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TENSOR PRODUCTS OF HILBERT MODULES OVER LOCALLY C^*-ALGEBRAS

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Abstract. In this paper the tensor products of Hilbert modules over locally C^*-algebras are defined and their properties are studied. Thus we show that most of the basic properties of the tensor products of Hilbert C^*-modules are also valid in the context of Hilbert modules over locally C^*-algebras.

Keywords: locally C^*-algebras, continuous $*$-morphism, inverse system of Hilbert C^*-modules, exterior tensor product of Hilbert modules, interior tensor product of Hilbert modules

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1. Introduction

Hilbert modules over locally C^*-algebras generalize the notion of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra. They were first considered independently by A. Mallios in [7] and N. C. Phillips in [8], where the latter showed that most of the basic properties of Hilbert C^*-modules are valid for Hilbert modules over locally C^*-algebras. The Hilbert modules over locally C^*-algebras are also studied in [4], [5] and elsewhere. Thus in [4] the present author proved a stabilization theorem for countably generated Hilbert modules over locally C^*-algebras and in [5] she proved a version of the classical KSGNS (Kasparov, Stinespring, Gel'fand, Segal, Naimark) construction in the context of Hilbert modules over locally C^*-algebras.

In this paper we will define the exterior tensor product and the interior tensor product of Hilbert modules over locally C^*-algebras and we will show that some properties of the tensor products of Hilbert C^*-modules are valid in the context of Hilbert modules over locally C^*-algebras.

2. Preliminaries

A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C^*-seminorm p on A.

If A is a locally C^*-algebra and $S(A)$ is the set of all continuous C^*-seminorms on A, then for each $p\in S(A)$, $A_p=A/\ker(p)$ is a C^*-algebra in the norm induced by p and $A=\varprojlim_p A_p$. The canonical map from A onto A_p, $p\in S(A)$, will be denoted by π_p, and the image of a under π_p will be denoted by a_p. The connecting maps of the inverse system $\{A_p\}_{p\in S(A)}$ will be denoted by π_{pq}, $q,p\in S(A)$, $p\geqslant q$.

A continuous $*$-morphism φ from A into $L(H)$, the C^*-algebra of all bounded linear operators on the Hilbert space H, is called a $*$-representation of A on H. If A and B are locally C^*-algebras we will denote by $A\otimes B$ the injective tensor product of A and B which is the completion of $A\otimes_{\mathrm{alg}} B$ in the topology induced by the family of C^*-seminorms $\{\vartheta_{(p,q)}\}_{(p,q)\in S(A)\times S(B)}$, where $\vartheta_{(p,q)}(c)=\sup\{\|((\varphi\otimes\psi)\circ(\pi_p\otimes\pi_q))(c)\|;\ \varphi$ is a $*$-representation of A_p and ψ is a $*$-representation of $B_q\}$. Moreover, $A\otimes B=\varprojlim_{(p,q)} A_p\otimes B_q$, where $A_p\otimes B_q$ is the injective tensor product of the C^*-algebras A_p and B_q (see [1]).

Now we recall some results about Hilbert modules over locally C^*-algebras from [8].

Definition 2.1. A pre-Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle\colon E\times E\to A$ which is $\mathbb{C}$- and A-linear in its second variable and satisfies the following relations:

(i) $\langle x,y\rangle^*=\langle y,x\rangle$ for every $x,y\in E$;

(ii) $\langle x,x\rangle\geqslant 0$ for every $x\in E$;

(iii) $\langle x,x\rangle=0$ if and only if $x=0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\|x\|_p=\sqrt{p(\langle x,x\rangle)}$, $x\in E$, $p\in S(A)$.

Given a Hilbert A-module E, then for $p\in S(A)$, $N_p^E=\{x\in E;\ p(\langle x,x\rangle)=0\}$ is a closed submodule of E and $E_p=E/N_p^E$ is a Hilbert A_p-module with $(x+N_p^E)\pi_p(a)=xa+N_p^E$ and $\langle x+N_p^E,y+N_p^E\rangle=\pi_p(\langle x,y\rangle)$. The canonical map from E onto E_p, $p\in S(A)$, will be denoted by σ_p^E, and the image of x under σ_p^E will be denoted by x_p.

For $p,q\in S(A)$, $p\geqslant q$ there is a canonical surjective linear map $\sigma_{pq}^E\colon E_p\to E_q$ such that $\sigma_{pq}^E(x_p)=x_q$, $x_p\in E_p$. Then $\{E_p;A_p;\ \sigma_{pq}^E,p\geqslant q,p,q\in S(A)\}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}^E(x_pa_p)=\sigma_{pq}^E(x_p)\pi_{pq}(a_p)$ for every $x_p\in E_p$ and for every $a_p\in A_p$; $\langle\sigma_{pq}^E(x_p),\sigma_{pq}^E(y_p)\rangle=\pi_{pq}(\langle x_p,y_p\rangle)$ for every

$x_p, y_p \in E_p$; $\sigma^E_{qr} \circ \sigma^E_{pq} = \sigma^E_{pr}$, $p \geqslant q \geqslant r$; $\sigma^E_{pp} = \mathrm{id}_{E_p}$, and $\varprojlim_p E_p$ is a Hilbert A-module with $((x_p)_p)((a_p)_p) = (x_p a_p)_p$ and $\langle (x_p)_p, (y_p)_p \rangle = (\langle x_p, y_p \rangle)_p$. Moreover, $\varprojlim_p E_p$ may be identified with E.

As in the case of the C^*-algebras, the set H_A of all sequences $(a_n)_n$ with a_n in A such that $\sum\limits_n a_n^* a_n$ converges in A is a Hilbert A-module with $((a_n)_n)b = (a_n b)_n$ and $\langle (a_n)_n, (b_n)_n \rangle = \sum\limits_n a_n^* b_n$. Moreover, for each $p \in S(A)$, $(H_A)_p = H_{A_p}$.

Given Hilbert A-modules E and F, a map $T\colon E \to F$ is adjointable if there is a map $T^*\colon F \to E$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in E$ and for all $y \in F$. Moreover, T is a $\mathbb{C}$- and A-linear continuous map. We denote by $L_A(E, F)$ the set of all adjointable maps from E into F and write $L_A(E)$ for $L_A(E, E)$.

For $p \in S(A)$, since $T\big(N^E_p\big) \subseteq N^F_p$ for all $T \in L_A(E, F)$, we can consider the linear map $(\pi_p)_*\colon L_A(E, F) \to L_{A_p}(E_p, F_p)$ defined by $(\pi_p)_*(T)\big(\sigma^E_p(x)\big) = \sigma^F_p(T(x))$, $T \in L(E, F)$, $x \in E$.

We topologize $L_A(E, F)$ via the seminorms $\widetilde{p}(T) = \|(\pi_p)_*(T)\|$, $T \in L_A(E, F)$, $p \in S(A)$. In this way $L_A(E, F)$ may be identified with $\varprojlim_p L_{A_p}(E_p, F_p)$ and $L_A(E)$ becomes a locally C^*-algebra. The connecting maps of the inverse system $\{L_{A_p}(E_p, F_p)\}_{p \in S(A)}$ will be denoted by $(\pi_{pq})_*$, $p, q \in S(A)$, $p \geqslant q$ and $(\pi_{pq})_*(T_p)\big(\sigma^E_q(x)\big) = \sigma^F_{pq}\big(T_p\big(\sigma^E_p(x)\big)\big)$, $T_p \in L_{A_p}(E_p, F_p)$, $x \in E$. For $x \in E$ and $y \in F$ we consider the rank one homomorphism $\theta_{y,x}$ from E into F defined by $\theta_{y,x}(z) = y\langle x, z \rangle$. Evidently, $\theta_{y,x} \in L_A(E, F)$ and $\theta^*_{y,x} = \theta_{x,y}$. We denote by $K_A(E, F)$ the closed linear subspace of $L_A(E, F)$ spanned by $\{\theta_{y,x};\ x \in E, y \in F\}$, and write $K_A(E)$ for $K_A(E, E)$. Moreover, $K_A(E, F)$ may be identified with $\varprojlim_p K_{A_p}(E_p, F_p)$.

We say that the Hilbert A-modules E and F are unitarily equivalent if there is a unitary element U in $L_A(E, F)$ (namely, $U^*U = \mathrm{id}_E$ and $UU^* = \mathrm{id}_F$).

3. Exterior tensor product

Let A and B be locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module. The algebraic tensor product $E \otimes_{\mathrm{alg}} F$ is a right-module over $A \otimes_{\mathrm{alg}} B$ in the obvious way: $(x \otimes y)(a \otimes b) = xa \otimes yb$, $x \in E$, $y \in F$, $a \in A$, $b \in B$.

We consider the map $\langle \cdot, \cdot \rangle\colon (E \otimes_{\mathrm{alg}} F) \times (E \otimes_{\mathrm{alg}} F) \to A \otimes_{\mathrm{alg}} B$ defined by

$$\left\langle \sum_{i=1}^n x_i \otimes z_i, \sum_{j=1}^m y_j \otimes t_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^m \langle x_i, y_j \rangle \otimes \langle z_i, t_j \rangle.$$

In the same way as in the case of the Hilbert C^*-modules (see, for example, [6, Chapter 4]), using [4, Theorem 6], we show that this map defines an inner product on $E \otimes_{\rm alg} F$. Since $A \otimes_{\rm alg} B$ is dense in $A \otimes B$, $E \otimes_{\rm alg} F$ becomes a pre-Hilbert $A \otimes B$-module. We denote by $E \otimes F$ the completion of $E \otimes_{\rm alg} F$. We call $E \otimes F$ the exterior tensor product of E and F.

Remark 3.1. If B is a locally C^*-algebra and H is a separable infinite dimensional Hilbert space, then exactly as in the case of the Hilbert C^*-modules we deduce that the Hilbert B-modules $H \otimes B$ and H_B are unitarily equivalent.

For $p \in S(A)$ and $q \in S(B)$ we denote by $E_p \otimes F_q$ the exterior tensor product of the Hilbert C^*-modules E_p and F_q.

Let $p_1, p_2 \in S(A)$, $p_1 \geqslant p_2$ and $q_1, q_2 \in S(B)$, $q_1 \geqslant q_2$. Then the linear map $\sigma^E_{p_1p_2} \otimes \sigma^F_{q_1q_2}\colon E_{p_1} \otimes_{\rm alg} F_{q_1} \to E_{p_2} \otimes_{\rm alg} F_{q_2}$ defined by $\bigl(\sigma^E_{p_1p_2} \otimes \sigma^F_{q_1q_2}\bigr)(x_{p_1} \otimes y_{q_1}) = \sigma^E_{p_1p_2}(x_{p_1}) \otimes \sigma^F_{q_1q_2}(y_{q_1})$ may be extended by continuity to a linear map $\sigma^E_{p_1p_2} \otimes \sigma^F_{q_1q_2}$ from $E_{p_1} \otimes F_{q_1}$ into $E_{p_2} \otimes F_{q_2}$. It is easy to verify that $\{E_p \otimes F_q;\ A_p \otimes B_q;\ \sigma^E_{p_1p_2} \otimes \sigma^F_{q_1q_2},\ p_1, p_2 \in S(A),\ p_1 \geqslant p_2,\ q_1, q_2 \in S(B),\ q_1 \geqslant q_2\}$ is an inverse system of Hilbert C^*-modules. We will show that the Hilbert $A \otimes B$-modules $E \otimes F$ and $\varprojlim_{(p,q)} (E_p \otimes F_q)$ are unitarily equivalent.

Proposition 3.2. *Let A, B, E and F be as above. Then the Hilbert $A \otimes B$-modules $E \otimes F$ and $\varprojlim_{(p,q)} (E_p \otimes F_q)$ are unitarily equivalent.*

Proof. First we will show that for each $p \in S(A)$ and $q \in S(B)$ the Hilbert $A_p \otimes B_q$-modules $(E \otimes F)_{(p,q)}$ and $E_p \otimes F_q$ are unitarily equivalent.

Let $p \in S(A)$ and $q \in S(B)$. Since

$$\begin{aligned}\vartheta_{(p,q)}(\langle x \otimes y, x \otimes y\rangle) &= \|\pi_p(\langle x, x\rangle) \otimes \pi_q(\langle y, y\rangle)\|_{A_p \otimes B_q}\\ &= \bigl\|\bigl\langle \sigma^E_p(x), \sigma^E_p(x)\bigr\rangle \otimes \bigl\langle \sigma^F_q(y), \sigma^F_q(y)\bigr\rangle\bigr\|_{A_p \otimes B_q}\\ &= \bigl\|\bigl\langle \sigma^E_p(x) \otimes \sigma^F_q(y), \sigma^E_p(x) \otimes \sigma^F_q(y)\bigr\rangle\bigr\|_{A_p \otimes B_q}\end{aligned}$$

for all $x \in E$ and $y \in F$, we can define a linear map $U_{(p,q)}\colon (E \otimes_{\rm alg} F)/N^{E\otimes F}_{(p,q)} \to E_p \otimes_{\rm alg} F_q$ by

$$U_{(p,q)}\bigl(x \otimes y + N^{E\otimes F}_{(p,q)}\bigr) = \sigma^E_p(x) \otimes \sigma^F_q(y).$$

Evidently $U_{(p,q)}$ is a surjective $A_p \otimes_{\rm alg} B_q$-linear map and

$$\left\|U_{(p,q)}\left(\sum_{i=1}^n x_i \otimes y_i + N^{E\otimes F}_{(p,q)}\right)\right\|_{E_p \otimes F_q} = \left\|\sum_{i=1}^n x_i \otimes y_i + N^{E\otimes F}_{(p,q)}\right\|_{(E\otimes F)_{(p,q)}}$$

for all $\sum\limits_{i=1}^n x_i \otimes y_i \in E \otimes_{\rm alg} F$. From these facts, taking into account that $A_p \otimes_{\rm alg} B_q$ is dense in $A_p \otimes B_q$; $(E \otimes_{\rm alg} F)/N^{E\otimes F}_{(p,q)}$ is dense in $(E \otimes F)_{(p,q)}$ and $E_p \otimes_{\rm alg} F_q$ is dense in $E_p \otimes F_q$, we conclude that $U_{(p,q)}$ may be extended by continuity to an isometric surjective $A_p \otimes B_q$-linear map $U_{(p,q)}$ from $(E \otimes F)_{(p,q)}$ onto $E_p \otimes F_q$. According to [6, Theorem 3.5], $U_{(p,q)}$ is a unitary element in $L_{A_p \otimes B_q}((E \otimes F)_{(p,q)}, E_p \otimes F_q)$.

It is easy to verify that $\big(\sigma^E_{p_1 p_2} \otimes \sigma^F_{q_1 q_2}\big) \circ U_{(p_1,q_1)} = U_{(p_2,q_2)} \circ \sigma^{E \otimes F}_{(p_1,q_1)(p_2,q_2)}$ and $(U_{(p_2,q_2)})^* \circ (\sigma^E_{p_1 p_2} \otimes \sigma^F_{q_1 q_2}) = \sigma^{E \otimes F}_{(p_1,q_1)(p_2,q_2)} \circ (U_{(p_1,q_1)})^*$ for all $p_1, p_2 \in S(A)$, $p_1 \geqslant p_2$ and $q_1, q_2 \in S(B)$, $q_1 \geqslant q_2$. Therefore $(U_{(p,q)})_{(p,q) \in S(A) \times S(B)}$ is an inverse system of adjointable maps of Hilbert C^*-modules.

Let $U = \lim\limits_{\longleftarrow (p,q)} U_{(p,q)}$. It is easy to see that U is an adjointable map from $\lim\limits_{\longleftarrow (p,q)} (E \otimes F)_{(p,q)}$ into $\lim\limits_{\longleftarrow (p,q)} (E_p \otimes F_q)$ and $U^* = \lim\limits_{\longleftarrow (p,q)} (U_{(p,q)})^*$. Therefore U is a unitary element in $L_{A \otimes B}\Big(\lim\limits_{\longleftarrow (p,q)} (E \otimes F)_{(p,q)}, \lim\limits_{\longleftarrow (p,q)} (E_p \otimes F_q)\Big)$ and Proposition 3.2 is proved. □

Using the above and [8, Theorem 4.2], we obtain:

Corollary 3.3. *Let A and B be locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module. Then the locally C^*-algebras $L_{A \otimes B}(E \otimes F)$ and $\lim\limits_{\longleftarrow (p,q)} L_{A_p \otimes B_q}(E_p \otimes F_q)$ as well as $K_{A \otimes B}(E \otimes F)$ and $\lim\limits_{\longleftarrow (p,q)} K_{A_p \otimes B_q}(E_p \otimes F_q)$ are isomorphic.*

Proposition 3.4. *Let A and B be locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module. Then there is a continuous $*$-morphism j from $L_A(E) \otimes L_B(F)$ into $L_{A \otimes B}(E \otimes F)$ such that*

$$j(T \otimes S)(x \otimes y) = Tx \otimes Sy, \quad T \in L_A(E), \ S \in L_B(F), \ x \in E, \ y \in F.$$

Moreover, j is injective and $j(K_A(E) \otimes K_B(F)) = K_{A \otimes B}(E \otimes F)$.

P r o o f. Let $p \in S(A)$ and $q \in S(B)$. Then, since A_p and B_q are C^*-algebras, E_p is a Hilbert A_p-module and F_q is a Hilbert B_q-module, there is an injective morphism of C^*-algebras $j_{(p,q)}$ from $L_{A_p}(E_p) \otimes L_{B_q}(F_q)$ into $L_{A_p \otimes B_q}(E_p \otimes F_q)$ such that

$$j_{(p,q)}(T_p \otimes S_q)(x_p \otimes y_q) = T_p x_p \otimes S_q y_q$$

for all $T_p \in L_{A_p}(E_p)$, $S_q \in L_{B_q}(F_q)$, $x_p \in E_p$, $y_q \in F_q$ and

$$j_{(p,q)}(K_{A_p}(E_p) \otimes K_{B_q}(F_q)) = K_{A_p \otimes B_q}(E_p \otimes F_q)$$

(see, for instance, [6, pp. 35–37]).

It is easy to verify that

$$j_{(p_2,q_2)} \circ ((\pi_{p_1 p_2})_* \otimes (\pi_{q_1 q_2})_*) = (\pi_{(p_1,q_1)(p_2,q_2)})_* \circ j_{(p_1,q_1)}$$

for all $p_1, p_2 \in S(A)$, $p_1 \geqslant p_2$ and $q_1, q_2 \in S(B)$, $q_1 \geqslant q_2$. Then $(j_{(p,q)})_{(p,q)\in S(A)\times S(B)}$ is an inverse system of morphisms of C^*-algebras. Let $j = \lim\limits_{\longleftarrow (p,q)} j_{(p,q)}$. Evidently j is an injective continuous $*$-morphism from $L_A(E)\otimes L_B(F)$ into $L_{A\otimes B}(E\otimes F)$ and

$$j(T\otimes S)(x\otimes y) = Tx\otimes Sy, \quad T\in L_A(E),\ \ S\in L_B(F),\ \ x\in E,\ \ y\in F.$$

Now, since

- for each $p\in S(A)$ and for each $q\in S(B)$,

$$j_{(p,q)}|_{K_{A_p}(E_p)\otimes K_{B_q}(F_q)}\colon\ K_{A_p}(E_p)\otimes K_{B_q}(F_q)\to K_{A_p\otimes B_q}(E_p\otimes F_q)$$

  is an isomorphism of C^*-algebras;
- $K_A(E)\otimes K_B(F) = \lim\limits_{\longleftarrow (p,q)} K_{A_p}(E_p)\otimes K_{B_q}(F_q)$

and

- $K_{A\otimes B}(E\otimes F) = \lim\limits_{\longleftarrow (p,q)} K_{A_p\otimes B_q}(E_p\otimes F_q)$,

we deduce that $j(K_A(E)\otimes K_B(F)) = K_{A\otimes B}(E\otimes F)$. □

4. Interior tensor product

Let A and B be locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi\colon A\to L_B(F)$ be a continuous $*$-morphism. We can regard F as a left A-module, the action being given by $(a,y)\to\Phi(a)y$, $a\in A$, $y\in F$, and we can form the algebraic tensor product of E and F over A, $E\otimes_A F$. It is the quotient of the vector space tensor product $E\otimes_{\mathrm{alg}} F$ by the vector subspace N_Φ generated by elements of the form $xa\otimes y - x\otimes\Phi(a)y$, $a\in A$, $x\in E$, $y\in F$. Now, $E\otimes_A F$ is a right B-module in the obvious way, the action of B being given by $(x\otimes y + N_\Phi, b)\to x\otimes yb + N_\Phi$, $b\in B$, $x\in E$, $y\in F$.

Exactly as in the case of the Hilbert C^*-modules, we show:

Proposition 4.1. *Let A, B, E, F and Φ be as above. Then $E\otimes_A F$ is a pre-Hilbert B-module with the inner product given by*

$$\langle x\otimes y, z\otimes t\rangle = \langle y, \Phi(\langle x,z\rangle)t\rangle, \quad x\in E,\ \ y\in F.$$

In the particular case when $F=B$, this proposition was proved in [8, pp. 181].

We denote by $E\otimes_\Phi F$ the completion of $E\otimes_A F$. We call $E\otimes_\Phi F$ the interior tensor product of E and F using Φ. For the element $x\otimes y + N_\Phi$ we use the notation $x\dot{\otimes} y$.

For each $q \in S(B)$, the map $\Phi_q\colon A \to L_{B_q}(F_q)$ defined by $\Phi_q = (\pi_q)_* \circ \Phi$ is a continuous $*$-morphism.

Let $q_1, q_2 \in S(B)$, $q_1 \geqslant q_2$. Define a linear map $\psi_{q_1 q_2}\colon E \otimes_{\text{alg}} F_{q_1} \to E \otimes_{\text{alg}} F_{q_2}$ by

$$\psi_{q_1 q_2}(x \otimes y_{q_1}) = x \otimes \sigma^F_{q_1 q_2}(y_{q_1}).$$

Since

$$\begin{aligned}
\langle \psi_{q_1 q_2}(x \otimes y_{q_1}), \psi_{q_1 q_2}(x \otimes y_{q_1}) \rangle &= \left\langle \sigma^F_{q_1 q_2}(y_{q_1}), \Phi_{q_2}(\langle x, x\rangle) \sigma^F_{q_1 q_2}(y_{q_1}) \right\rangle \\
&= \left\langle \sigma^F_{q_1 q_2}(y_{q_1}), (\pi_{q_2})_*(\Phi(\langle x, x\rangle)) \sigma^F_{q_1 q_2}(y_{q_1}) \right\rangle \\
&= \left\langle \sigma^F_{q_1 q_2}(y_{q_1}), \sigma^F_{q_1 q_2}((\pi_{q_1})_*(\Phi(\langle x, x\rangle)) y_{q_1}) \right\rangle \\
&= \pi_{q_1 q_2}(\langle y_{q_1}, \Phi_{q_1}(\langle x, x\rangle) y_{q_1} \rangle) \\
&= \pi_{q_1 q_2}(\langle x \otimes y_{q_1}, x \otimes y_{q_1} \rangle)
\end{aligned}$$

for all $x \in E$ and $y_{q_1} \in F_{q_1}$, $\psi_{q_1 q_2}$ may be extended to a linear map $\psi_{q_1 q_2}\colon E \otimes_{\Phi_{q_1}} F_{q_1} \to E \otimes_{\Phi_{q_2}} F_{q_2}$ such that

$$\psi_{q_1 q_2}(x \dot{\otimes} y_{q_1}) = x \dot{\otimes} \sigma^F_{q_1 q_2}(y_{q_1}).$$

Proposition 4.2. *Let A, B, E, F and Φ be as above. Then*

$$\{E \otimes_{\Phi_q} F_q;\ B_q;\ \psi_{q_1 q_2}, q_1 \geqslant q_2, q_1, q_2 \in S(B)\}$$

is an inverse system of Hilbert C^*-modules, and the Hilbert B-modules $E \otimes_\Phi F$ and $\varprojlim_q (E \otimes_{\Phi_q} F_q)$ are unitarily equivalent.

Proof. The fact that $\{E \otimes_{\Phi_q} F_q;\ B_q \psi_{q_1 q_2},\ q_1 \geqslant q_2,\ q_1, q_2 \in S(B)\}$ is an inverse system of Hilbert C^*-modules is a simple verification.

To show that the Hilbert B-modules $E \otimes_\Phi F$ and $\varprojlim_q (E \otimes_{\Phi_q} F_q)$ are unitarily equivalent, first we will show that for each $q \in S(B)$ the Hilbert B_q-modules $(E \otimes_\Phi F)_q$ and $E \otimes_{\Phi_q} F_q$ are unitarily equivalent.

Let $q \in S(B)$. Define a linear map $U_q\colon E \otimes_{\text{alg}} F \to E \otimes_{\text{alg}} F_q$ by

$$U_q(x \otimes y) = x \otimes \sigma^F_q(y),\ x \in E,\ y \in F.$$

Since

$$\begin{aligned}
\langle U_q(x \otimes y), U_q(x \otimes y) \rangle &= \left\langle \sigma^F_q(y), \Phi_q(\langle x, x\rangle) \sigma^F_q(y) \right\rangle \\
&= \left\langle \sigma^F_q(y), (\pi_q)_*(\Phi(\langle x, x\rangle)) \sigma^F_q(y) \right\rangle \\
&= \left\langle \sigma^F_q(y), \sigma^F_q(\Phi(\langle x, x\rangle) y) \right\rangle \\
&= \pi_q(\langle x \otimes y, x \otimes y \rangle)
\end{aligned}$$

for all $x \in E$ and $y \in F$, U_q may be extended by continuity to an isometric B_q-linear map $U_q\colon (E \otimes_\Phi F)_q \to E \otimes_{\Phi_q} F_q$ such that

$$U_q(x\dot{\otimes}y) = x\dot{\otimes}\sigma_q^F(y), \quad x \in E, \; y \in F$$

and, moreover, it is surjective. Then according to [6, Theorem 3.5], U_q is a unitary element in $L_{B_q}((E \otimes_\Phi F)_q, E \otimes_{\Phi_q} F_q)$. It is easy to verify that $(U_q)_{q \in S(B)}$ is an inverse system of adjointable maps of Hilbert C^*-modules.

Let $U = \varprojlim_q U_q$. A simple calculation shows that U is a unitary element in $L_B\Big(\varprojlim_q (E \otimes_\Phi F)_q, \varprojlim_q (E \otimes_{\Phi_q} F_q)\Big)$. Therefore the Hilbert B-modules $E \otimes_\Phi F$ and $\varprojlim_q (E \otimes_{\Phi_q} F_q)$ are unitarily equivalent. □

Corollary 4.3. *Let A, B, E, F and Φ be as above. Then the locally C^*-algebras $L_B(E \otimes_\Phi F)$ and $\varprojlim_q L_{B_q}(E \otimes_{\Phi_q} F_q)$ as well as $K_B(E \otimes_\Phi F)$ and $\varprojlim_q K_{B_q}(E \otimes_{\Phi_q} F_q)$ are isomorphic.*

Proposition 4.4. *Let A and B be locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi\colon A \to L_B(F)$ be a continuous $*$-morphism.*

1. *Then there is a continuous $*$-morphism $\Phi_*\colon L_A(E) \to L_B(E \otimes_\Phi F)$ such that*

$$\Phi_*(T)(x \mathbin{\dot{\otimes}} y) = T(x) \mathbin{\dot{\otimes}} y, \quad x \in E, \; y \in F, \; T \in L_A(E).$$

   Moreover, if Φ is injective, then Φ_* is injective.

2. *If $\Phi(A) \subseteq K_B(F)$, then $\Phi_*(K_A(E)) \subseteq K_B(E \otimes_\Phi F)$. Moreover, if $\Phi(A)$ is dense in $K_A(F)$, then $\Phi_*(K_A(E))$ is dense in $K_B(E \otimes_\Phi F)$.*

Proof. First we suppose that B is a C^*-algebra.

(1) The continuity of Φ implies that there is a continuous $*$-morphism $\Psi_p\colon A_p \to L_B(F)$ such that $\Psi_p \circ \pi_p = \Phi$. Then, since A_p and B are C^*-algebras and $\Psi_p\colon A_p \to L_B(F)$ is a morphism of C^*-algebras, there is a morphism of C^*-algebras $(\Psi_p)_*\colon L_{A_p}(E_p) \to L_B(E_p \otimes_{\Psi_p} F)$ such that $(\Psi_p)_*(T_p)\big(\sigma_p^E(x) \mathbin{\dot{\otimes}} y\big) = T_p\big(\sigma_p^E(x)\big) \mathbin{\dot{\otimes}} y$ (see, for instance, [6, pp. 42–43]). It is easy to verify that the linear map $U\colon E \otimes_\Phi F \to E_p \otimes_{\Psi_p} F$ defined by $U(x\dot{\otimes}y) = \sigma_p^E(x)\dot{\otimes}y$ is a unitary element in $L_B(E \otimes_\Phi F, E_p \otimes_{\Psi_p} F)$ and the map $\Phi_*\colon L_A(E) \to L_B(E \otimes_\Phi F)$ defined by $\Phi_*(T) = U^* \circ (\Psi_p)_*((\pi_p)_*(T)) \circ U$ is a continuous $*$-morphism and

$$\Phi_*(T)(x \mathbin{\dot{\otimes}} y) = T(x) \mathbin{\dot{\otimes}} y, \quad x \in E, \; y \in F, \; T \in L_A(E).$$

If Φ is injective, then it is easy to see that Φ_* is injective.

(2) If $\Phi(A) \subseteq K_B(F)$, then $\Psi_p(A_p) \subseteq K_B(F)$ and according to [6, Proposition 4.7], $(\Psi_p)_*(K_{A_p}(E_p)) \subseteq K_B(E_p \otimes_{\Psi_p} F)$. Since $(\pi_p)_*(K_A(E)) = K_{A_p}(E_p)$, it is easy to see that $\Phi_*(K_A(E)) \subseteq K_B(E \otimes_\Phi F)$.

If $\Phi(A)$ is dense in $K_A(F)$ then $\Psi_p(A_p) = K_B(F)$ and according to [6, Proposition 4.7], $(\Psi_p)_*(K_{A_p}(E_p)) = K_B(E_p \otimes_{\Psi_p} F)$. Therefore $\Phi_*(K_A(E)) = K_B(E \otimes_\Phi F)$.

Now we will suppose that B is an arbitrary locally C^*-algebra.

(1) For each $q \in S(B)$ we consider the map $\Phi_q \colon A \to L_{B_q}(F_q)$ defined by $\Phi_q = (\pi_q)_* \circ \Phi$. Evidently Φ_q is a continuous $*$-morphism and according to the first half of this proof, there is a continuous $*$-morphism $(\Phi_q)_* \colon L_A(E) \to L_{B_q}(E \otimes_{\Phi_q} F_q)$ such that

$$(\Phi_q)_*(T)\big(x \dot{\otimes} \sigma_q^F(y)\big) = T(x) \dot{\otimes} \sigma_q^F(y), \quad x \in E,\ y \in F,\ T \in L_A(E).$$

It is easy to see that $(\pi_{q_1 q_2})_* \circ (\Phi_{q_1})_* = (\Phi_{q_2})_*$ for all $q_1, q_2 \in S(B)$, $q_1 \geqslant q_2$. Therefore there is a continuous $*$-morphism $\Psi \colon L_A(E) \to \varprojlim_q L_{B_q}(E \otimes_{\Phi_q} F_q)$ such that $(\pi_q)_* \circ \Psi = (\Phi_q)_*$ for all $q \in S(B)$. Identifying the Hilbert B-modules $E \otimes_\Phi F$ and $\varprojlim_q (E \otimes_{\Phi_q} F_q)$ (cf. Proposition 4.2) and the locally C^*-algebras $K_B(E \otimes_\Phi F)$ and $\varprojlim_q L_{B_q}(E \otimes_{\Phi_q} F_q)$ (cf. Corollary 4.3) we can identify the continuous $*$-morphism Ψ with a continuous $*$-morphism $\Phi_* \colon L_A(E) \to L_B(E \otimes_\Phi F)$. It is easy to see that $\Phi_*(T)(x \dot{\otimes} y) = T(x) \dot{\otimes} y$, $x \in E$, $y \in F$, $T \in L_A(E)$. Also it is easy to verify that if Φ is injective, then Φ_* is injective.

(2) If $\Phi(A) \subseteq K_B(F)$, then $\Phi_q(A) \subseteq K_{B_q}(F_q)$ for each $q \in S(B)$, and according to the first part of this proof, $(\Phi_q)_*(K_A(E)) \subseteq K_{B_q}(E \otimes_{\Phi_q} F_q)$. This implies that $\Phi_*(K_A(E)) \subseteq K_B(E \otimes_\Phi F)$, since $K_B(E \otimes_\Phi F) = \varprojlim_q K_{B_q}(E \otimes_{\Phi_q} F_q)$ and $(\pi_q)_* \circ \Phi_* = (\Phi_q)_*$ for each $q \in S(B)$.

If $\Phi(A)$ is dense in $K_A(F)$ then for each $q \in S(B)$, $\Phi_q(A)$ is dense in $K_{B_q}(F_q)$ and according to the first half of this proof, $(\Phi_q)_*(K_A(E))$ is dense in $K_{B_q}(E \otimes_{\Phi_q} F_q)$. Thus we have

$$\overline{(\Phi_q)_*(K_A(E))} = \varprojlim_q \overline{(\Phi_q)_*(K_A(E))} = \varprojlim_q K_{B_q}(E \otimes_{\Phi_q} F_q) = K_B(E \otimes_\Phi F).$$

□

Remark 4.5. In the case when B is a C^*-algebra and $F = B$, the proposition was proved in [8, pp. 184–185].

Corollary 4.6. *Let A and B be locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi\colon A \to L_B(F)$ be a continuous $*$-morphism such that $\Phi(A) = K_B(F)$. If for each $q \in S(B)$ there is $p_q \in S(A)$ such that $\widetilde{q}(\Phi(a)) = p_q(a)$ for all $a \in A$ and if $\{p_q;\ q \in S(B)\}$ is a cofinal subset of $S(A)$, then $\Phi_*(K_A(E)) = K_B(E \otimes_\Phi F)$.*

Proof. According to Proposition 4.4 (2), $\overline{\Phi_*(K_A(E))} = K_B(E \otimes_\Phi F)$. We will show that $\Phi_*(K_A(E))$ is closed. Let $q \in S(A)$. We know that there is $p_q \in S(A)$ such that $\widetilde{q}(\Phi(a)) = p_q(a)$ for all $a \in A$. Therefore there is a continuous $*$-morphism $\Phi_{p_q}\colon A_{p_q} \to L_{B_q}(F_q)$ such that $\Phi_{p_q} \circ \pi_{p_q} = (\pi_q)_* \circ \Phi$. Moreover, $\Phi_{p_q}(A_{p_q}) = K_{B_q}(F_q)$ and then according to [6, Proposition 4.7], $\|(\Phi_{p_q})_*(T)\| = \|T\|$ for all T in $K(E_{p_q})$. It is easy to verify that $(\Phi_{p_q})_* \circ (\pi_{p_q})_* = (\pi_q)_* \circ (\Phi)_*$. Then for each $T \in K_A(E)$ we have

$$\widetilde{q}((\Phi)_*(T)) = \|(\pi_q)_*((\Phi)_*(T))\| = \|(\Phi_{p_q})_*((\pi_{p_q})_*(T))\| = \|(\pi_{p_q})_*(T)\| = \widetilde{p_q}(T).$$

From this, since $\{p_q;\ q \in S(B)\}$ is a cofinal subset of $S(A)$, it follows that $\Phi_*(K_A(E))$ is closed. □

Proposition 4.7. *Let A and B be locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi\colon A \to L_B(F)$ be a continuous $*$-morphism such that $\Phi(A)F$ is dense in F. Then the Hilbert B-modules $H_A \otimes_\Phi F$ and $H \otimes F$, where H is a separable infinite dimensional Hilbert space (as well as $A \otimes_\Phi F$ and F) are unitarily equivalent.*

Proof. First we suppose that B is a C^*-algebra.

The continuity of Φ implies that there is a continuous $*$-morphism $\Psi_p\colon A_p \to L_B(F)$ such that $\Psi_p \circ \pi_p = \Phi$. Since π_p is surjective, $\Psi_p(A_p)F$ is dense in F. Then, since A_p and B are C^*-algebras and $\Psi_p\colon A_p \to L_B(F)$ is a morphism of C^*-algebras such that $\Psi_p(A_p)F$ is dense in F, the Hilbert C^*-modules $H_{A_p} \otimes_{\Psi_p} F$ and $H \otimes F$ (as well as $A_p \otimes_{\Psi_p} F$ and F) are unitarily equivalent (see, for instance, [6, pp. 41–42]).

On the other hand, we know that the Hilbert C^*-modules $H_A \otimes_\Phi F$ and $H_{A_p} \otimes_{\Psi_p} F$ (as well as $A \otimes_\Phi F$ and $A_p \otimes_{\Psi_p} F$) are unitarily equivalent (see the proof of the Proposition 4.4). Therefore the proposition is proved in this case.

Now we suppose that B is an arbitrary locally C^*-algebra.

For each $q \in S(B)$, $\Phi_q(A)F_q$ is dense in F_q, where Φ_q is a continuous $*$-morphism from A into $L_{B_q}(F_q)$ defined by $\Phi_q = (\pi_q)_* \circ \Phi$, since $\Phi_q(A)F_q = (\pi_q)_*(\Phi(A))F_q = \sigma_q^F(\Phi(A)F)$ and $\Phi(A)F$ is dense in F. Then, according to the first half of this proof, the Hilbert C^*-modules $H_A \otimes_{\Phi_q} F_q$ and $H \otimes F_q$ (as well as $A \otimes_{\Phi_q} F_q$ and F_q) are unitarily equivalent. It is easy to see that the Hilbert B-modules $\varprojlim_q (H_A \otimes_{\Phi_q} F_q)$

and $\varprojlim_q (H \otimes F_q)$ (as well as $\varprojlim_q (A \otimes_{\Phi_q} F_q)$ and $\varprojlim_q F_q$) are unitarily equivalent and thus the proposition is proved. □

Remark 4.8. Putting $F = B$ in Proposition 4.7 and using Remark 3.1 we deduce that the Hilbert B-modules $H_A \otimes_\Phi B$ and H_B (as well as $A \otimes_\Phi B$ and B) are unitarily equivalent.

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