Shahabaddin Ebrahimi Atani

On prime modules over pullback rings

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 3, 781–789

Persistent URL: `http://dml.cz/dmlcz/127929`

Czechoslovak Mathematical Journal, 54 (129) (2004), 781–789

ON PRIME MODULES OVER PULLBACK RINGS

Shahabaddin Ebrahimi Atani, Rasht

(Received December 17, 2001)

Abstract. First, we give a complete description of the indecomposable prime modules over a Dedekind domain. Second, if R is the pullback, in the sense of [9], of two local Dedekind domains then we classify indecomposable prime R-modules and establish a connection between the prime modules and the pure-injective modules (also representable modules) over such rings.

Keywords: indecomposable prime modules, pullback rings, separated modules

MSC 2000: 13C05, 13C13, 16D70

1. Introduction

In this paper all rings are commutative rings with identity and all modules are unital. Several authors have extended the notions of prime ideals to modules (see [1], [2], [12] and [13], for example). Let R be a domain which is not a field. Then R is a prime R-module, but it is not pure-injective (even it is not secondary) and also if p is a fixed prime integer then $E(\mathbb{Z}/p\mathbb{Z})$, the injective hull of the $\mathbb{Z}$-module $\mathbb{Z}/P\mathbb{Z}$, is not prime, but it is pure-injective and representable (see [12, Section 2] and [6, 2.8]).

Let R be the pullback of two local Dedekind domains over a common factor field. The present author classified the indecomposable pure-injective modules (and also indecomposable representable modules) with finite-dimensional top (for any module M we define its top as $M/\operatorname{Rad}(R)M$) over R (see [4], [7]) and showed that, over R, every indecomposable representable module with finite-dimensional top is pure-injective ([7, 3.9]). Here, in fact, we follow the idea of these papers and classify the indecomposable prime modules, and we show that they are pure-injective (see 3.6).

Now we define the concepts that we will need. The first definitions and facts that we quote are from [9].

Let $v_1\colon R_1 \to \overline{R}$ and $v_2\colon R_2 \to \overline{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\overline{R}$. Denote the *pullback*

$$R = \{(r_1, r_2) \in R_1 \oplus R_2\colon\ v_1(r_1) = v_2(r_2)\} \tag{1}$$

by $(R_1 \xrightarrow{v_1} \overline{R} \xleftarrow{v_2} R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \overline{R}) = P = P_1 \times P_2$, $R/P \cong \overline{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). In particular, R is a commutative noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$). Furthermore, for $i \neq j$, the sequence $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [9]).

An R-module S is called *separated* if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$). Equivalently, S is separated if it is the pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [9, Corollary 3.3] and $S \leqslant (S/P_2S) \oplus (S/P_1S)$. Also S is separated if and only if $P_1S \cap P_2S = 0$ [9, Lemma 2.9]. A *separated representation* of an R-module M is an R-module epimorphism $\varphi\colon\ S \to M$ such that S is separated and such that, if φ admits a factorization $\varphi\colon\ S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one.

Let N be an R-submodule of M. Then N is pure in M if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called *relatively divisible* in M (or an RD-submodule of M) if $rN = N \cap rM$ for all $r \in R$. An important property of Dedekind domains is that N is pure in M if and only if N is an RD-submodule of M [8, Theorem 4.5]. A module I is pure-injective if and only if any (infinite) system of equations (allowing infinitely many indeterminates) in I which is finitely solvable in I is solvable in I [15, 2.8].

An R-module M is secondary if $M \neq 0$ and for each $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent. If this is the case, then $J = \mathrm{Rad}(\mathrm{Ann}_R M)$, the radical of $\mathrm{Ann}_R M$, is prime, and we say that M is J-secondary. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such representation exists, we shall say M is representable (see [14]).

Recall that a commutative ring is local if it has a unique maximal ideal. If R is commutative and S is a multiplicative subset of R then we denote by R_S the localization of R with respect to S. If P is a prime ideal and $S = R - P$ we also write R_S as R_P.

If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R\colon\ rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M, $\mathrm{Ann}(M)$. A proper submodule N of a module M over a ring R is said to be a prime submodule if for each $r \in R$ the homothety $M/N \xrightarrow{r.} M/N$ is either injective or zero, so $(0 : M/N) = P$ is a prime ideal of R, and N is said to be a P-prime submodule. We say that M is a prime module if the zero submodule of M is a prime submodule of M, so N is a prime submodule of M if and only if M/N is a prime module. The following lemmas are well-known, but we write them here for the sake of references.

Lemma 1.1 [16, 17]**.** *Let R be a commutative ring. For any module M, the following properties are equivalent:*

(i) *M is prime.*

(ii) *If N is a non-zero submodule of M then $(0 : N) = (0 : M)$.*

(iii) *If $rm = 0$ ($r \in R$, $m \in M$) then either $m = 0$ or $rM = 0$.*

Lemma 1.2 [12, Theorem 1]**.** *Let R be any ring and M any module. Then a submodule N of M is prime if and only if $(N : M) = Q$ is a prime ideal of R and the R/Q-module M/N is torsion-free.*

2. Prime modules over a Dedekind domain

The aim of this section is to classify the prime modules over a Dedekind domain. First we reduce the problem to the local case.

Lemma 2.1. *Let R and R' be any commutative rings, $f\colon R \to R'$ a ring homomorphism, and M an R'-module.*

(i) *If M is a Q-prime R'-module, then M is $f^{-1}(Q)$-prime as an R-module.*

(ii) *If f is surjective and M is prime as an R-module, then M is prime as an R'-module.*

P r o o f. The proof is completely straightforward. □

Lemma 2.2. *Let M be a Q-prime module over a commutative ring R, and let I be an ideal in R such that $I \subseteq Q$. Then M is Q/I-prime as an R/I-module.*

P r o o f. Suppose that $(r + I)m = 0$, where $r \in R$ and $m \in M$, so $rm = 0$. Then either $m = 0$ or $rM = 0$, and hence either $m = 0$ or $(r + I)M = rM = 0$, as required. □

Lemma 2.3. *Let M be a faithful prime module over a domain R. Then $M = QM$ for each maximal ideal Q of R.*

Proof. By 1.2, M is torsion-free and either $QM = M$ or QM is a prime R-submodule of M. If QM is prime in M then by [13, Result 2], $(QM : M) = 0$, a contradiction, so $QM = M$. $\square$

Lemma 2.4. *Let M be a Q-prime module over a commutative ring R. Then $M_Q \neq 0$ and M_Q is Q_Q-prime as an R_Q-module.*

Proof. Let M be Q-prime and $M_Q = 0$. Then for $0 \neq m \in M$ there exists $0 \neq q \notin Q$ such that $qm = 0$. It follows that $m = 0$ since M is prime, a contradiction.

As $QM = 0$ we have $Q_Q M_Q = 0$. Suppose that $(r/q)(m/q') = 0$ and $m/q' \neq 0$. Then there exists $t \notin Q$ such that $trm = 0$. If $r \notin Q$ then $m = 0$, a contradiction. So $r \in Q$, and hence $(r/q)M_Q = 0$ as required. $\square$

Lemma 2.5. *Let R be a Dedekind domain, and let M be a faithful prime module over R. Then R does not occur among the direct summands of M.*

Proof. Suppose that Q is a maximal ideal of R. Then $QM = M$ by 2.3. Let $M = R \oplus T$ for some non-zero submodule T, and let $\pi\colon M \to R$ be the natural projection. Then $QR = \pi(QM) = \pi(M) = R$, a contradiction, as required. $\square$

Lemma 2.6. *Let R be a local Dedekind domain with a maximal ideal Q, and let M be a faithful prime R-module. Then the completion $\hat{R}$ of R in the Q-adic topology, does not occur among the direct summands of M.*

Proof. Let $M = \hat{R} \oplus N$ for some non-zero submodule N of M. Then $\hat{R}$ is a pure submodule of M, so R is pure in M since R is pure in $\hat{R}$ (see [15, p. 48] and [3, pp. 18–22]). As $QM = M$, by the purity of R we have $Q = QR = R \cap QM = R$, a contradiction, as required. $\square$

Proposition 2.7. *Let M be a prime module over a Dedekind domain R. Then there is a prime ideal Q such that the R-module structure of M extends naturally to a structure of M as a prime module over the localization R_Q of R at Q.*

Proof. Let M be a $0 \neq Q$-prime R-module. By 2.4, it is enough to show that every element of $R - Q$ acts invertibly on M. Suppose that $r \in R - Q$. Then the homothety $M \xrightarrow{r.} M$ is injective. Since $(0 : M) = Q$ is a maximal ideal in R, so M is a vector space over R/Q, and hence $(r + Q)M = rM = M$ as requried.

If $(0 : M) = 0$ then by 1.2, M is torsion-free. In fact, if M is a torsion-free R-module, then M can be regarded as a submodule of its localization at the zero ideal. $\square$

Remark 1. Now, by 2.7, we can assume that R is a local Dedekind domain with a unique maximal ideal $Q = Rq$. Clearly, R is indecomposable as an R-module. Here is the list of indecomposable R-modules (see [4, 1.3]):

(1) R;
(2) $\hat{R}$, the completion of R at Q;
(3) R/Q;
(4) $Q(R)$, the quotient field of R.

Proposition 2.8. *Let R be a local Dedekind domain with a maximal ideal $Q = Rq$.*

(i) *If M is a $0 \neq Q$-prime R-module then M is a direct sum of copies of the module as described in* (3) *of the Remark* 1.
(ii) *If $M \neq R, \hat{R}$ is a* (0)*-prime R-module then M is a direct sum of copies of the module as described in* (4) *of the Remark* 1.

Proof. (i) Let T denote an indecomposable direct summand of M. Suppose that $0 \neq t \in T$. Then $(0 : t) = Q$ since any submodule T of M is Q-prime, so $Rt \cong R/Q$. As $qt = 0$, t is not divisible by q in T, so Rt is pure in T. But Rt is itself pure-injective, so Rt is a direct summand of T, and hence $Rt \cong T$ as required.

(ii) Let N denote an indecomposable summand of M. Then by 2.5 and 2.6 we have $N \neq R, \hat{R}$ and $QN = N$. If $0 \neq t \in N$ then $(0 : t) = 0$. From $Q^n N = N$, $n \geqslant 1$, we obtain that t is divisible by every power q^n of q. Thus t is uniquely divisible by every non-zero element of $Q(R)$. So the morphism from the module $Q(R)$ to N given by taking b to bt is well-defined and an isomorphism. Thus we have a copy of the injective module $Q(R)$ embedded in N which must, therefore, be isomorphic to $Q(R)$ (see [6, 2.7]). □

Lemma 2.9. *Let R be a local Dedekind domain with a maximal ideal $Q = Rq$. Then any R-module M in* (1)–(4) *of the Remark* 1 *is prime.*

Proof. Clearly, R/Q is prime. Since R and $\hat{R}$ are torsion-free, so they are prime by 1.2. As $\operatorname{Spec}(Q(R)) = \{(0)\}$, so $Q(R)$ is prime by [13, Theorem 1]. □

Theorem 2.10. *Let R be a local Dedekind domain R with a maximal ideal $Q = Rq$. Let M be an indecomposable prime R-module. Then M is isomorphic to one of the modules listed in Lemma* 2.9.

Proof. We can assume that $M \neq R, \hat{R}$ (because they are indecomposable prime R-modules). If M is an indecomposable (0)-prime R-module then M satisfies the case (4) by 2.8. If M is an indecomposable $0 \neq Q$-prime R-module then M satisfies the case (3) by 2.8. □

Corollary 2.11. *Let R be a Dedekind domain. Then every prime R-module different from R is pure-injective.*

P r o o f. This follows from 2.9 and [4, 1.3]. □

3. The separated case

Throughout this section we shall assume unless otherwise stated, that

$$R = (R_1 \to \overline{R} \leftarrow R_2)$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 respectively, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \oplus R_2/P_2 \oplus R/P \oplus \overline{R}$ is a field.

The purpose of this section is to give a complete description of the prime R-modules where R is the pullback ring as described in (1).

Proposition 3.1. *Let R be a pullback ring as described in* (1). *Then every prime R-module is separated; in fact, if $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ is a separated representation of M with M prime then $S \cong M$.*

P r o o f. Note that the only prime ideals of R are $P_1 \oplus 0$, $0 \oplus P_2$ and P. If $(0 : M) = P_1 \oplus 0$ then $(P_1 \oplus 0)M \cap (0 \oplus P_2)M = 0$, and similarly for $0 \oplus P_2$. If $(0 : M) = P$ then $(P_1 \oplus 0)M \subseteq PM = 0$, so $(P_1 \oplus 0)M \cap (0 \oplus P_2)M = 0$. Thus M is separated by [9, 2.9].

Suppose that M is a prime R-module. Then there is a factorization $\varphi : S \xrightarrow{\varphi} M \xrightarrow{i} M$ (i is the inclusion mapping) with M separated. So $\varphi : S \to M$ is one-to-one, hence $M \cong S$ (see [9, 2.3]). □

Now, by 3.1, it is enough to give a complete description of the prime separated R-modules. According to the definition in Section 1, we have to consider separated R-modules of the form $S = (S_1 \to \overline{S} \leftarrow S_2)$ where, for any $i = 1, 2$, S_i is a module over R_i. Throughout this section, we tacitly assume to deal with R-modules satisfying these assumptions.

Remark 2. Let R be the pullback ring as described in (1), and let $S = (S_1 \to \overline{S} \leftarrow S_2)$ be a separated R-module. Suppose that π_i is the projection map of R onto R_i. Since for each i, $i = 1, 2$, $\pi_i : R \to R_i$ is a ring homomorphism, so if S_i is a Q_i-prime R_i-module then S_i is $\pi_i^{-1}(Q_i)$-prime as an R-module by 2.1.

Proposition 3.2. *Let R be a pullback ring as described in* (1), *and let $S = (S/P_2S = S_1 \overset{f_1}{\twoheadrightarrow} \overline{S} = S/PS \overset{f_2}{\twoheadleftarrow} S_2 = S/P_1S)$ be a separated R-module. Then*

(i) *S is a P-prime R-module if and only if for each i, $i = 1, 2$, $0 \neq S_i$ is a P_i-prime R_i-module.*

(ii) *S is a $(P_1 \oplus 0)$-prime R-module if and only if $S_1 = 0$ and S_2 is a (0)-prime R_2-module.*

(iii) *S is a $(0 \oplus P_2)$-prime R-module if and only if $S_2 = 0$ and S_1 is a (0)-prime R_1-module.*

P r o o f. (i) Suppose that for each i, $i = 1, 2$, $0 \neq S_i$ is a P_i-prime R_i-module. If $rs = (r_1, r_2)(s_1, s_2) = 0$ with $r \in R$ and $s \in S$ then $r_is_i = 0$, $i = 1, 2$, so either $s_i = 0$ or $r_iS_i = 0$.

If $r_1S_1 = 0$ then $r_1 \in P_1$ and $r_2 \in P_2$ (since $v_1(r_1) = v_2(r_2) = 0$), so $r_2S_2 = 0$. It follows that $rS \subseteq r(S_1 \oplus S_2) = 0$.

If $r_1S_1 \neq 0$ then $s_1 = 0$ and $r_1 \notin P_1$, so $r_2S_2 \neq 0$ (since $r_2 \notin P_2$). Thus $s_2 = 0$, and hence S is a P-prime R-module.

Conversely, suppose that S is a P-prime R-module. By 2.2, S_1 is a $P_1 \cong P/(0 \oplus P_2)$-prime $R/(0 \oplus P_2) \cong R_1$-module. Similarly, S_2 is a P_2-prime R_2-module.

(ii) If S is a $(P_1 \oplus 0)$-prime R-module then $(P_1 \oplus 0)S = 0$ and $(0 \oplus P_2)S \neq 0$ since $(0 \oplus P_1) \cap (0 \oplus P_2) = 0$. Suppose that $s_1 \in S_1$. Then there exists $s_2 \in S_2$ such that $f_1(s_1) = f_2(s_2)$, so $(s_1, s_2) \in S$. From $(P_1 \oplus 0)(s_1, s_2) = 0$, we have $P_1s_1 = 0$. It follows that $P(s_1, 0) = 0$, so $s_1 = 0$, and hence $S_1 = 0$. Moreover, by 2.2, $S/(0 \oplus P_2)S = S_2$ is a $(0) \cong (P_1 \oplus 0)/(P_1 \oplus 0)S$-prime $R/(P_1 \oplus 0) \cong R_2$-module. This proves one half of the case (ii), and the other half is obvious by Remark 2.

(iii) This proof is similar to that of case (ii) and we omit it. □

Remark 3. Let R be the pullback ring as described in (1). Here is the list of *indecomposable* separated R-modules (see [4, 2.7]):

(1) $S = (Q(R_1) \twoheadrightarrow 0 \twoheadleftarrow 0)$;

(2) $(0 \twoheadrightarrow 0 \twoheadleftarrow Q(R_2))$;

(3) R/P.

Proposition 3.3. *Let R be a pullback ring as described in* (1).

(i) *If S is a P-prime separated R-module then S is a direct sum of copies of the module described in* (3) *of the Remark* 3.

(ii) *If S is a $(0 \oplus P_2)$-prime R-module then S is a direct sum of copies of the module described in* (1) *of the Remark* 3.

(iii) *If S is a $(P_1 \oplus 0)$-prime R-module then S is a direct sum of copies of the module described in* (2) *of the Remark* 3.

P r o o f. (i) Let T denote an indecomposable summand of S. So T is separated and we can write $T = (T_1 \to \overline{T} \leftarrow T_2)$. Since T is P-prime (because every submodule of a P-prime module is P-prime), $PT = 0$, so $PT \neq T$ (that is, $\overline{T} \neq 0$). Thus by 3.2 (i), for each i, T_i is a P_i-prime R_i-module and $P_1T_1 = 0$, $P_2T_2 = 0$.

For $t \in T$, let $o(t)$ denote the least positive integer m such that $P^m t = 0$. Now choose $t \in T_1 \cup T_2$ with $\bar{t} \neq 0$ and such that $o(t)$ is minimal (given that $\bar{t} \neq 0$). Clearly, there is a $t = (t_1, t_2)$ such that $o(t) = 1$, $o(t_1) = 1$ and $o(t_2) = 1$. Then R_it_i is pure in T_i, $i = 1, 2$ (see [4, 2.9]). Moreover, $R_1t_1 \cong R_1/\operatorname{Ann}_{R_1} t_1 \cong R_1/P_1$ is a direct summand of T_1 since R_1t_1 is pure-injective. Similarly, $R_2t_2 \cong R_2/P_2$ is pure in T_2, and hence it is a direct summand of T_2 . Let $\overline{M}$ be the $\overline{R}$-subspace of $\overline{T}$ generated by $\bar{t}$. Then $\overline{M} \cong \overline{R}$. Let $M = (R_1t_1 = M_1 \longrightarrow \overline{M} \longleftarrow M_2 = R_2t_2)$. Then M is a direct summand of T; this implies that $T = M$, and T belongs to the case (3) (see [4, 2.9]).

(ii) Let S be a separated $(0 \oplus P_2)$-prime R-module. Then by 3.2 (iii), $S_2 = 0 = \overline{S}$ and S_1 is a (0)-prime R_1-module, so S_1 is a direct sum of copies of $Q(R_1)$ by 2.8 (ii), as required.

(iii) This proof is similar to that of case (ii) and we omit it. $\square$

Lemma 3.4. *Let R be a pullback ring as described in* (1). *Then any R-module S in* (1)–(3) *is prime.*

P r o o f. It is clear that the modules described in (1) are $(0 \oplus P_2)$-prime and the modules described in (2) are $(P_1 \oplus 0)$-prime. To see this consider, for example, $Q(R_1)$ (that is, $(Q(R_1) \to 0 \leftarrow 0)$). By 2.9, since $Q(R_1)$ is a 0-prime R_1-module, so by Remark 2, it is a $(0 \oplus P_2) = \pi_1^{-1}(0)$-prime R-module. By 3.2, the module as described in (3) is P-prime since the R_i-modules occuring as components are P_i-prime R_i-modules. $\square$

Theorem 3.5. *Let R be a pullback ring as described in* (1). *Let S be an indecomposable prime separated R-module. Then S is isomorphic to one of the modules listed in Lemma* 3.4.

P r o o f. Let $S = (S_1 \to \overline{S} \leftarrow S_2)$ be an indecomposable prime separated R-module such that, for $i = 1, 2$, S_i is a module over R_i. Note that the only prime ideals of R are P, $(0 \oplus P_2)$, and $(P_1 \oplus 0)$.

If S is $(P_1 \oplus 0)$-prime then $PS = (0 \oplus P_2)S = S$. Similarly for $0 \oplus P_2$. Then $S = PS = P_1S_1 \oplus P_2S_2 = S_1 \oplus S_2$, so $S = S_1$ or S_2. Thus S is an indecomposable prime R_i-module for some i and, since $PS = S$, it is of type (1) or (2) from the above list by 3.3. So we may assume that S is P-prime. Since S is an indecomposable P-prime R-module, S belongs to the case (3) by 3.3. $\square$

Corollary 3.6. *Let R be a pullback ring as described in* (1). *Then every prime R-module is pure-injective.*

P r o o f. This follows from [4, 2.9]. □

Corollary 3.7. *Let R be a pullback ring as decribed in* (1). *Then every prime R-module is representable.*

P r o o f. This follows from [7, 2.5]. □

Remark 4. Recall that for a given field k, the infinite-dimensional k-algebra $R = k[x, y : xy = 0]_{(x,y)}$ is the pullback $(k[x]_{(x)} \to k \leftarrow k[y]_{(y)})$ of the local Dedekind domains $k[x]_{(x)}, k[y]_{(y)}$. This paper includes the classification of indecomposable primes over $k[x, y : xy = 0]_{(x,y)}$.

References

[1] *J. Dauns*: Prime modules. Reine angew Math. *298* (1978), 156–181.
[2] *J. Dauns*: Prime modules and one-sided ideals Proc. Third Oklahoma Conf. Ring Theory and Algebra. 1980, pp. 301–303.
[3] *J. Dauns*: Modules and Rings. Cambridge University Press, Cambridge, 1994.
[4] *S. Ebrahimi Atani*: On pure-injective modules over pullback rings. Comm. Algebra *28* (2000), 4037–4069.
[5] *S. Ebrahimi Atani*: I-Torsion-free modules over pullback rings. Honam Math. J. *22* (2000), 1–7.
[6] *S. Ebrahimi Atani*: On secondary modules over Dedekind domains. South-east Asian Bulletin *25* (2001), 1–6.
[7] *S. Ebrahimi Atani*: On secondary modules over pullback rings. Comm. Algebra *30* (2002), 2675–2685.
[8] *L. Fuchs and L. Salce*: Modules over Valuation Domains. Lecture Notes in Pure and Applied Mathematics, 1985.
[9] *L. Levy*: Modules over pullbacks and subdirect sums. J. Algebra *71* (1981), 50–61.
[10] *L. Levy*: Mixed modules over $\mathbb{Z}G$, G cyclic of prime order, and over related Dedekind pullbacks. J. Algebra *71* (1981), 62–114.
[11] *L. Levy*: Modules over Dedekind-like rings. J. Algebra *93* (1981), 1–116.
[12] *C. P. Lu*: Prime submodules of modules. Comm. Univ. Sancti. Paulli *33* (1984), 61–69.
[13] *C. P. Lu*: Spectra of modules. Comm. Algebra *23* (1995), 3741–3752.
[14] *I. G. Macdonald*: Secondary representation of modules over a commutative ring. Symposia Matematica 11 (Istituto Nazionale di alta Matematica), Roma, 1973, pp. 23–43.
[15] *M. Prest*: Model Theory and Modules. London Math. Soc. Lecture Note Series. Cambridge University Press, 1988.
[16] *Y. Tiras, A. Tercan and A. Harmanci*: Prime modules. Honam Math. J. *18* (1996), 5–15.
[17] *S. Yassemi*: The dual notion of prime submodules. Arch. Math. *37* (2001), 273–278.

Author's address: Guilan University, Department of Mathematics, P. O. Box 1914, Rasht, Iran, e-mail: `ebrahimi@guilan.ac.ir`.