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ON DOMINATION NUMBER OF 4-REGULAR GRAPHS

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Abstract. Let G be a simple graph. A subset $S \subseteq V$ is a dominating set of G, if for any vertex $v \in V - S$ there exists a vertex $u \in S$ such that $uv \in E(G)$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. In this paper we prove that if G is a 4-regular graph with order n, then $\gamma(G) \leqslant \frac{4}{11}n$.

Keywords: regular graph, dominating set, domination number

MSC 2000: 05C69

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph. For a vertex $v \in V(G)$, denote by $N(v)$ the open neighborhood of v. Let $N[v] = N(v) \cup \{v\}$. Denote by $\delta(G)$ the minimum degree of G. For a subset S of $V(G)$, denote by $G[S]$ the subgraph induced by S. A subset $S \subseteq V$ is a dominating set of G, if for any vertex $u \in V - S$ there exists a vertex $v \in S$ such that $uv \in E(G)$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set S of G is a γ-set if $|S| = \gamma(G)$. Some bounds on $\gamma(G)$ with minimum degree conditions have been obtained as follows.

Theorem 1 [3]. *If a graph G has no isolated vertices, then $\gamma(G) \leqslant n/2$.*

McGuaig and Shepherd made another improvement on the upper bound. Let $\mathscr{A}$ be the collection of graphs in Figure 1.

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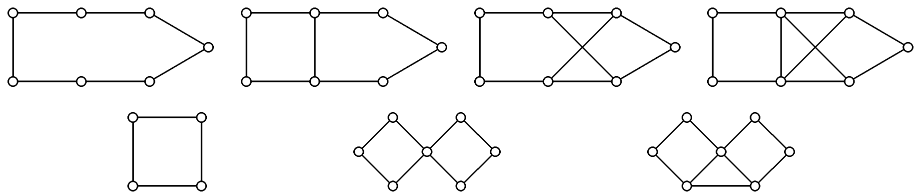

Figure 1. Graphs in family $\mathscr{A}$

Theorem 2 [2]. *If G is a connected graph with $\delta(G) \geqslant 2$ and $G \notin \mathscr{A}$, then $\gamma(G) \leqslant 2n/5$.*

Reed again improved the bound by increasing the minimum degree requirement.

Theorem 3 [4]. *If G is a connected graph with $\delta(G) \geqslant 3$, then $\gamma(G) \leqslant 3n/8$.*

Motivated by the above conclusions, Haynes et al. [1] conjectured that

Conjecture 1 [1]. *For any graph G with $\delta(G) \geqslant k$, $\gamma(G) \leqslant k(3k-1)^{-1}n$.*

The question still remains open for graphs G having $4 \leqslant \delta(G) \leqslant 6$. In the next section, we will prove that $\gamma(G) \leqslant 4n/11$ for any 4-regular graph G with order n.

2. Main results

First, we give some definitions and symbols needed for the proof of Theorem 4.

Let S be a γ-set of G, let $N_i(S) = \{u \in V - S \colon |N(u) \cap S| = i\}$ where $1 \leqslant i \leqslant 4$. For any vertex $v \in S$, let $N_i(v, S) = N(v) \cap N_i(S)$. Denote by $\lambda(S)$ the number of isolates in $G[S]$. Let $\mu(S) = |N_1(S)|$ and $\eta(S) = |N_2(S)|$. Let $J_0 = \{v \in S \colon |N_1(v,S)| = 0\}$, $J_1 = \{v \in S \colon |N_1(v,S)| = 1\}$ and $J_2 = \{v \in S \colon |N_1(v,S)| \geqslant 2\}$. Let $B = \{v \in J_0 \colon N(v) \cap N_3(S) \neq \emptyset\}$ and $R = \{u \in N_3(S) \colon N(u) \cap B \neq \emptyset\}$. For any vertex $v \in J_1$ there exists only one vertex $u \in V - S$ such that $u \in N_1(v, S)$; we write $P(v)$ for u.

For any two vertex subsets $C, D \subseteq V$, we denote the set of edges between C and D by $E[C, D]$.

Theorem 4. *If G is a 4-regular graph with order n, then $\gamma(G) \leqslant \frac{4}{11}n$.*

P r o o f. Among all γ-sets of G, let S be chosen so that

(1) $\lambda(S)$ is maximized;

(2) subject to (1), $\mu(S)$ is minimized;

(3) subject to (2), $\eta(S)$ is minimized.

Before proceeding further, we prove the following claims.

Claim 1. *Each vertex $v \in J_0 \cup J_1$ is an isolate in $G[S]$.*

P r o o f. Suppose to the contrary that v is not isolated in $G[S]$. If $v \in J_0$, then $S' = S - \{v\}$ is a domination set of G. This contradicts the fact that S is a γ-set of G. If $v \in J_1$, then $S' = (S - \{v\}) \cup \{P(v)\}$ is a γ-set of G with $\lambda(S') > \lambda(S)$. This contradicts our choice of S. □

Claim 2. *For any vertex $v \in J_1$, if $|N_2(v,S)| = 0$ then $|N(P(v)) \cap N_1(S)| = 0$.*

P r o o f. Suppose to the contrary that $|N(P(v)) \cap N_1(S)| > 0$, then $S' = (S - \{v\}) \cup \{P(v)\}$ is also a γ-set of G with $\lambda(S') = \lambda(S)$, $\mu(S') < \mu(S)$, a contradiction. □

Claim 3. *For any $u \in V - S$, if $v_1, v_2 \in N(u) \cap J_0$ then $|N_2(v_1,S) \cap N_2(v_2,S)| \geqslant 2$.*

P r o o f. Suppose to the contrary that $|N_2(v_1,S) \cap N_2(v_2,S)| < 2$. Then if $|N_2(v_1,S) \cap N_2(v_2,S)| = 0$, then $S' = (S - \{v_1,v_2\}) \cup \{u\}$ is a dominating set of G with $|S'| < |S|$, a contradiction. If $|N_2(v_1,S) \cap N_2(v_2,S)| = 1$ then $S' = (S - \{v_1,v_2\}) \cup (N_2(v_1,S) \cap N_2(v_2,S))$ is a dominating set of G with $|S'| < |S|$, a contradiction. □

Claim 4. *Assume that $v \in J_1$ and $|N_2(v,S)| = 0$. For $1 \leqslant t \leqslant 3$, if $|N_4(v,S)| = t$, then $|N(P(v)) \cap N_3(S)| \geqslant t$.*

P r o o f. Suppose to the contrary that $|N(P(v)) \cap N_3(S)| < t$. Then we have $|N(P(v)) \cap N_2(S)| = 3 - |N(P(v)) \cap N_3(S)| > 3 - t$. Thus $S' = (S - \{v\}) \cup \{P(v)\}$ is a γ-set of G with $\lambda(S') = \lambda(S), \mu(S') = \mu(S)$ and $\eta(S') < \eta(S)$. This contradicts the choice of S. □

Now we define a function $f\colon E[V - S, V] \to \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ as follows.

For any $v \in S$, define

$$f(uv) = \begin{cases} 1, & u \in N_1(v,S), \\ \frac{1}{2}, & u \in N_2(v,S), \\ \frac{1}{4}, & u \in N_3(v,S) \cup N_4(v,S), \\ 0, & \text{otherwise.} \end{cases}$$

For any $u \in N_3(S)$, define

$$f(uw) = \begin{cases} \frac{1}{2}, & w \in N_3(S), \\ \frac{1}{4}, & w \in V - N_3(S). \end{cases}$$

For any $u \in V - S - N_3(S)$, define

$$f(uw) = \begin{cases} \frac{1}{4}, & w \in N_3(S), \\ 0, & w \in V - S - N_3(S). \end{cases}$$

In order to prove the theorem, note that

$$n - |S| = |V - S| = \sum_{uv \in E[V-S,V]} f(uv),$$

so we need only to prove that

$$\sum_{uv \in E[V-S,V]} f(uv) \geqslant \frac{7}{4}|S|.$$

If we can find a function $g\colon\ E[V - S, S]$ satisfying the conditions

$$\sum_{uv \in E[V-S,V]} f(uv) \geqslant \sum_{uv \in E[V-S,S]} g(uv), \tag{1}$$

$$\sum_{uv \in E[V-S,S]} g(uv) \geqslant \frac{7}{4}|S|, \tag{2}$$

the conclusion will follow immediately.

For convenience, for any $v \in S$ we define $h(v) = \sum\limits_{u \in N(v) \cap (V-S)} g(uv)$.

Note that

$$\sum_{uv \in E[V-S,S]} g(uv) = \sum_{v \in S} \Bigg(\sum_{u \in N(v) \cap (V-S)} g(uv) \Bigg) = \sum_{v \in S} h(v).$$

If the following condition holds, then condition (2) holds as well:

$$\text{For any vertex} \quad v \in S, \quad h(v) \geqslant \frac{7}{4}.$$

In the following, we will define a function $g\colon\ E[V - S, S] \to \{0, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}\}$ satisfying conditions (1) and (3).

For any vertex $v \in J_2$ and $uv \in E[V - S, S]$, define

$$g(uv) = \begin{cases} 1, & u \in N_1(v, S), \\ 0, & \text{otherwise}. \end{cases}$$

Assuming that $w_1, w_2 \in N_1(v, S)$, we have $h(v) \geqslant g(w_1 v) + g(w_2 v) = 2$.

For any vertex $v \in J_1$ and $uv \in E[V-S, S]$, define

$$g(uv) = \begin{cases} 1, & u \in N_1(v, S), \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Thus we have

$$h(v) = g(P(v)v) + \sum_{u \in N(v) - \{P(v)\}} g(uv) = 1 + \frac{3}{4} = \frac{7}{4}.$$

For any $v \in J_0$ and $uv \in E[V-S, S]$, if $u \in N_2(v, S)$, define

$$g(uv) = f(uv) = \frac{1}{2}.$$

Before proceeding further, we introduce the following notation:
Let $K = \{v \in J_0 \colon N(v) \cap N(J_0 - \{v\}) = \emptyset\}$ and $L = J_0 - K$.
Denote

$$\begin{aligned} M_1 = \{y \in N(K) \cap N_4(S) | N(y) - K \subseteq J_1 \\ \text{and for any vertex } x \in N(y) - K,\ N_2(x, S) = \emptyset\}, \\ Q_1 = \{y \in N(K) \cap N_4(S) | N(y) - K \subseteq J_1 \\ \text{and there exist two vertices } x_1, x_2 \in N(y) - K \end{aligned}$$

such that $N_2(x_1, S) = \emptyset$ and $N_2(x_2, S) = \emptyset$ and an other vertex $x_3 \in N(y) - K$ such that $N_2(x_3, S) \neq \emptyset\}$.

Now, we consider the following two cases.

Case 1. $v \in K$.

Case 1.1. $|N(v) \cap Q_1| \leqslant 1$.

Case 1.1.1. $N(v) \cap M_1 = \emptyset$.

For any $u \in N(v) - Q_1 - N_2(S)$, if $u \in N_3(S)$ then there exists a vertex $x \in V - S$ such that $ux \in E(G)$. Define $g(uv) = f(uv) + f(ux) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. If $u \in N_4(S)$, then $(N(u) - \{v\}) \subseteq J_1 \cup J_2$. If $(N(u) - \{v\}) \cap J_2 \neq \emptyset$, then there exists a vertex $x \in N(u) \cap J_2$. Define $g(uv) = f(uv) + f(ux) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Otherwise, there exist two vertices $x_1, x_2 \in N(u) - \{v\}$ such that $N_2(x_1, S) \neq \emptyset$ and $N_2(x_2, S) \neq \emptyset$. Assume that $w_1 \in N_2(x_1, S)$ and $w_2 \in N_2(x_2, S)$ and define

$$g(uv) = f(uv) + \frac{1}{2}\Big(f(w_1x_1) - \frac{1}{4}\Big) + \frac{1}{2}\Big(f(w_2x_2) - \frac{1}{4}\Big) = \frac{1}{2}.$$

If $N(v)\cap Q_1\neq\emptyset$, for any vertex $u\in N(v)\cap Q_1$ define $g(uv)=f(uv)=\frac{1}{4}$. Thus

$$h(v)=\sum_{u\in N(v)\cap(V-S)} g(uv)\geqslant \sum_{u\in N(v)\cap(V-S-Q_1)} g(uv)+\sum_{u\in N(v)\cap Q_1} g(uv)\geqslant\frac{7}{4}.$$

Case 1.1.2. $N(v)\cap M_1\neq\emptyset$.

There exists a vertex $u\in N(v)\cap M_1$. For any vertex $x\in N(u)-\{v\}$, by Claim 4, we can select a vertex $z\in N(P(x))\cap N_3(S)$. We claim that $z\in N_3(S)-(R-N(v))$. Suppose to the contrary that $z\in R-N(v)$, then there exists a vertex $b\in B$ such that $bz\in E(G)$ and $b\neq v$. Since $N(v)\cap N(b)=\emptyset$, we let $S'=(S-\{b,v,x\})\cup\{z,u\}$. Then S' is a dominating set of G with cardinality less than S, a contradiction. Assume that $N(u)-\{v\}=\{x_1,x_2,x_3\}$. Then for $1\leqslant i\leqslant 3$ there exist $z_i\in N(P(x_i))\cap(N_3(S)-(R-N(v)))$. Assume $N(v)=\{u_1,u_2,u_3,u_4\}$. For $i=1,2,3$, define

$$g(u_iv)=f(u_iv)+f(z_iP(x_i))\geqslant\frac{1}{2}.$$

Moreover, define $g(u_4v)=f(u_4v)\geqslant\frac{1}{4}$. Thus we have

$$h(v)=\sum_{i=1}^{4} g(u_iv)\geqslant\frac{7}{4}.$$

Case 1.2. $|N(v)\cap Q_1|\geqslant 2$.

Then there exist two vertices $u,u'\in N(v)\cap Q_1$. Further, there exist $x_1,x_2\in N(u)\cap J_1$ and $x'_1,x'_2\in N(u')\cap J_1$ such that $N_2(x_1,S)=\emptyset, N_2(x_2,S)=\emptyset, N_2(x'_1,S)=\emptyset$ and $N_2(x'_2,S)=\emptyset$. By Claim 4 there exist $z_i\in N(P(x_i))\cap(N_3(S)-(R-N(v)))$ and $z'_i\in N(P(x'_i))\cap(N_3(S)-(R-N(v)))$ where $i=1,2$. Assume $N(v)=\{u_1,u_2,u_3,u_4\}$. For $i=1,2$, define

$$g(u_iv)=f(u_iv)+f(z_iP(x_i))\geqslant\frac{1}{2}.$$

Moreover, define $g(u_4v)=f(u_4v)+f(z'_1P(x'_1))\geqslant\frac{1}{2}$ and $g(u_4v)=f(u_4v)$. Thus we have

$$h(v)=\sum_{i=1}^{4} g(u_iv)\geqslant\frac{7}{4}.$$

Case 2. $v \in L$.

By the definition of L and Claim 3 there exists $v' \in L$ such that $|N_2(v,S) \cap N_2(v',S)| \geqslant 2$.

If $|N_2(v,S) \cap N_2(v',S)| = 4$, then

$$h(v) = \sum_{u \in N(v)} g(uv) = 4 \times \frac{1}{2} = 2 \quad \text{and} \quad h(v') = \sum_{u' \in N(v')} g(u'v') = 4 \times \frac{1}{2} = 2.$$

If $|N_2(v,S) \cap N_2(v',S)| = 3$, then for $u \in N(v) - N_2(v,S)$ and $u' \in N(v') - N_2(v',S)$, define $g(uv) = f(uv) = \frac{1}{4}$ and $g(u'v') = f(u'v') = \frac{1}{4}$, thus

$$h(v) = \sum_{y \in N(v)} g(yv) = 3\times\frac{1}{2}+\frac{1}{4} = \frac{7}{4} \quad \text{and} \quad h(v') = \sum_{u' \in N(v')} g(u'v') = 3\times\frac{1}{2}+\frac{1}{4} = \frac{7}{4}.$$

If $|N_2(v,S) \cap N_2(v',S)| = 2$, then we assume that $u_1, u_2 \in N_2(v,S) \cap N_2(v',S)$ and distinguish the following cases.

Case 2.1. $|(N(v) \cup N(v')) \cap (N_3(S) \cup N(J_2))| \geqslant 2$.

Case 2.1.1. $|(N(v) \cup N(v')) \cap N(J_2)| \geqslant 2$.

Without loss of generality, there exist $y \in N(v) \cap N(J_2)$ and $y' \in N(v') \cap N(J_2)$. Then there exist vertices $x, x' \in J_2$ such that $xy \in E(G)$ and $x'y' \in E(G)$. Define $g(yv) = f(yv) + f(yx) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $g(y'v') = f(y'v') + f(y'x') = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. For $z \in N(v) - \{u_1, u_2, y\}$ and $z' \in N(v') - \{u_1, u_2, y'\}$, define $g(zv) = f(zv) = \frac{1}{4}$ and $g(z'v') = f(z'v') = \frac{1}{4}$. Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(yv) + g(zv) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(yv') + g(zv') = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}.$$

Case 2.1.2. $|(N(v) \cup N(v')) \cap N_3(S)| \geqslant 2$.

Without loss of generality, there exist vertices $y \in N(v) \cap N_3(S)$ and $y' \in N(v') \cap N_3(S)$. Then there exist vertices $x, x' \in V - S$ such that $xy \in E(G)$ and $x'y' \in E(G)$. If $x \in N_3(S)$, define $g(yv) = f(yv) + \frac{1}{2}f(yx) = \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$. Otherwise, define $g(yv) = f(yv) + f(yx) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Similarly, we can define $g(y'v') = \frac{1}{2}$. For $z \in N(v) - \{u_1, u_2, y\}$ and $z' \in N(v') - \{u_1, u_2, y'\}$, define $g(zv) = f(zv) = \frac{1}{4}$ and $g(z'v') = f(z'v') = \frac{1}{4}$. Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(yv) + g(zv) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(yv') + g(zv') = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}.$$

Case 2.3. *$|(N(v) \cup N(v')) \cap N(J_2)| = 1$ and $|(N(v) \cup N(v')) \cap N_3(S)| = 1$.*

Without loss of generality, there exist vertices $y \in N(v) \cap N_3(S)$ and $y' \in N(v') \cap N(J_2)$. Then there exist a vertex $x \in V - S$ such that $xy \in E(G)$ and a vertex $x' \in J_2$ such that $x'y' \in E(G)$. If $x \in N_3(S)$, define $g(yv) = f(yv) + \frac{1}{2}f(yx) = \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$. Otherwise, define $g(yv) = f(yv) + f(yx) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Define $g(y'v') = f(y'v') + f(y'x') = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. For $z \in N(v) - \{u_1, u_2, y\}$ and $z' \in N(v') - \{u_1, u_2, y'\}$, define $g(zv) = f(zv) = \frac{1}{4}$ and $g(z'v') = f(z'v') = \frac{1}{4}$. Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(yv) + g(zv) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(y'v') + g(z'v') = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}.$$

Case 2.2. *$|(N(v) \cup N(v')) \cap (N_3(S) \cup N(J_2))| \leqslant 1$.*

Denote

$$\begin{aligned} M_2 = \{&y \in N(L) \cap N_4(S) | N(y) - L \subseteq J_1 \text{ and there exist two vertices} \\ &x_1, x_2 \in N(y) - L \text{ such that } N_2(x_1, S) = \emptyset,\ N_2(x_2, S) = \emptyset\}, \\ Q_2 = \{&y \in N(L) \cap N_4(S) | N(y) - L \subseteq J_1 \text{ and } |N(y) - L| = 2 \\ &\text{and for } x_1, x_2 \in N(y) - L,\ N_2(x_1, S) = \emptyset,\ N_2(x_2, S) \neq \emptyset\}. \end{aligned}$$

Case 2.2.1. *$|N(v) \cup N(v') \cap Q_2| \leqslant 1$.*

Case 2.2.1.1. *$(N(v) \cup N(v')) \cap M_2 = \emptyset$.*

If $N(v) \cap Q_2 \neq \emptyset$, assume $u_3 \in N(v) \cap Q_2$ and $u_4 \in N(v) - \{u_1, u_2, u_3\}$ and $u_5 \in N(v') - \{u_1, u_2, u_3\}$. Then, without loss of generality, there exist vertices x_1, x_2, x_3 and x_4 such that $x_1, x_2 \in N(u_3) - L$, $x_3, x_4 \in N(u_4) - L$ and $N_2(x_1, S) = \emptyset, N_2(x_2, S) \neq \emptyset, N_2(x_3, S) \neq \emptyset$ and $N_2(x_4, S) \neq \emptyset$. Assume $w_i \in N_2(x_i, S)$ for $i = 2, 3, 4$. By Claim 4 we can select $z_1 \in (N(P(x_1)) \cap (N_3(S) - (R - (N(v) \cap N(v'))))$. Define $g(u_3v') = f(u_3v) + f(z_1x_1) = \frac{1}{2}, g(u_4v) = f(u_4v) = \frac{1}{4}$ and $g(u_3v) = f(u_3v) = \frac{1}{4}, g(u_5v') = f(u_5v') + \frac{1}{2} \times (f(w_2x_2) - \frac{1}{4}) + \frac{1}{2} \times (f(w_3x_3) - \frac{1}{4}) + \frac{1}{2} \times (f(w_4x_4) - \frac{1}{4}) \geqslant \frac{1}{2}$.

Therefore, we have

$$h(v)=g(u_1v)+g(u_2v)+g(u_3v)+g(u_4v)\geqslant\frac{7}{4},$$
$$h(v')=g(u_1v')+g(u_2v')+g(u_3v')+g(u_5v')\geqslant\frac{7}{4}.$$

If $N(v)\cap Q_2=\emptyset$, assume $u_3,u_4\in N(v)-\{u_1,u_2\}$ and $u_5,u_6\in N(v')-\{u_1,u_2\}$. Then, without loss of generality, there exist vertices x_1,x_2,x_3 and x_4 such that $x_1,x_2\in N(u_3)-L$, $x_3,x_4\in N(u_4)-L$ and $N_2(x_1,S)\neq\emptyset, N_2(x_2,S)\neq\emptyset, N_2(x_3,S)\neq\emptyset$ and $N_2(x_4,S)\neq\emptyset$. Assume $w_i\in N_2(x_i,S)$ for $i=1,2,3,4$.

Define $g(u_3v)=f(u_3v)+\frac{1}{2}\times(f(w_1x_1)-\frac{1}{4})+\frac{1}{2}\times(f(w_2x_2)-\frac{1}{4})=\frac{1}{2}, g(u_4v)=f(u_4v)=\frac{1}{4}$ and $g(u_5v')=f(u_5v')+\frac{1}{2}\times(f(w_3x_3)-\frac{1}{4})+\frac{1}{2}\times(f(w_4x_4)-\frac{1}{4})=\frac{1}{2}, g(u_6v')=f(u_6v')=\frac{1}{4}$. Therefore, we have

$$h(v)=g(u_1v)+g(u_2v)+g(u_3v)+g(u_4v)\geqslant\frac{7}{4},$$

and

$$h(v')=g(u_1v')+g(u_2v')+g(u_5v')+g(u_6v')\geqslant\frac{7}{4}.$$

Case 2.2.1.2. $(N(v)\cup N(v'))\cap M_2\neq\emptyset$.

Assume $u\in(N(v)\cup N(v'))\cap M_2$, then there exist two vertices $x_1,x_2\in N(u)-\{v,v'\}\subseteq J_1$ such that $N_2(x_1,S)=\emptyset, N_2(x_2,S)=\emptyset$. By the same argument as in case 1, we can select two vertices $z_1\in N(P(x_1))\cap N_3(S), z_2\in N(P(x_2))\cap N_3(S)$; we claim that $z_1,z_2\in N_3(S)-(R-(N(v)\cup N(v')))$. Without loss of generality, suppose to the contrary that there exists a vertex $b\in B$ such that $bz_1\in E(G)$ and $b\notin\{v,v'\}$. Since $N(v)\cap N(b)=\emptyset$, $S'=S-\{b,v,x_1\}\cup\{z_1,u\}$ is a dominating set of G with cardinality less than S, a contradiction. Assume that $y_1,y_2\in N(v)-\{u_1,u_2\}$, $y_1',y_2'\in N(v')-\{u_1,u_2\}$, define $g(y_1v)=f(y_1v)+f(z_1P(x_1))\geqslant\frac{1}{2}, g(y_2v)=f(y_2v)\geqslant\frac{1}{4}$ and $g(y_1'v')=f(y_1'v')+f(z_2P(x_2))\geqslant\frac{1}{2}, g(y_2'v')=f(y_2'v')\geqslant\frac{1}{4}$. Thus we have

$$h(v)=g(u_1v)+g(u_2v)+g(y_1v)+g(y_2v)\geqslant\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}$$

and

$$h(v')=g(u_1v')+g(u_2v')+g(y_1'v')+g(y_2'v')\geqslant\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}.$$

Case 2.2.2. $|N(v) \cup N(v') \cap Q_2| \geqslant 2$.

Then there exist two distinct vertices y, $y' \in (N(v) \cup N(v')) \cap Q_2$, so there exist $x_1, x_2 \in (N(y) \cup N(y')) \cap J_1$ such that $N_2(x_1, S) = \emptyset$ and $N_2(x_2, S) = \emptyset$. (Note that this is possible for $x_1 = x_2$). By Claim 4, we can select $z_1 \in (N(P(x_1)) \cap (N_3(S) - (R - (N(v) \cap N(v')))), z_2 \in N(P(x_2)) \cap (N_3(S) - (R - (N(v) \cap N(v'))))$. We can argue in the same way as before, and conclude that $h(v) \geqslant \frac{7}{4}$ and $h(v') \geqslant \frac{7}{4}$.

Thus we complete the definition of the function g. It is easy to find that g satisfies conditions (1) and (3). This completes the proof of the theorem. □

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