Czechoslovak Mathematical Journal

T. S. Kopaliani

On some structural properties of Banach function spaces and boundedness of certain integral operators

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 3, 791–805

Persistent URL: `http://dml.cz/dmlcz/127930`

Czechoslovak Mathematical Journal, 54 (129) (2004), 791–805

ON SOME STRUCTURAL PROPERTIES OF BANACH FUNCTION SPACES AND BOUNDEDNESS OF CERTAIN INTEGRAL OPERATORS

T. S. Kopaliani, Tbilisi

(Received January 15, 2002)

Abstract. In this paper the notions of uniformly upper and uniformly lower ℓ-estimates for Banach function spaces are introduced. Further, the pair (X, Y) of Banach function spaces is characterized, where X and Y satisfy uniformly a lower ℓ-estimate and uniformly an upper ℓ-estimate, respectively. The integral operator from X into Y of the form

$$Kf(x) = \varphi(x)\int_0^x k(x,y)f(y)\psi(y)\,\mathrm{d}y$$

is studied, where k, φ, ψ are prescribed functions under some local integrability conditions, the kernel k is non-negative and is assumed to satisfy certain additional conditions, notably one of monotone type.

Keywords: Banach function space, uniformly upper, uniformly lower ℓ-estimate, Hardy type operator

MSC 2000: 42B20, 42B25

1. Notation and basic facts

Let (Ω, μ) be a complete σ-finite measure space. By $S = S(\Omega, \mu)$ we denote the collection of all real-valued measurable functions on Ω.

Recall that we say that a Banach space X the elements of which are equivalence classes (modulo equality a.e.) of measurable functions in (Ω, μ) is a *Banach function space* (BFS) if:

1) the norm $\|f\|_X$ is defined for every μ-measurable function f and $f \in X$ if and only if $\|f\|_X < \infty$; $\|f\|_X = 0$ if and only if $f = 0$ a.e.;

2) $\|f\|_X = \big\||f|\big\|_X$ for all $f \in X$;
3) if $0 \leqslant f \leqslant g$ a.e., then $\|f\|_X \leqslant \|g\|_X$;
4) if $0 \leqslant f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$ (Fatou property);
5) if E is a measurable subset of Ω such that $\mu(E) < \infty$, then $\|\aleph_E\|_X < \infty$ (where $\aleph_E$ is the characteristic function of the set E);
6) for every measurable set E, $\mu(E) < \infty$, there is a constant $C_E > 0$ such that $\int_E f(x)\,\mathrm{d}x \leqslant C_E\|f\|_X$.

Given a Banach function space X we can always consider its associate space X' consisting of those $g \in S$ that $f \cdot g \in L^1$ for every $f \in X$ with the usual order and the norm $\|g\|_{X'} = \sup\{\|f \cdot g\|_{L^1}\colon\ \|f\|_X \leqslant 1\}$. X' is a BFS in (Ω, μ) and a closed norming subspace of X^* (norming means: $\|f\|_X = \sup\{\|f \cdot g\|_{L^1}\colon\ \|g\|_{X'} \leqslant 1\}$ for all $f \in X$).

Let X be a BFS and ω a weight, i.e., a positive measurable function on Ω. By X_ω we denote the BFS $\{f \in S\colon\ f\omega \in X\}$ equipped with the norm $\|f\|_{X_\omega} = \|f\omega\|_X$. (For more details and proofs of results about BFSs (Banach lattices) we refer to [1], [2].)

In the paper we study Hardy type operators $K\colon\ X \to Y$ of the form

$$Kf(x) = \varphi(x)\int_0^x k(x,y)f(y)\psi(y)\,\mathrm{d}y.$$

Here X, Y are BFSs on $\Omega = [0, +\infty)$, μ is the usual Lebesgue measure, φ, ψ are measurable positive functions on $[0, +\infty)$, the kernel k is a positive measurable function on the set $\{(x,y) \mid x > y > 0\}$ such that

$$d^{-1}(k(x,z) + k(z,y)) \leqslant k(x,y) \leqslant d(k(x,z) + k(z,y))$$

for some constant $d \geqslant 1$ and for all x, y, z with $x \geqslant z \geqslant y \geqslant 0$. (For Lebesgue spaces, Orlicz and Orlicz-Lorentz spaces see [5], [7], [11].)

Beside the classical Hardy operator, examples of Hardy type operators are: the Riemann-Liouville fractional integral operator $k(x,y) = (x-y)^\gamma$ with $\gamma > 0$, the logarithmic kernel operator with $k(x,y) = \log^\beta(x/y)$, $\beta > 0$, and $k(x,y) = \left(\int_y^x h(s)\,\mathrm{d}s\right)^\gamma$ with $\gamma > 0$, $h \in S$, $h \geqslant 0$ a.e.

By Π_* (by Π^*) we denote the family of sequences $\Pi = \{I_i\}$ where I_i are intervals in $\mathbb{R}^+ = [0, +\infty)$ (measurable subsets in $\mathbb{R}^+$, $\mu(I_i) > 0$) such that $\mathbb{R}^+ = \bigcup_i I_i$ and $I_i \cap I_j = \emptyset$ for $i \neq j$. We ignore the difference in notation caused by a null set.

Everywhere in the sequel ℓ_Π is a Banach sequential space (BSS), meaning that axioms 1)–6) are completed in relation to discrete measure, and e_k denotes the standard basis in ℓ_Π.

We introduce the following notation.

Definition 1. Let $\ell = \{\ell_\Pi\}_{\Pi\in\Pi^*}$ (or, respectively, $\ell = \{\ell_\Pi\}_{\Pi\in\Pi_*}$) be a family of BSSs. A BFS X is said to satisfy a uniformly upper (lower) ℓ-estimate for Π^* (for Π_*) if there exists a constant $C < \infty$ such that for every $f \in X$ and $\Pi \in \Pi^*$ ($\Pi \in \Pi_*$) we have

$$\|f\|_X \leqslant C \Big\| \sum_{I_i\in\Pi} \|f\aleph_{I_i}\|_X e_i \Big\|_{\ell_\Pi} \tag{1}$$

$$\Big(\|f\|_X \geqslant C \Big\| \sum_{I_i\in\Pi} \|f\aleph_{I_i}\|_X e_i \Big\|_{\ell_\Pi} \Big). \tag{2}$$

Note that if $\ell_{\Pi_1} = \ell_{\Pi_2} = \ell_p$ for all $\Pi_1, \Pi_2 \in \Pi^*$ and $1 < p < \infty$, then conditions (1) and (2) are the well-known upper and lower p-estimates for X (see [2]). The notions of uniformly upper (lower) ℓ-estimates, when $\ell_{\Pi_1} = \ell_{\Pi_2}$ for all $\Pi_1, \Pi_2 \in \Pi^*$ (or $\Pi_1, \Pi_2 \in \Pi_*$) were introduced by Berezhnoi (see [9]). Note also that, following [9], in this case a BFS X is said to be ℓ-convex or ℓ-concave.

Definition 2. A pair (X, Y) of BFSs is said to have the property $G(\Pi^*)$ (property $G(\Pi_*)$) if there exists a constant C such that

$$\sum_{I_i\in\Pi} \|f\aleph_{I_i}\|_X \cdot \|g\aleph_{I_i}\|_{Y'} \leqslant C\|f\|_X \cdot \|g\|_{Y'}$$

for any sequence $\Pi = \{I_i\}$, $\Pi \in \Pi^*$ ($\Pi \in \Pi_*$) and every $f \in X$, $g \in Y'$.

Definition 2 was introduced by Berezhnoi (see [10]). Let us remark that a pair (L_p, L_q) possesses the property $G(\Pi_*)$ if and only if $p \leqslant q$.

Let X, Y be BFSs on (Ω_1, μ_1) and (Ω_2, μ_2), respectively. Under the spaces with the mixed norm $X[Y]$, $Y[X]$ we mean the spaces consisting of all $k \in S(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$ such that $\|k(t,\cdot)\|_Y \in X$ and $\|k(\cdot,s)\|_X \in Y$ with norms

$$\|k\|_{X[Y]} = \big\| \|k(t,\cdot)\|_Y \big\|_X, \quad \|k\|_{Y[X]} = \big\| \|k(\cdot,s)\|_X \big\|_Y.$$

It is known that $X[Y]$, $Y[X]$ are BFSs on $\Omega_1 \times \Omega_2$. (For more details we refer to [3].) In the general case the spaces $X[Y]$ and $Y[X]$ are not isomorphic. Moreover, Bukhvalov has proved the following theorem (see [3], [8]).

The generalization of Kolmogorov-Nagumo's theorem. *Let (X, Y) be a pair of BFSs on (Ω_1, μ_1) and (Ω_2, μ_2), respectively. (But such that it does not satisfy the Fatou property.) Suppose that for every choice of functions $\{f_i\}_{i=1}^n$ in X*

with pair-wise disjoint supports, and every choice of functions $\{g_i\}_{i=1}^n$ *in* Y *with pair-wise disjoint supports we have*

$$(3)\qquad \Big\|\sum_{k=1}^{n} f_i(t)g_i(s)\Big\|_{X[Y]} \sim \Big\|\sum_{k=1}^{n} f_i(t)g_i(s)\Big\|_{Y[X]}.$$

Then there exist $p\in[1,\infty)$ *and weights* ω_1 *on* Ω_1 *and* ω_2 *on* Ω_2 *such that* $X = L^p_{\omega_1}(\mathrm{d}\mu_1)$, $Y = L^p_{\omega_2}(\mathrm{d}\mu_2)$ *(in the sense order isomorphic) or both* X, Y *are AM spaces.*

Definition 3. A pair (X,Y) of BFSs is said to have the property $K(\Pi^*)$ (property $K(\Pi_*)$) if there exists a constant C such that

$$\Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)g(s)\aleph_{I_i}(s)\Big\|_{Y[X]} \leqslant C\Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)g(s)\aleph_{I_i}(s)\Big\|_{X[Y]}$$

for all sequences $\Pi\in\Pi^*$ $(\Pi\in\Pi_*)$ and every $f\in X$, $g\in Y$.

Note that if we have a continuous embedding $X[Y]\subset Y[X]$, then the pair (X,Y) of BFSs satisfies the property $K(\Pi^*)$. For example, $L_1[Y]\subseteq Y[L_1]$ (generalized Minkowski's inequality). Let us remark that a pair (L_p, L_q) satisfies property $K(\Pi_*)$ if and only if $p\leqslant q$. It is well known that if X, Y are order continuous BFSs, then $X[Y] = X\otimes_m Y$. (For the definition of this tensor product see [3], [5].) The problem of embedding the tensor product of function spaces into another function space of the same type has interesting applications in the theory of integral operators.

2. The main result

First we discuss the connections between the notions just introduced.

We start with the following observation which is easy to prove analogously to the corresponding facts for upper and lower p-estimates (see [2]). Thus, we consider Theorem 1 proved.

Theorem 1. *Let* $\{\ell_\Pi\}_{\Pi\in\Pi^*}$ *(or* $\ell=\{\ell_\Pi\}_{\Pi\in\Pi_*}$*) be a family of BFSs. A BFS* X *satisfies a uniformly lower (upper)* ℓ*-estimate, if and only if its dual* X' *satisfies the uniformly upper (lower)* ℓ'*-estimate where* $\ell'=\{\ell'_\Pi\}_{\Pi\in\Pi^*}$ $(\ell'=\{\ell'_\Pi\}_{\Pi\in\Pi_*})$.

The main results concerning the notions introduced above are summarized in

Theorem 2. *Let (X,Y) be a pair of BFSs on $\mathbb{R}^+$. Then the following assertions are equivalent:*

1) *A pair (X,Y) of BFSs possesses property $G(\Pi_*)$ (property $G(\Pi^*)$).*
2) *A pair (X,Y) of BFSs possesses property $K(\Pi_*)$ (property $K(\Pi^*)$).*
3) *There is a family $\ell = \{\ell_\Pi\}_{\Pi\in\Pi_*}$ (family $\ell = \{\ell_\Pi\}_{\Pi\in\Pi^*}$) of BSSs such that X satisfies a uniformly lower ℓ-estimate and Y satisfies a uniformly upper ℓ-estimate.*

Different conditions for a pair (X,Y) of BFSs to have property $G(\Pi_*)$ in terms of ℓ-concavity and ℓ-convexity (in that case $\ell_{\Pi_1} = \ell_{\Pi_2}$ for any $\Pi_1, \Pi_2 \in \Pi_*$) can be found in [9]. Here (X,Y) is a pair of symmetric spaces (Lebesgue, Lorentz, Marcinkewicz).

The next theorem characterizes the L_p spaces ($1 \leqslant p < \infty$).

Theorem 3. *Let X be an order continuous BFS on $\mathbb{R}^+$. Then the following assertions are equivalent:*

1) *There is a family $\ell = \{\ell_\Pi\}_{\Pi\in\Pi^*}$ of BSSs such that X satisfies a uniformly lower ℓ-estimate and a uniformly upper ℓ-estimate.*
2) *A pair (X,X) of BFSs has property $G(\Pi^*)$.*
3) *X is order isomorphic to L^p_ω for some weight ω and p ($1 \leqslant p < \infty$).*

Theorem 4. *Let X, Y be order continuous BFSs on $\mathbb{R}^+$. Then the following assertions are equivalent:*

1) *Pairs (X,Y) and (Y,X) of BFSs possess property $K(\Pi^*)$.*
2) *X and Y are order isomorphic to $L^p_{\omega_1}$ and $L^p_{\omega_2}$, respectively, for some weights ω_1, ω_2 and p ($1 \leqslant p < \infty$).*

Note that if in Theorem 3 $\ell_{\Pi_1} = \ell_{\Pi_2}$ for any $\Pi_1, \Pi_2 \in \Pi^*$, then the implication 1) ⇒ 3) is easily obtained from the result of L. Tzafriri (see [2, Theorem I.b.12]). Note also that in Theorem 4 the implication 1) ⇒ 2) is not obtained from the generalized theorem of Kolmogorov-Nagumo. (In general, $\operatorname{supp} f_i \neq \operatorname{supp} g_i$ in (3).)

The following theorem characterizes the properties of boundedness of the map K acting between BFSs when the pair (X,Y) has property $G(\Pi_*)$.

Theorem 5. *Let a pair (X,Y) of BFSs have property $G(\Pi_*)$. Then $K\colon X \to Y$ is bounded if and only if*

$$\sup_{t>0} \|\aleph_{[t,\infty)}\varphi\|_Y \cdot \|\aleph_{[0,t)}(\cdot)k(t,\cdot)\psi(\cdot)\|_{X'} < \infty$$

and

$$\sup_{t>0} \|\aleph_{[t,\infty)}k(\cdot,t)\varphi(\cdot)\|_Y \cdot \|\aleph_{[0,t)}\psi\|_{X'} < \infty.$$

Note that Theorem 5 has a natural analogue for the dual operator $K^*\colon Y' \to X'$,

$$K^*g(x) = \psi(x)\int_x^\infty k(y,x)g(y)\varphi(y)\,\mathrm{d}y.$$

In the case when X is ℓ-concave and Y is ℓ-convex, Theorem 5 was proved by Stepanov and Lomakina (see [6]). (The case $k(x,y)=1$ was investigated by Berezhnoi in [9].)

Remark. Analogously we can consider the case $\Omega = [0,1]$.

3. Proof of theorems

In what follows C denotes a positive constant different from line to line and independent of the function f.

Proof of Theorem 2. Here we present only the case when the family of covering sequences is in Π_* (for $\Pi \in \Pi^*$ the proof is similar).

For a fixed $\Pi = \{I_i\} \in \Pi_*$ we introduce the sequence space ℓ^X_Π with the norm

$$\Big\|\sum_i e_i a_i\Big\|_{\ell^X_\Pi} = \sup\Big\|\sum_{I_i\in\Pi} a_i f_i \aleph_{I_i}\Big\|_X,$$

where the supremum is taken over all possible sequences of functions $\{f_i\}$, $\|f_i\|_X \leqslant 1$. (Similarly we introduce the space ℓ^Y_Π.) It is easy to see that ℓ^X_Π is a BSS and $\ell^1 \subset \ell^X_\Pi \subset \ell^\infty$. Obviously, X satisfies a uniformly upper ℓ-estimate, where $\ell = \{\ell^X_\Pi\}_{\Pi\in\Pi_*}$.

Let a pair (X,Y) of BFSs have property $K(\Pi_*)$. For $f \in X$ and any sequence of functions $\{g_i\}$, $\|g_i\|_Y \leqslant 1$, we have

$$\begin{aligned}\Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)g_i(s)\aleph_{I_i}(s)\Big\|_{Y[X]} &= \Big\|\sum_{I_i\in\Pi} \|f\aleph_{I_i}\|_X g_i(s)\aleph_{I_i}(s)\Big\|_Y\\ &\leqslant C\Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)g_i(s)\aleph_{I_i}(s)\Big\|_{X[Y]}\\ &= C\Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)\|g_i\aleph_{I_i}\|_Y\Big\|_X \leqslant C\|f\|_X.\end{aligned}$$

It follows immediately that X satisfies the uniformly lower ℓ-estimate, where $\ell = \{\ell^Y_\Pi\}_{\Pi\in\Pi_*}$. This completes the proof of the implication 2) $\Rightarrow$ 3). Conversely, if

X satisfies a uniformly lower ℓ-estimate and Y satisfies a uniformly upper ℓ-estimate for some family BSSs $\ell = \{\ell_\Pi\}_{\Pi\in\Pi_*}$, we have

$$\begin{aligned}\Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)g(s)\aleph_{I_i}(s)\Big\|_{Y[X]} &= \Big\|\sum_{I_i\in\Pi} \|f\aleph_{I_i}\|_X g(s)\aleph_{I_i}(s)\Big\|_Y \leqslant C\Big\|\sum_{I_i\in\Pi} e_i\|f\aleph_{I_i}\|_X\cdot\|g\aleph_{I_i}\|_Y\Big\|_{\ell_\Pi}\\ &\leqslant C\Big\|\sum_{I_i\in\Pi}\|g_i\aleph_{I_i}\|_Y\cdot f(t)\aleph_{I_i}(t)\Big\|_X = C\Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)g(s)\aleph_{I_i}(s)\Big\|_{X[Y]},\end{aligned}$$

and the equivalence 2) $\Leftrightarrow$ 3) is proved.

Suppose that 3) holds. By duality (Theorem 1) it follows that Y' satisfies a uniformly lower ℓ'-estimate, where $\ell' = \{\ell'_\Pi\}_{\Pi\in\Pi_*}$. Applying Hölder's inequality we obtain that the pair (X, Y) of BFSs possesses property $G(\Pi_*)$.

Finally, we must prove 1) $\Rightarrow$ 3). For fixed $f \in X$ and any sequence of functions $\{g_i\}$, $\|g_i\|_Y \leqslant 1$, we have

$$\begin{aligned}\Big\|\sum_{I_i\in\Pi}\|f\aleph_{I_i}\|_X g_i(s)\aleph_{I_i}(s)\Big\|_Y &= \sup_{\|g\|_{Y'}\leqslant 1}\int_{\mathbb{R}^+}\sum_{I_i\in\Pi}\|f\aleph_{I_i}\|_X g_i(t)\aleph_{I_i}(t)g(t)\,\mathrm{d}t\\ &\leqslant \sup_{\|g\|_{Y'}\leqslant 1}\sum_{I_i\in\Pi}\|f\aleph_{I_i}\|_X\cdot\|g_i\aleph_{I_i}\|_Y\cdot\|g\aleph_{I_i}\|_{Y'}\\ &\leqslant \sup_{\|g\|_{Y'}\leqslant 1}\sum_{I_i\in\Pi}\|f\aleph_{I_i}\|_X\cdot\|g\aleph_{I_i}\|_{Y'} \leqslant C\|f\|_X.\end{aligned}$$

Consequently, X satisfies a uniformly lower ℓ-estimate, where $\ell = \{\ell^Y_\Pi\}_{\Pi\in\Pi_*}$. This completes the proof of 1) $\Rightarrow$ 3). □

Remark. Let X simultaneously satisfy uniformly upper and lower ℓ-estimates, $\ell = \{\ell_\Pi\}_{\Pi\in\Pi_*}$. Then for any $f \in X$

$$\frac{1}{C}\|f\|_X \leqslant \Big\|\sum_{I_i\in\Pi}\frac{\|f\aleph_{I_i}\|_X}{\|\aleph_{I_i}\|_X}\cdot\aleph_{I_i}\Big\|_X \leqslant C\|f\|_X. \tag{4}$$

It follows from Theorem 2 that

$$\|f\|_X = \Big\|\sum_{I_i\in\Pi} f(t)\aleph_{I_i}(t)\Big\|\frac{\aleph_{I_i}}{\|\aleph_{I_i}\|_X}\Big\|_X\Big\|_X \leqslant C\Big\|\sum_{I_i\in\Pi}\frac{\|f\aleph_{I_i}\|_X}{\|\aleph_{I_i}\|_X}\cdot\aleph_{I_i}\Big\|_X.$$

In a similar way we obtain the right inequality of (4).

P r o o f of Theorem 3. The fact 1) $\Leftrightarrow$ 2) is a direct consequence of Theorem 2. Implications 3) $\Rightarrow$ 1) and 3) $\Rightarrow$ 2) are obvious. We will show 1) $\Rightarrow$ 3).

First we recall some standard notation (see [2]). A closed linear subspace X_0 of a Banach space X is said to be a complemented subspace if there is a projection from X onto X_0, or what is the same, if there exists a closed linear subspace X_1 of X such that $X = X_0 \oplus X_1$. By a sublattice of a BFS X we mean a norm closed linear subspace X_0 of X such that $\max(x(t), y(t))$ belongs to X_0 whenever $x, y \in X_0$. The key point in the proof of implication 1) $\Rightarrow$ 3) consists in the fact that every sublattice of X is complemented. (The existence of projections on every sublattice implies that the space is L^p $(1 \leqslant p < \infty)$ or of c_0 type. For more details and proofs of results of J. Lindenstrauss and L. Tzafriri we refer to [2].)

Let P_0 denote the canonical embedding of X into X^{**}. It should be noted that $P_0(X)$ is a complemented sublattice of X^{**}.

Let X_0 be a sublattice of X. For every finite set $A = \{f_i\}_{i=1}^n$ of disjoint positive functions with norm one in X_0 there exists a set $A' = \{g_i\}_{i=1}^n$ of disjoint functions with norm one in $X^* = X'$ such that $\operatorname{supp} f_i = \operatorname{supp} g_i$, $\langle f_i, g_i\rangle = 1$ for any i.

There is a positive projection P_A from X onto $\operatorname{span}\{f_i,\ i = 1, 2, \ldots, n\}$, defined by

$$P_A f(x) = \sum_{i=1}^n \left(\int_{\mathbb{R}^+} f(s) g_i(s)\,\mathrm{d}s\right) \cdot f_i(t), \quad f \in E.$$

Applying Hölder's inequality and Theorem 2 we obtain

$$\begin{aligned} \|P_A f\|_X = & \sup_{\|g\|_{X'} \leqslant 1} \int_{\mathbb{R}^+} P_A f(x) \cdot g(x)\,\mathrm{d}x \\ & \leqslant \sum_{i=1}^n \|f \aleph_{\operatorname{supp} g_i}\|_X \cdot \|g \aleph_{\operatorname{supp} g_i}\|_{X'} \leqslant C \|f\|_X. \end{aligned}$$

We partially order the set $\overline{A}$ of a finite set of disjoint positive vectors with norm one in X_0 by $\{y_i\}_{i=1}^n < \{z_j\}_{j=1}^m$ if $\operatorname{span}\{y_i,\ i = 1, 2, \ldots, n\} \subseteq \operatorname{span}\{z_j,\ j = 1, 2, \ldots, m\}$.

Now we consider each P_A as an operator from X into X^{**}. For fixed $f \in X$ and every $A \in \overline{A}$, the function $P_A f$ belongs to the $W^*(X^{**}, X^*)$ compact subset $\{y \colon \|y\|_{X^{**}} \leqslant C \cdot \|f\|_X\}$ in X^{**}. Hence, by Tichonoff's theorem, the net $\{P_A\}_{A \in \overline{A}}$ of operators from X into X^{**} has a subnet which converges to the same limit point P (in the topology of point-wise convergence on X taking in X^{**} the $W^*(X^{**}, X^*)$ topology).

It follows immediately that $P_0 P$ is a positive projection from X onto X_0. (Note that for any fixed $\varepsilon > 0$ and $f \in X_0$ there are functions $\{f_i\}_{i=1}^N$ in X_0 with pair-wise disjoint supports such that $\left\| f - \sum\limits_{i=1}^N f_i \right\|_X < \varepsilon$. For more details about Freudenthal's spectral theorem see [2], [3].) This completes the proof. $\square$

P r o o f of Theorem 4. Implication 2) $\Rightarrow$ 1) is obvious. We will show 1) $\Rightarrow$ 2). There is a family $\ell = \{\ell_\Pi\}_{\Pi\in\Pi^*}$ of BSSs such that X satisfies a uniformly lower ℓ-estimate and a uniformly upper ℓ-estimate, namely, we can use BSSs with the norm

$$\Bigl\|\sum_i a_i e_i\Bigr\|_{\ell_\Pi} = \Bigl\|\sum_{I_i\in\Pi} \frac{a_i}{\|\aleph_{I_i}\|_Y}\aleph_{I_i}\Bigr\|_Y$$

and, consequently, X is order isomorphic to $L^{p_1}_{\omega_1}$ for some p_1 $(1 \leqslant p_1 < \infty)$ and a weight ω_1. In a similar way we conclude that Y is order isomorphic to $L^{p_2}_{\omega_2}$ for some p_2 $(1 \leqslant p_2 < \infty)$ and a weight ω_2. Obviously, $p_1 = p_2$. □

P r o o f of Theorem 5. It is clear that the continuity of $K\colon X \to Y$ is equivalent to the continuity of $K_0\colon X_0 \to Y_0$, where $X_0 = X_{\psi^{-1}}$, $Y_0 = Y_\varphi$, and

$$K_0 f(x) = \int_0^x k(x,t)\,\mathrm{d}t.$$

Note also that if a pair (X, Y) of BFSs has property $G(\Pi_*)$, then (X_0, Y_0) has property $G(\Pi_*)$ too. Consequently, without loss of generality we can assume that $\psi = \varphi = 1$.

Without loss of generality suppose that f is nonnegative with compact support. Following the procedure introduced in [7] (see also [9]), select a monotone sequence $\{x_i\} \subset \mathbb{R}^+$, $-\infty < i \leqslant N \leqslant +\infty$ such that

$$\begin{aligned} K_0 f(x) \leqslant C\biggl(&\sum_{-\infty\leqslant i\leqslant N} \aleph_{(x_i,x_{i+1})}(x)\int_{x_{i-1}}^{x_i} k(x_i,t)f(t)\,\mathrm{d}t \\ &+ \sum_{-\infty\leqslant i\leqslant N} k(x_i,x_{i-1})\int_0^{x_{i-1}} f(t)\,\mathrm{d}t\cdot\aleph_{(x_i,x_{i+1})}(x)\biggr) = C(F_1(x)+F_2(x)). \end{aligned}$$

Applying Hölder's inequality we obtain

$$\begin{aligned} &\int_{x_i}^{x_{i+1}} g(t)\,\mathrm{d}t\cdot\int_{x_{i-1}}^{x_i} k(x_i,t)f(t)\,\mathrm{d}t \\ &\qquad\leqslant \|g\aleph_{(x_i,x_{i+1})}\|_{Y'}\cdot\|\aleph_{(x_i,x_{i+1})}\|_Y\cdot\|k(x_i,\cdot)\aleph_{(x_{i-1},x_i)}\|_{X'}\cdot\|f\aleph_{(x_{i-1},x_i)}\|_X \\ &\qquad\leqslant C\|g\aleph_{(x_i,x_{i+1})}\|_{Y'}\cdot\|f\aleph_{(x_{i-1},x_i)}\|_X. \end{aligned}$$

Substituting these estimates into the formula for $\|F_1\|_Y$ we obtain

$$\begin{aligned} \|F_1\|_Y &= \sup_{\|g\|_{Y'}\leqslant 1}\sum_i \int_{x_i}^{x_{i+1}} g(t)\,\mathrm{d}t\cdot\int_{x_{i-1}}^{x_i} k(x_i,t)f(t)\,\mathrm{d}t \\ &\leqslant \sup_{\|g\|_{Y'}\leqslant 1}\biggl(\sum_{-\infty<2i\leqslant N}\|g\aleph_{(x_i,x_{i+1})}\|_{Y'}\cdot\|f\aleph_{(x_{i-1},x_i)}\|_X \\ &\qquad+ \sum_{-\infty<2i+1\leqslant N}\|g\aleph_{(x_i,x_{i+1})}\|_{Y'}\cdot\|f\aleph_{(x_{i-1},x_i)}\|_X\biggr) \leqslant C\|f\|_X. \end{aligned}$$

To estimate $\|F_2\|_Y$ we note that for a fixed strictly increasing sequence $\{x_i\}$ $(-\infty < i < +\infty)$ the inequality

$$\left\| \sum_i k(x_i, x_{i-1}) \int_0^{x_{i-1}} f(t)\,\mathrm{d}t \cdot \aleph_{(x_i, x_{i+1})} \right\|_Y \leqslant C\|f\|_Y \tag{5}$$

is valid if

$$\sup_i \left\| \sum_{j \geqslant i} \aleph_{(x_j, x_{j+1})} k(x_j, x_{j-1}) \right\|_Y \cdot \|\aleph_{(0, x_{i-1})}\|_{X'} < \infty. \tag{6}$$

The proof of (6) $\Rightarrow$ (5) is based on the fact that the function $\int_0^x f(t)\,\mathrm{d}t$ is non-decreasing. Let m be an integer such that $\|f\|_{L^1} \in (2^m, 2^{m+1}]$. Then there is an increasing sequence $\{t_i\}$ $(-\infty < i \leqslant m)$ such that $2^i = \int_0^{t_i} f(t)\,\mathrm{d}t = \int_{t_i}^{t_{i+1}} f(t)\,\mathrm{d}t$ for $k \leqslant m-1$ and $2^m = \int_0^{t_m} f(t)\,\mathrm{d}t$.

It is clear that $\int_0^{x_{i-1}} f(t)\,\mathrm{d}t \asymp \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t$ for $x_{i-1} \in (t_k, t_{k+1}]$. Substituting this estimate into the formula for $\|F_2\|_Y$ and applying the above method (see the calculation of the norm $\|F_1\|_Y$) we can prove implication (6) $\Rightarrow$ (5).

It follows from the inequality $k(x_i, x_{i-1}) \leqslant d^2 k(x,t)$ for $x \geqslant x_i \geqslant x_{i-1} \geqslant t$ that

$$\left\| \sum_{j \geqslant i} \aleph_{(x_j, x_{j+1})} k(x_j, x_{j-1}) \right\|_Y \cdot \|\aleph_{(0, x_{i-1})}\|_{X'} \leqslant C \sup_{t>0} \|\aleph_{[t,\infty)} k(\cdot, t)\|_Y \cdot \|\aleph_{(0,t]}\|_{X'} < \infty,$$

which completes the proof of the sufficiency part.

The necessity can be obtained in a similar way as for the Lebesgue space (see [7]) and we omit it here. □

4. Examples

Let p be a fixed μ-measurable function on Ω, $1 \leqslant p(t) \leqslant +\infty$. Put $\Omega_\infty = \{t \colon p(t) = +\infty\}$, $\Omega_0 = \Omega \setminus \Omega_\infty$. The BFS $L^{p(t)}$ is defined by the norm

$$\|f\|_{L^{p(t)}} = \inf \left\{ \lambda > 0 \colon \int_{\Omega_0} \left| \frac{f(t)}{\lambda} \right|^{p(t)} \mathrm{d}\mu(t) \leqslant 1 \right\} + \|f \aleph_{\Omega_\infty}\|_{L^\infty}.$$

It is well known that (see [13]) $(L^{p(t)})'$ is isomorphic to the space $L^{q(t)}$, where $p(t)^{-1} + q(t)^{-1} = 1$. Moreover, the norm is order continuous if and only if $p \in L^\infty$. The spaces $L^{p(t)}$ are of Musielak-Orlicz type. The concept of Musielak-Orlicz spaces was introduced in [4].

Below we consider the case $\Omega = [0,1]$ and the μ-Lebesgue measure. Let $P_{[0,1]}$ denote the set of functions $p \in C([0,1])$, $\|p\|_C \geqslant 1$ such that for all $t_1, t_2 \in [0,1]$

$$\bigl|(p(t_1) - p(t_2)) \ln |t_1 - t_2|\bigr| \leqslant C.$$

Example 1. Let $p_1, p_2 \in P_{[0,1]}$ and $p_1(t) \leqslant p_2(t)$ for all $t \in [0,1]$. Then the pair $(L^{p_1(t)}, L^{p_2(t)})$ of BFSs has property $G(\Pi_*)$.

Proof. First we prove that for fixed $p \in P_{[0,1]}$,

$$\Bigl\| \sum_{I_i \in \Pi} \frac{\|f \aleph_{I_i}\|_{L^{p(t)}}}{\|\aleph_{I_i}\|_{L^{p(t)}}} \cdot \aleph_{I_i} \Bigr\|_{L^{p(t)}} \asymp \|f\|_{L^{p(t)}}. \tag{7}$$

We need the following lemma (see [12], [14]).

Lemma. *Let p_1, p_2 be fixed measurable functions on $[0,1]$, $1 \leqslant p_1(t) \leqslant p_2(t) \leqslant C < +\infty$ a.e. Then for every $f \in L^{p_2(t)}$ the inequality*

$$\|f\|_{L^{p_1(t)}} \leqslant 2 \|f\|_{L^{p_2(t)}} \tag{8}$$

is valid.

To prove the inequalities (7) let us first consider the case $p(t) > 1$ on $[0,1]$. Below we will use the following notation:

$$\underline{p}(I) = \min_{t \in I} p(t), \quad \bar{p}(I) = \max_{t \in I} p(t), \quad \bar{p}_\Pi(t) = \sum_{I_i \in \Pi} \bar{p}(I_i) \aleph_{I_i}(t).$$

From (8) it follows that

$$\frac{1}{2} |I|^{1/\underline{p}(I)} \leqslant \|\aleph_I\|_{L^{p(t)}} \leqslant 2 |I|^{1/\bar{p}(I)}.$$

Under our assumptions we have for some constant C and every interval $I \subset [0,1]$

$$|I|^{1/\bar{p}(I) - 1/\underline{p}(I)} = \bigl(\exp\bigl(-\log |I| \cdot (\bar{p}(I) - \underline{p}(I))\bigr)\bigr)^{1/(\underline{p}(I)\bar{p}(I))}.$$

Consequently,

$$|I|^{1/\underline{p}(I)} \asymp \|\aleph_I\|_{L^{p(t)}} \asymp |I|^{1/\bar{p}(I)}. \tag{9}$$

From the definition of the norm, it is obvious that if $\|f \aleph_I\|_{L^{p(t)}} \leqslant 1$, then

$$\|f \aleph_I\|_{L^{p(t)}} \leqslant \Bigl(\int_I |f(t)|^{p(t)} \, \mathrm{d}t \Bigr)^{1/\bar{p}(I)}. \tag{10}$$

Combining the estimates (9), (10) for $\|f\|_{L^{p(t)}} \leqslant 1$ we have

$$\int_{[0,1]} \Bigl(\sum_{I_i \in \Pi} \frac{\|f\aleph_{I_i}\|_{L^{p(t)}}}{\|\aleph_{I_i}\|_{L^{p(t)}}} \cdot \aleph_{I_i} \Bigr)^{\bar{p}_\Pi(t)} \mathrm{d}t \leqslant C \sum_{I_i \in \Pi} \int_{I_i} \frac{\int_{I_i} |f(t)|^{p(t)}\,\mathrm{d}t}{|I_i|}\,\mathrm{d}t$$
$$\leqslant C \sum_{I_i \in \Pi} \int_{I_i} |f(t)|^{p(t)}\,\mathrm{d}t = C \int_{[0,1]} |f(t)|^{p(t)}\,\mathrm{d}t \leqslant C.$$

Consequently, for any $f \in L^{p(t)}$ we have

$$\Bigl\| \sum_{I_i \in \Pi} \frac{\|f\aleph_{I_i}\|_{L^{p(t)}}}{\|\aleph_{I_i}\|_{L^{p(t)}}} \aleph_{I_i} \Bigr\|_{L^{p(t)}} \leqslant 2 \Bigl\| \sum_{I_i \in \Pi} \frac{\|f\aleph_{I_i}\|_{L^{p(t)}}}{\|\aleph_{I_i}\|_{L^{p(t)}}} \aleph_{I_i} \Bigr\|_{L^{\bar{p}_\Pi(t)}} \leqslant C\|f\|_{L^{p(t)}}. \tag{11}$$

Analogously, for $q(t)$ we have

$$\Bigl\| \sum_{I_i \in \Pi} \frac{\|g\aleph_{I_i}\|_{L^{q(t)}}}{\|\aleph_{I_i}\|_{L^{q(t)}}} \aleph_{I_i} \Bigr\|_{L^{q(t)}} \leqslant C\|g\|_{L^{q(t)}}. \tag{12}$$

Using the estimates (11), (12) we obtain

$$\begin{aligned}
\|f\|_{L^{p(t)}} &= \sup_{\|g\|_{L^{q(t)}} \leqslant 1} \int_{[0,1]} f(t)g(t)\,\mathrm{d}t \leqslant \sup_{\|g\|_{L^{q(t)}} \leqslant 1} \sum_{I_i \in \Pi} \int_{I_i} f(t)g(t)\,\mathrm{d}t \\
&\leqslant \sup_{\|g\|_{L^{q(t)}} \leqslant 1} \sum_{I_i \in \Pi} \|f\aleph_{I_i}\|_{L^{p(t)}} \cdot \|g\aleph_{I_i}\|_{L^{q(t)}} \\
&\leqslant \sup_{\|g\|_{L^{q(t)}} \leqslant 1} \int_{[0,1]} \sum_{I_i \in \Pi} \frac{\|f\aleph_{I_i}\|_{L^{p(t)}} \|g\aleph_{I_i}\|_{L^{q(t)}}}{\|\aleph_{I_i}\|_{L^{p(t)}} \|\aleph_{I_i}\|_{L^{q(t)}}} \aleph_{I_i} \\
&\leqslant \sup_{\|g\|_{L^{q(t)}} \leqslant 1} \Bigl\| \sum_{I_i \in \Pi} \frac{\|f\aleph_{I_i}\|_{L^{p(t)}}}{\|\aleph_{I_i}\|_{L^{p(t)}}} \aleph_{I_i} \Bigr\|_{L^{p(t)}} \cdot \Bigl\| \sum_{I_i \in \Pi} \frac{\|g\aleph_{I_i}\|_{L^{q(t)}}}{\|\aleph_{I_i}\|_{L^{q(t)}}} \aleph_{I_i} \Bigr\|_{L^{q(t)}} \\
&\leqslant C \Bigl\| \sum_{I_i \in \Pi} \frac{\|f\aleph_{I_i}\|_{L^{p(t)}}}{\|\aleph_{I_i}\|_{L^{p(t)}}} \aleph_{I_i} \Bigr\|_{L^{p(t)}}.
\end{aligned}$$

The proof of inequalities (7) in the case $\underline{p}([0,1]) > 1$ is complete.

Case $\underline{p}([0,1]) = 1$. Fix $k \in \mathbb{N}$ and put

$$p_k(t) = \begin{cases} 1 + \dfrac{1}{k}, & \text{when } t \in \{x\colon p(x) \leqslant 1 + 1/k\}, \\ p(t), & \text{when } t \in \{x\colon p(x) > 1 + 1/k\}. \end{cases}$$

It is clear that inequalities (7) are valid for $L^{p_k(t)}$, and a simple limiting argument gives the desired result.

By $\ell_\Pi^{p_k(t)}$ $(k=1,2)$ we denote sequence spaces with the norm

$$\Big\|\sum_i a_i e_i\Big\|_{\ell_\Pi^{p_k(t)}} = \Big\|\sum_{I_i\in\Pi} \frac{a_i}{\|\aleph_{I_i}\|_{L^{p_k(t)}}}\,\aleph_{I_i}\Big\|_{L^{p_k(t)}}.$$

Let $\|\sum a_i e_i\|_{\ell_\Pi^{p_1(t)}} = 1$, then

$$\int_{[0,1]}\Big(\sum_{I_i\in\Pi}\frac{|a_i|}{\|\aleph_{I_i}\|_{L^{p_1(t)}}}\,\aleph_{I_i}\Big)^{p_1(t)}\,\mathrm{d}t = 1$$

and consequently

$$\sum_{I_i\in\Pi}\frac{1}{|I_i|}\int_{I_i} a_i^{p_1(t)}\,\mathrm{d}t \leqslant C.$$

Using the last inequality and the fact that $|a_i|\leqslant 1$, we have

$$\sum_{I_i\in\Pi}\frac{1}{|I_i|}\int_{I_i} a_i^{p_2(t)}\,\mathrm{d}t \leqslant C$$

and consequently

$$\Big\|\sum_i a_i e_i\Big\|_{\ell_\Pi^{p_2(t)}} \leqslant C.$$

It follows that for any $\Pi\in\Pi_*$

$$\Big\|\sum_i a_i e_i\Big\|_{\ell_\Pi^{p_2(t)}} \leqslant C\Big\|\sum_i a_i e_i\Big\|_{\ell_\Pi^{p_1(t)}}. \tag{13}$$

Using (7), (13) we have that $L^{p_1(t)}$ satisfies a uniformly lower $\{\ell_\Pi^{p_1(t)}\}_{\Pi\in\Pi_*}$-estimate and $L^{p_2(t)}$ satisfies a uniformly upper $\{\ell_\Pi^{p_1(t)}\}_{\Pi\in\Pi_*}$-estimate. □

Case $\Omega=[0,\infty)$. Let $P_{[0,\infty)}$ denote the set of functions defined on $[0,\infty)$ of the form $p(2/\pi\arctan t)$ where $p\in P_{[0,1]}$.

Example 2. Let $p_1,p_2\in P_{[0,\infty)}$ and $p_1(t)\leqslant p_2(t)$ for all $t\in[0,+\infty)$. Then the pair $(L^{p_1(t)},L^{p_2(t)})$ of BFSs possesses property $G(\Pi_*)$.

Proof. For any $p\in P_{[0,1]}$ the spaces $L^{p(t)}$ and $L_\omega^{p(\ell(t))}$ are isomorphic, where $\ell(t)=2/\pi\arctan t$, $t\in[0,\infty)$, and $\omega(t)=(2/\pi(1+t^2)^{-1})^{1/p(\ell(t))}$. (Note that the measure spaces $([0,1],\mathrm{d}t)$, $([0,\infty),2(\pi(1+t^2))^{-1}\,\mathrm{d}t)$ are isomorphic.)

Note also that the pair (X,Y) of BFSs possesses property $G(\Pi_*)$ if and only if the pair $(X_{\omega_1},Y_{\omega_2})$ possesses property $G(\Pi_*)$ for some weights ω_1, ω_2. This completes the proof. □

Remark. Below we will construct a function $p(t) \in C([0,1])$ such that the pair $(L^{p(t)}, L^{p(t)})$ of BFSs does possess property $G(\Pi_*)$.

Let us start by defining some subsets of $[0,1]$. Let us put

$$Q_m^k = \Big(4^{-m-1} + \frac{3(k-1)}{16m}\,4^{-m}, 4^{-m-1} + \frac{3k}{16m}\,4^{-m}\Big),\ \ m \in \mathbb{N},\ \ k = 1,2,\dots,4m;$$
$$O_m^k = \Big(4^{-m-1} + \frac{3(k-1)}{4m}\,4^{-m}, 4^{-m-1} + \frac{3k}{4m}\,4^{-m}\Big),\ \ m \in \mathbb{N},\ \ k = 1,2,\dots,m.$$

Let $\{p_m^1\}_{m\in\mathbb{N}}$ be a convergent sequence of numbers with $p_m^1 \geqslant 2$, $m \in \mathbb{N}$. Let us define a new sequence of numbers $\{p_m^2\}_{m\in\mathbb{N}}$ in the following way: $p_m^2 = p_m^1 - (\ln(m+5))^{-\alpha}$, where α is a number from the interval $(0,1)$.

Let us construct a function $p \in C([0,1])$, $p(t) > 1$, such that $p(t) = p_m^1$ with $x \in Q_m^{4l+1}$, $m \in \mathbb{N}$, $l = 0,1,\dots,m-1$, and $p(t) = p_m^2$ with $x \in Q_m^{4l+3}$, $m \in \mathbb{N}$, $l = 0,1,\dots,m-1$. (The possibility of such construction is obvious.)

For the functions

$$f_m(t) = \sum_{l=0}^{m-1} \aleph_{Q_m^{4l+1}}(t), \quad g_m(t) = \sum_{l=0}^{m-1} \aleph_{Q_m^{4l+3}}(t)$$

we have

$$A_m = \sum_{k=1}^{m} \|f_m \aleph_{O_m^k}\|_{L^{p(t)}} \cdot \|g_m \aleph_{O_m^k}\|_{L^{q(t)}} = m|Q_m^1|^{1/p_m^1} \cdot |Q_m^1|^{1-1/p_m^2}$$

and

$$B_m = \|f_m\|_{L^{p(t)}} \cdot \|g_m\|_{L^{q(t)}} = (m|Q_m^1|)^{1/p_m^1} \cdot (m|Q_m^1|)^{1-1/p_m^2}.$$

It is clear that
$$\frac{A_m}{B_m} = m^{1/p_m^2 - 1/p_m^1} \to \infty \quad \text{as} \quad m \to \infty.$$

Consequently, the pair $(L^{p(t)}, L^{p(t)})$ does not possess property $G(\Pi_*)$.

References

[1] *C. Bennett and R. Sharpley*: Interpolation of Operators. Acad. Press, Boston, 1988.

[2] *J. Lindenstrauss and L. Tzafriri*: Classical Banach Spaces. II. Function Spaces. Springer-Verlag, 1979.

[3] *A. V. Bukhvalov, V. B. Korotkov, A. G. Kusraev, S. S. Kutateladze and B. M. Makarov*: Vector Lattices and Integral Operators. Nauka, Novosibirsk, 1992. (In Russian.)

[4] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Math. 1034. Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[5] *V. D. Stepanov*: Nonlinear Analysis. Function Spaces and Applications 5. Olympia Press, 1994, pp. 139–176.

[6] *E. N. Lomakina and V. D. Stepanov*: On Hardy type operators in Banach function spaces on half-line. Dokl. Akad. Nauk *359* (1998), 21–23. (In Russian.)

[7] *P. Oinarov*: Two-side estimates of the norm of some classes of integral operators. Trudy Mat. Inst. Steklov. *204* (1993), 240–250. (In Russian.)

[8] *A. V. Bukhvalov*: Generalization of Kolmogorov-Nagumo's theorem on tensor product. Kachestv. pribl. metod. issledov. operator. uravnen. *4* (1979), 48–65. (In Russian.)

[9] *E. I. Berezhnoi*: Sharp estimates for operators on cones in ideal spaces. Trudy Mat. Inst. Steklov. *204* (1993), 3–36. (In Russian.)

[10] *E. I. Berezhnoi*: Two-weighted estimations for the Hardy–Littlwood maximal function in ideal Banach spaces. Proc. Amer. Math. Soc. *127* (1999), 79–87.

[11] *Q. Lai*: Weighted modular inequalities for Hardy type operators. Proc. London Math. Soc. *79* (1999), 649–672.

[12] *I. I. Sharafutdinov*: On the basisity of the Haar system in $L^{p(t)}([0,1])$ spaces. Mat. Sbornik *130* (1986), 275–283. (In Russian.)

[13] *I. I. Sharafutdinov*: The topology of the space $L^{p(t)}([0,1])$. Mat. Zametki *26* (1976), 613–632. (In Russian.)

[14] *O. Kováčik and J. Rákosník*: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. *41* (1991), 592–618.

[15] *H. H. Schefer*: Banach Lattices and Positive Operators. Springer-Verlag, Berlin-Heidelberg-New York, 1974.

Author's address: Department of Mechanics and Mathematics, Tbilisi State University, 1 Chavchavadze Ave., Tbilisi 380028, Georgia, e-mail: t_kopaliani@hotmail.com.