{Z}_{2k-1}, \mathbf{Z}_{2k}) \\ &= f(\mathbf{Z}_1, \mathbf{Z}_3)f(\mathbf{Z}_2, \mathbf{Z}_4)\ldots f(\mathbf{Z}_{2k-1}, \mathbf{Z}_{2k}) \\ &\quad + f(\mathbf{Z}_1, \mathbf{Z}_4)f(\mathbf{Z}_3, \mathbf{Z}_2)\ldots f(\mathbf{Z}_{2k-1}, \mathbf{Z}_{2k}) + \ldots \\ &\quad + f(\mathbf{Z}_1, \mathbf{Z}_{2k-1})f(\mathbf{Z}_3, \mathbf{Z}_4)\ldots f(\mathbf{Z}_2, \mathbf{Z}_{2k}) \\ &\quad + f(\mathbf{Z}_1, \mathbf{Z}_{2k})f(\mathbf{Z}_3, \mathbf{Z}_4)\ldots f(\mathbf{Z}_{2k-1}, \mathbf{Z}_2).\end{aligned} \tag{3.17}$$

If we substitute $\mathbf{Z}_i = \mathbf{U}$ $(1 \leqslant i \leqslant 2k)$, the above equation becomes

$$f(\mathbf{U}, \mathbf{U}) \equiv \mathbf{O}. \tag{3.18}$$

For any nonzero component of a nontrivial solution of the equation (3.17) (denoted again by f) there exists at least one pair of constant complex vectors $(\mathbf{A}, \mathbf{B})$ such that $f(\mathbf{A}, \mathbf{B}) \neq 0$.

Putting in the functional equation (3.17)

$$\mathbf{Z}_{2i-1} = \mathbf{A} \quad (1 \leqslant i \leqslant k), \quad \mathbf{Z}_{2j} = \mathbf{B} \quad (2 \leqslant j \leqslant k), \quad \mathbf{Z}_2 = \mathbf{U},$$

it takes the form

$$f^{k-1}(\mathbf{A},\mathbf{B})[f(\mathbf{U},\mathbf{B})+f(\mathbf{B},\mathbf{U})]=0,$$

from which it follows that

$$f(\mathbf{U},\mathbf{B})=-f(\mathbf{B},\mathbf{U}). \tag{3.19}$$

For $\mathbf{Z}_{2i-1}=\mathbf{A}$, $\mathbf{Z}_{2i}=\mathbf{B}$ $(1\leqslant i\leqslant k-1)$, $\mathbf{Z}_{2k-1}=\mathbf{U}$, $\mathbf{Z}_{2k}=\mathbf{V}$, the functional equation (3.17) by virtue of (3.19) yields

$$f^{k-1}(\mathbf{A},\mathbf{B})f(\mathbf{U},\mathbf{V})=f^{k-2}(\mathbf{A},\mathbf{B})[f(\mathbf{A},\mathbf{U})f(\mathbf{B},\mathbf{V})-f(\mathbf{A},\mathbf{V})f(\mathbf{B},\mathbf{U})].$$

If we introduce the notation

$$\frac{f(\mathbf{A},\mathbf{U})}{f(\mathbf{A},\mathbf{B})}=H_1(\mathbf{U}),\quad f(\mathbf{B},\mathbf{U})=H_2(\mathbf{U}),$$

we obtain that the function

$$f(\mathbf{U},\mathbf{V})=\begin{vmatrix}H_1(\mathbf{U}) & H_1(\mathbf{V})\\ H_2(\mathbf{U}) & H_2(\mathbf{V})\end{vmatrix}$$

is the general solution of the functional equation (3.2) for $n=2$ because it includes the trivial solution $f(\mathbf{X},\mathbf{Y})=0$.

Now, we suppose that Theorem 3.1 holds for $n-1$, i.e., the general solution of the functional equation

$$\begin{aligned}(k-1)&F(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{k(n-1)})\\ &=\sum_{r=n}^{k(n-1)}\Psi_{n-1,r}F(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{n-1},\mathbf{Z}_n,\dots,\mathbf{Z}_{k(n-1)}),\end{aligned} \tag{3.20}$$

where

$$F(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{k(n-1)})=\prod_{i=0}^{k-1}f(\mathbf{Z}_{(n-1)i+1},\mathbf{Z}_{(n-1)i+2},\dots,\mathbf{Z}_{(n-1)i+n-1}), \tag{3.21}$$

is given by

$$f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_{n-1})=\begin{vmatrix}H_1(\mathbf{U}_1) & H_1(\mathbf{U}_2) & \dots & H_1(\mathbf{U}_{n-1})\\ H_2(\mathbf{U}_1) & H_2(\mathbf{U}_2) & \dots & H_2(\mathbf{U}_{n-1})\\ \vdots & & & \\ H_{n-1}(\mathbf{U}_1) & H_{n-1}(\mathbf{U}_2) & \dots & H_{n-1}(\mathbf{U}_{n-1})\end{vmatrix}. \tag{3.22}$$

Let $f(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \neq 0$ (here, as usual, f denotes a nonzero component of a nontrivial solution). If we put

$$\mathbf{Z}_{ni+1} = \mathbf{A}_n \quad (0 \leqslant i \leqslant k-1),$$
$$f(\mathbf{A}_n, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_n) = g(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_n) \quad (f\colon \mathscr{V}^n \to \mathbb{C}, \quad g\colon \mathscr{V}^{n-1} \to \mathbb{C}),$$

we obtain from (3.2) according to Lemma 3.2

$$\begin{aligned}(3.23)\quad &(k-1)g(\mathbf{Z}_2, \ldots, \mathbf{Z}_{n-1}, \mathbf{Z}_n)g(\mathbf{Z}_{n+2}, \mathbf{Z}_{n+3}, \ldots, \mathbf{Z}_{2n}) \ldots g(\mathbf{Z}_{(k-1)n+2}, \ldots, \mathbf{Z}_{kn})\\ &= g(\mathbf{Z}_2, \ldots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+2})g(\mathbf{Z}_n, \mathbf{Z}_{n+3}, \ldots, \mathbf{Z}_{2n}) \ldots g(\mathbf{Z}_{(k-1)n+2}, \ldots, \mathbf{Z}_{kn})\\ &\quad + g(\mathbf{Z}_2, \ldots, \mathbf{Z}_{n-1}, \mathbf{Z}_{n+3})g(\mathbf{Z}_{n+2}, \mathbf{Z}_n, \mathbf{Z}_{n+4}, \ldots, \mathbf{Z}_{2n}) \ldots g(\mathbf{Z}_{(k-1)n+2}, \ldots, \mathbf{Z}_{kn})\\ &\quad + \ldots\\ &\quad + g(\mathbf{Z}_2, \ldots, \mathbf{Z}_{n-1}, \mathbf{Z}_{kn})g(\mathbf{Z}_{n+2}, \ldots, \mathbf{Z}_{2n}) \ldots g(\mathbf{Z}_{(k-1)n+2}, \ldots, \mathbf{Z}_{kn-1}, \mathbf{Z}_n).\end{aligned}$$

According to the induction hypothesis, we obtain that the general solution of the equation (3.23) is given by

$$(3.24)\quad g(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{n-1}) = f(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{n-1}) = \begin{vmatrix} H_1(\mathbf{U}_1) & H_1(\mathbf{U}_2) & \ldots & H_1(\mathbf{U}_{n-1})\\ H_2(\mathbf{U}_1) & H_2(\mathbf{U}_2) & \ldots & H_2(\mathbf{U}_{n-1})\\ \vdots & & & \\ H_{n-1}(\mathbf{U}_1) & H_{n-1}(\mathbf{U}_2) & \ldots & H_{n-1}(\mathbf{U}_{n-1}) \end{vmatrix},$$

where $H_i\colon \mathscr{V} \to \mathbb{C}$ $(1 \leqslant i \leqslant n-1)$ are arbitrary functions.

If we put in the equation (3.2)

$$\mathbf{Z}_i = \mathbf{A}_i \quad (1 \leqslant i \leqslant n-1), \quad \mathbf{Z}_n = \mathbf{U}_1,$$
$$\mathbf{Z}_{nj+m} = \mathbf{A}_m \quad (1 \leqslant j \leqslant k-2;\ 1 \leqslant m \leqslant n),$$
$$\mathbf{Z}_{(k-1)n+1} = \mathbf{A}_n, \quad \mathbf{Z}_{(k-1)n+r} = \mathbf{U}_r \quad (2 \leqslant r \leqslant n),$$

we obtain

$$\begin{aligned}(3.25)\quad &f(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1}, \mathbf{A}_n)f(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n)\\ &= f(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}, \mathbf{U}_1)f(\mathbf{A}_n, \mathbf{U}_2, \ldots, \mathbf{U}_n)\\ &\quad - f(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}, \mathbf{U}_2)f(\mathbf{A}_n, \mathbf{U}_1, \mathbf{U}_3, \ldots, \mathbf{U}_n) - \ldots\\ &\quad - f(\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}, \mathbf{U}_n)f(\mathbf{A}_n, \mathbf{U}_2, \ldots, \mathbf{U}_{n-1}, \mathbf{U}_1).\end{aligned}$$

On the basis of the equality (3.24) we obtain

$$\begin{aligned}(3.26)\quad &f(\mathbf{A}_n, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_i, \mathbf{U}_{i+1}, \ldots, \mathbf{U}_{j-1}, \mathbf{U}_j, \mathbf{U}_{j+1}, \ldots, \mathbf{U}_{n-1})\\ &= -f(\mathbf{A}_n, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_j, \mathbf{U}_{i+1}, \ldots, \mathbf{U}_{j-1}, \mathbf{U}_i, \mathbf{U}_{j+1}, \ldots, \mathbf{U}_{n-1})\\ &\qquad (1 \leqslant i < j \leqslant n-1).\end{aligned}$$

By using the equalities (3.24), (3.25) and (3.26) along with the notation

$$\frac{f(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1}, \mathbf{U})}{f(\mathbf{A}_1, \ldots, \mathbf{A}_n)} = H_n(\mathbf{U}),$$

we obtain that $f(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n)$ has the form (3.4).

It remains still to show that every function of the form (3.4) is really a solution of the equation (3.2). For this purpose, we consider the determinant

$$D(j) = \begin{vmatrix} H_1(\mathbf{Z}_1) & H_2(\mathbf{Z}_1) & \ldots & H_n(\mathbf{Z}_1) & 0 & 0 & \ldots & 0 \\ H_1(\mathbf{Z}_2) & H_2(\mathbf{Z}_2) & \ldots & H_n(\mathbf{Z}_2) & 0 & 0 & \ldots & 0 \\ \vdots & & & & & & & \\ H_1(\mathbf{Z}_{n-1}) & H_2(\mathbf{Z}_{n-1}) & \ldots & H_n(\mathbf{Z}_{n-1}) & 0 & 0 & \ldots & 0 \\ H_1(\mathbf{Z}_n) & H_2(\mathbf{Z}_n) & \ldots & H_n(\mathbf{Z}_n) & H_1(\mathbf{Z}_n) & H_2(\mathbf{Z}_n) & \ldots & H_n(\mathbf{Z}_n) \\ H_1(\mathbf{Z}_{nj+1}) & H_2(\mathbf{Z}_{nj+1}) & \ldots & H_n(\mathbf{Z}_{nj+1}) & H_1(\mathbf{Z}_{nj+1}) & H_2(\mathbf{Z}_{nj+1}) & \ldots & H_n(\mathbf{Z}_{nj+1}) \\ \vdots & & & & & & & \\ H_1(\mathbf{Z}_{nj+n}) & H_2(\mathbf{Z}_{nj+n}) & \ldots & H_n(\mathbf{Z}_{nj+n}) & H_1(\mathbf{Z}_{nj+n}) & H_2(\mathbf{Z}_{nj+n}) & \ldots & H_n(\mathbf{Z}_{nj+n}) \end{vmatrix}$$

for $1 \leqslant j \leqslant k-1$. Applying the Laplace Theorem to the first $n-1$ rows of $D(j)$, we see that each of the corresponding minors has two identical columns, hence it vanishes. Consequently,

$$D(j) = 0. \tag{3.27}$$

According to (3.27), we conclude that the following identity holds:

$$\sum_{j=1}^{k-1} D(j) \prod_{\substack{i=1 \\ i \neq j}}^{k-1} \begin{vmatrix} H_1(\mathbf{Z}_{ni+1}) & H_1(\mathbf{Z}_{ni+2}) & \ldots & H_1(\mathbf{Z}_{ni+n}) \\ H_2(\mathbf{Z}_{ni+1}) & H_2(\mathbf{Z}_{ni+2}) & \ldots & H_2(\mathbf{Z}_{ni+n}) \\ \vdots & & & \\ H_n(\mathbf{Z}_{ni+1}) & H_n(\mathbf{Z}_{ni+2}) & \ldots & H_n(\mathbf{Z}_{ni+n}) \end{vmatrix} = 0. \tag{3.28}$$

Applying further the Laplace Theorem to the first n rows of $D(j)$, simple manipulations show that the function (3.4) is really a solution of the equation (3.2). This completes the proof of Theorem 3.1 for $a = k-1$.

Now we pass to the proof of Theorem 3.2 for $a = n(k-1)$.

First, we suppose that $f(\mathbf{U}, \mathbf{U}, \ldots, \mathbf{U}) \not\equiv \mathbf{O}$. Henceforth we denote by f any component of the function $f\colon \mathscr{V}^n \to \mathscr{V}$ for which $f(\mathbf{U}, \mathbf{U}, \ldots, \mathbf{U}) \not\equiv 0$. For such a component there exists at least one constant complex vector $\mathcal{C}$ for which $f(\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}) \neq 0$.

If we put $\mathbf{Z}_{n+i} = \mathbf{U}$ and substitute the rest of the variables by $\mathcal{C}$, from (3.2) we obtain

$$f(\mathcal{C}, \ldots, \mathcal{C}, \mathbf{U}, \mathcal{C}, \ldots, \mathcal{C}) = f(\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}, \mathbf{U}). \tag{3.29}$$

By putting $\mathbf{Z}_{n+i} = \mathbf{U}$, $\mathbf{Z}_{n+j} = \mathbf{V}$ $(j > i)$ and substituting the rest of the variables by $\mathcal{C}$, from the equation (3.2) by virtue of (3.29) we obtain

$$(3.30) \qquad f(\mathcal{C},\dots,\mathcal{C},\mathbf{U},\dots,\mathbf{V},\dots,\mathcal{C}) = \frac{f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U})f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{V})}{f(\mathcal{C},\mathcal{C},\dots,\mathcal{C})}.$$

We assume that

$$(3.31) \quad f(\mathcal{C},\dots,\mathbf{U}_1,\dots,\mathcal{C},\mathbf{U}_2,\dots,\mathcal{C},\dots,\mathbf{U}_k,\mathcal{C},\dots,\mathcal{C}) = \frac{\prod_{i=1}^{k} f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_i)}{f^{k-1}(\mathcal{C},\mathcal{C},\dots,\mathcal{C})}.$$

If we put $\mathbf{Z}_{n+i_1} = \mathbf{U}_1$, $\mathbf{Z}_{n+i_2} = \mathbf{U}_2,\dots,\mathbf{Z}_{n+i_k} = \mathbf{U}_k$, $\mathbf{Z}_{n+i_{k+1}} = \mathbf{U}_{k+1}$ and substitute the rest of the variables by $\mathcal{C}$, then the equation (3.2) becomes

$$\begin{aligned}(3.32) \quad & kf(\mathcal{C},\mathcal{C},\dots,\mathcal{C})f(\mathcal{C},\dots,\mathbf{U}_1,\dots,\mathcal{C},\mathbf{U}_2,\dots,\mathbf{U}_k,\mathcal{C},\dots,\mathbf{U}_{k+1},\dots,\mathcal{C}) \\ &= f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_1)f(\mathcal{C},\dots,\mathcal{C},\dots,\mathbf{U}_2,\dots,\mathbf{U}_k,\dots,\mathbf{U}_{k+1},\dots,\mathcal{C}) \\ &\quad + f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_2)f(\mathcal{C},\dots,\mathbf{U}_1,\dots,\mathbf{U}_k,\dots,\mathbf{U}_{k+1},\dots,\mathcal{C}) + \dots \\ &\quad + f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_{k+1})f(\mathcal{C},\dots,\mathbf{U}_1,\dots,\mathbf{U}_2,\dots,\mathbf{U}_k,\dots,\mathcal{C},\dots,\mathcal{C}).\end{aligned}$$

On the basis of the induction hypothesis (3.31), it follows from (3.32) that

$$(3.33) \quad f(\mathcal{C},\dots,\mathbf{U}_1,\dots,\mathbf{U}_2,\dots,\mathbf{U}_k,\dots,\mathbf{U}_{k+1},\dots,\mathcal{C}) = \frac{\prod_{i=1}^{k+1} f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_i)}{f^{k}(\mathcal{C},\mathcal{C},\dots,\mathcal{C})}.$$

Therefore, we have proved by mathematical induction that the formula (3.33) holds for every $k < n$.

By putting $\mathbf{Z}_i = \mathcal{C}$ $(1 \leqslant i \leqslant n)$, $\mathbf{Z}_{n+j} = \mathbf{U}_j$ $(1 \leqslant j \leqslant n)$, $\mathbf{Z}_{2n+s} = \mathcal{C}$ $(1 \leqslant s \leqslant n(k-2))$, from the equation (3.2) we obtain

$$\begin{aligned}(3.34) \qquad & nf(\mathcal{C},\mathcal{C},\dots,\mathcal{C})f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n) \\ &= f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_1)f(\mathcal{C},\mathbf{U}_2,\mathbf{U}_3,\dots,\mathbf{U}_n) \\ &\quad + f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_2)f(\mathbf{U}_1,\mathcal{C},\mathbf{U}_3,\mathbf{U}_4,\dots,\mathbf{U}_n) + \dots \\ &\quad + f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_n)f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_{n-1},\mathcal{C}).\end{aligned}$$

On the basis of the equality (3.33), the last equality (3.34) becomes

$$(3.35) \qquad f(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n) = \frac{\prod_{i=1}^{n} f(\mathcal{C},\mathcal{C},\dots,\mathcal{C},\mathbf{U}_i)}{f^{n-1}(\mathcal{C},\mathcal{C},\dots,\mathcal{C})}.$$

By introducing the notation $K(\mathbf{U}) = f(\mathcal{C},\dots,\mathcal{C},\mathbf{U})/\sqrt[n]{f^{n-1}(\mathcal{C},\mathcal{C},\dots,\mathcal{C})}$, we obtain that the function f in the case $a = n(k-1)$ really has the form (3.5).

Now, we suppose that $f(\mathbf{U},\mathbf{U},\dots,\mathbf{U}) \equiv 0$. Next, we will need the following result.

Lemma 3.3. *If $f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n)$ is a nonzero solution of the equation (3.2) for $a = n(k-1)$ such that $f(\mathbf{U}, \mathbf{U}, \dots, \mathbf{U}) \equiv 0$, and at least one of the variables $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-1}$ is equal to $\mathbf{U}_n$, then*

$$f(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) \equiv 0.$$

Proof. Let $E_{n-1} = \{1, 2, 3, \dots, n-1\}$, and let S_m $(1 \leqslant m \leqslant n-1)$ be a subset of the set E_{n-1} which contains m elements. We have $f(\mathbf{U}_n, \mathbf{U}_n, \dots, \mathbf{U}_n) \equiv 0$.

Now, we suppose that

$$f(\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{n-1}, \mathbf{U}_n) = 0 \tag{3.36}$$

holds for each of the $\binom{n-1}{m}$ sets S_m, where

$$\mathbf{V}_i = \begin{cases} \mathbf{U}_n, & i \in S_m, \\ \mathbf{Y}_i, & i \in E_{n-1} \setminus S_m. \end{cases}$$

We will prove that

$$f(\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_{n-1}, \mathbf{U}_n) = 0 \tag{3.37}$$

holds for each of the $\binom{n-2}{m-1}$ sets S_{m-1}, where

$$\mathbf{W}_i = \begin{cases} \mathbf{U}_n, & i \in S_{m-1}, \\ \mathbf{Y}_i, & i \in E_{n-1} \setminus S_{m-1}. \end{cases}$$

By substituting $\mathbf{Z}_{nm+i} = \mathbf{Z}_i$ $(0 \leqslant m \leqslant k-1)$ in the equation (3.2) and putting $\mathbf{Z}_i = \mathbf{W}_i$, on the basis of the assumption (3.36) we obtain

$$(k-1)(n-m)f^k(\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_{n-1}, \mathbf{U}_n) = 0,$$

from which we deduce (3.37) because $k \geqslant 2$ and $m < n$.

Therefore, Lemma 3.3 is proved by induction. □

By putting $\mathbf{Z}_{nm+i} = \mathbf{U}_i$ $(0 \leqslant m \leqslant k-1)$ and according to Lemma 3.3, from the equation (3.2) we obtain

$$(k-1)(n-1)f^k(\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_n) = 0.$$

Since $k > 1$ and $n > 1$, we conclude that the function f has the form (3.5).

We may easily check that the functions (3.5) satisfy the functional equation (3.2). We can do this by a direct substitution of (3.5) into (3.2). This completes the proof of Theorem 3.1 for the case $a = n(k-1)$.

Now, we will pass to the proof of Theorem 3.1 for the case $a \neq r(k-1)$ $(1 \leqslant r \leqslant n)$.

In this case, Lemma 3.3 also holds.

By putting $\mathbf{Z}_{nm+i} = \mathbf{Z}_i$ $(0 \leqslant m \leqslant k-1)$ and using Lemma 3.3, from the equation (3.2) we obtain

$$[a-(k-1)]f^k(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) = \mathbf{O}.$$

Since $a \neq k-1$, from the above equality we immediately deduce the statement of Theorem 3.1 for the case $a \neq r(k-1)$ $(1 \leqslant r \leqslant n)$.

Therefore, Theorem 3.1 has been completely proved. □

We have not been able to solve equation (3.2) for $a = r(k-1)$ $(2 \leqslant r \leqslant n-1)$.

Remark 3.2. The function

$$f(\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n) = \begin{vmatrix} H_1(\mathbf{U}_1) & H_1(\mathbf{U}_2) & \ldots & H_1(\mathbf{U}_s) \\ H_2(\mathbf{U}_1) & H_2(\mathbf{U}_2) & \ldots & H_2(\mathbf{U}_s) \\ \vdots & & & \\ H_s(\mathbf{U}_1) & H_s(\mathbf{U}_2) & \ldots & H_s(\mathbf{U}_s) \end{vmatrix} \prod_{i=s+1}^{n} H_1(\mathbf{U}_i), \tag{3.38}$$

where $s = n-r+1$, and H_i $(1 \leqslant i \leqslant n-r+1)$ are arbitrary functions, is a solution of the equation (3.2) for $a = r(k-1)$ but the question of generality of this solution remains open.

Also, for $r = n$ the function (3.38) becomes (3.5). If we assume that

$$\prod_{i=n+1}^{n} H_1(\mathbf{Z}_i) = \mathbf{I},$$

then the function (3.38) for $r = 1$ becomes (3.4).

All this suggests to put forth the following hypothesis.

Hypothesis 3.4. *The general solution of the functional equation* (3.2) *in the case $a = r(k-1)$ $(1 \leqslant r \leqslant n)$ is given by the formula* (3.38).

This paper makes an entity with the previous papers [2], [3], [4].

Acknowledgement. I wish to express sincere thanks to Prof. V. Covachev from Institute of Mathematics & Informatics, Bulgarian Academy of Sciences (at present at the College of Science, Sultan Qaboos University, Oman). His suggestions and remarks have been of much assistance in the preparation of this paper. I am also grateful to the referee whose comments have much helped to improve the quality of the paper.

References

[1] *J. Aczel*: Lectures on Functional Equations and Their Applications. Academic Press, New York-London, 1966.

[2] *I. B. Risteski, K. G. Trenčevski and V. C. Covachev*: A simple functional operator. New York J. Math. *5* (1999), 139–142.

[3] *I. B. Risteski, K. G. Trenčevski and V. C. Covachev*: Cyclic complex vector functional equations. Appl. Sci. *2* (2000), 13–18.

[4] *I. B. Risteski, K. G. Trenčevski and V. C. Covachev*: On a linear vector functional equation. In: Some Problems of Applied Mathematics (A. Ashyralyev, H. A. Yurtsever, eds.). Fatih University Publications, Istanbul, 2000, pp. 174–184.

Author's address: 2 Milepost Place # 606, Toronto, M4H 1C7, Ontario, Canada, e-mail: iceristeski@hotmail.com.