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RIEMANN-TYPE DEFINITION OF THE IMPROPER INTEGRALS

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Abstract. Riemann-type definitions of the Riemann improper integral and of the Lebesgue improper integral are obtained from McShane's definition of the Lebesgue integral by imposing a Kurzweil-Henstock's condition on McShane's partitions.

Keywords: McShane's partition, Kurzweil-Henstock's partition

MSC 2000: 26A36, 26A39

1. Introduction

Let $F\colon [a,b] \to \mathbb{R}$ be a differentiable function and let f be its derivative. The problem of recovering F from f is called *the problem of primitives*.

In 1912, the problem of primitives was solved by A. Denjoy with an integration process (called *totalization*) that includes the Lebesgue integral and the Lebesgue improper integral. Equivalent solutions are due to O. Perron, J. Kurzweil and R. Henstock (see for example [4], [6], [7], and [8]).

In 1986, A. M. Bruckner, R. J. Fleissner and J. Foran [3] remarked that the solution provided by Denjoy, Perron, Kurzweil and Henstock possesses a generality which is not needed for the problem of primitives. In fact the function $F(x) = x \sin(1/x^2)$ for $x \in (0,1]$ and $F(0) = 0$ is ACG^* (i.e. a primitive for the Denjoy-Perron-Kurzweil-Henstock integral) but, for any absolutely continuous function G, the function $F - G$ is not differentiable at $x = 0$.

Note that the function $f(x) = F'(x)$ for $x \in (0,1]$ and $f(0) = 0$ is a Riemann improper integrable function. So the minimal integral which includes Lebesgue integrable functions and derivatives (defined descriptively in [3]) does not contain the

Riemann improper integral. Now this minimal integral can be obtained from Mc-Shane's definition of the Lebesgue integral (see for example [4], [5], and [7]) by imposing a mild regularity condition on McShane's partitions (see [1] and [2]).

In this note we prove that the Riemann improper integral and the Lebesgue improper integral can be also obtained from McShane's definition of the Lebesgue integral by imposing a Kurzweil-Henstock's condition on McShane's partitions.

2. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$. If $E \subset \mathbb{R}$ then $|E|$ denotes the Lebesgue measure of E, and $\overline{E}$ its closure. In this paper $[a, b]$ is a fixed interval of $\mathbb{R}$. By *McShane's partition* of $[a, b]$ we mean any finite collection $\{(A_h, x_h)\}_{h=1}^p$ of pairwise disjoint intervals A_h and points $x_h \in [a, b]$ such that $[a, b] = \bigcup_h A_h$. We say that a McShane's partition $\{(A_h, x_h)\}_{h=1}^p$ satisfies the Kurzweil-Henstock condition (br. K-H condition) on a set $E \subset [a, b]$ whenever $x_h \in A_h$ if $x_h \in E$.

A *McShane's partition* satisfying the K-H condition on $[a, b]$ is called a *Kurzweil-Henstock's partition* of $[a, b]$.

Let δ be a gauge (i.e. a positive function) on $[a, b]$. A *McShane's partition* of $[a, b]$, say $\{(A_h, x_h)\}_{h=1}^p$, is said to be δ-fine whenever $A_h \subset (x_h - \delta(x_h), x_h + \delta(x_h))$, for each h.

We recall that a function $f \colon [a, b] \to \mathbb{R}$ is said to be McShane (resp. Kurzweil-Henstock) integrable on $[a, b]$ if there is a real number I satisfying the following condition: for each $\varepsilon > 0$ there exists a gauge δ such that

$$(1) \qquad \left| \sum_{h=1}^p f(x_h)|A_h| - I \right| < \varepsilon,$$

for each δ-fine McShane's (resp. Kurzweil-Henstock's) partition P of $[a, b]$ (see [4], [5], [6] and [7]).

I is called the McShane (resp. Kurzweil-Henstock) integral of f on $[a, b]$. The McShane integral is equivalent to the Lebesgue integral, and the Kurzweil-Henstock integral is equivalent to the Denjoy-Perron integral (see [4], [7]).

In this paper the Kurzweil-Henstock integral of f on $[a, b]$ is denoted by $\int_a^b f$. Since the Riemann integral, the Riemann improper integral, the Lebesgue integral and the Lebesgue improper integral are contained in the Kurzweil-Henstock integral, then by $\int_a^b f$ we also denote each of the mentioned integrals on $[a, b]$.

For simplicity, in the sequel we set $\sum_P f = \sum_{h=1}^p f(x_h)|A_h|$.

We also recall that a function $f \colon [a, b] \to \mathbb{R}$ is said to be Lebesgue (resp. Riemann) improper integrable on $[a, b]$ if there exist $a_i \in [a, b]$, $i = 0, 1, \ldots, k$ with $a = a_0 < a_1 < \ldots < a_k = b$ such that f is Lebesgue (resp. Riemann) integrable on each compact subinterval $[\alpha, \beta]$ of (a_i, a_{i+1}), and the limits

$$
(2) \qquad\qquad \lim_{\substack{\alpha \searrow a_i \\ \beta \nearrow a_{i+1}}} \int_\alpha^\beta f
$$

exist finite, for $i = 0, 1, \ldots, k - 1$.

3. The Lebesgue improper integral

In this section we prove

Theorem 1. *A function $f \colon [a, b] \to \mathbb{R}$ is Lebesgue improper integrable on $[a, b]$ if and only if there exists a finite set $E \subset [a, b]$ and a real number I such that:*

(LI) *for each $\varepsilon > 0$ there exists a gauge δ so that $\left| \sum_P f - I \right| < \varepsilon$ for each δ-fine McShane's partition P of $[a, b]$ satisfying the K-H condition on E.*

P r o o f.   Assume that f is Lebesgue improper integrable on $[a, b]$. Then there exist $a = a_0 < a_1 < \ldots < a_k = b$, such that f is Lebesgue integrable on each compact subinterval $[\alpha, \beta]$ of (a_i, a_{i+1}), and the limits (2) exist finite for each i.

Given $i \in \{0, 1, \ldots, k - 1\}$, let $\{\alpha_{i,j}\}_{j \in \mathbb{Z}} \subset (a_i, a_{i+1})$ be an increasing sequence such that $\lim_{j \to -\infty} \alpha_{i,j} = a_i$ and $\lim_{j \to \infty} \alpha_{i,j} = a_{i+1}$.

Fixed $\varepsilon > 0$, $i \in \{0, 1, \ldots, k - 1\}$ and $j \in \mathbb{Z}$, by Henstock's lemma there exists a gauge $\delta_{i,j}$ on $(\alpha_{i,j}, \alpha_{i,j+1})$ such that

$$
(3) \qquad\qquad \sum_{(A, x) \in P_{i,j}} \left| f(x) |A| - \int_A f \right| < \frac{\varepsilon}{k 2^{|j| + 3}}
$$

for each $\delta_{i,j}$-fine McShane's partition $P_{i,j}$ of $(\alpha_{i,j}, \alpha_{i,j+1})$.

For $x \neq a_i$, $0 \leqslant i \leqslant k$ we set

$$
\delta(x) = \begin{cases} \min\{\delta_{i,j}(x), x - \alpha_{i,j}, \alpha_{i,j+1} - x\}, & \text{if } \alpha_{i,j} < x < \alpha_{i,j+1}; \\ \min\{\delta_{i,j-1}(x), \delta_{i,j}(x)\}, & \text{if } x = \alpha_{i,j}. \end{cases}
$$

Moreover, for $x = a_i$, $0 \leqslant i \leqslant k$, we define $\delta(a_i) > 0$ such that

$$
(4) \qquad\qquad \delta(a_i) |f(a_i)| < \frac{\varepsilon}{8(k + 1)},
$$

and

(5)
$$\left| \int_{a_i}^{\alpha} f \right| < \frac{\varepsilon}{8(k+1)}$$

for each $\alpha \in [a, b]$ with $|\alpha - a_i| < \delta(a_i)$.

Let $\{(A_h, x_h)\}_{h=1}^{p}$ be a δ-fine McShane's partition of $[a, b]$ satisfying the K-H condition on $\{a_0, \ldots, a_k\}$. Then, by (4) and (5) we have

$$\sum_{x_h \in \{a_0, \ldots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| \leqslant \sum_{x_h \in \{a_0, \ldots, a_k\}} \left(|f(x_h)| \, |A_h| + \left| \int_{A_h} f \right| \right)$$

$$\leqslant \sum_{x_h \in \{a_0, \ldots, a_k\}} \left(2 \cdot \frac{\varepsilon}{8(k+1)} + 2 \cdot \frac{\varepsilon}{8(k+1)} \right)$$

$$\leqslant (k+1) \cdot \left(\frac{\varepsilon}{4(k+1)} + \frac{\varepsilon}{4(k+1)} \right) = \frac{\varepsilon}{2}.$$

So, by (3) we get

$$\left| \sum_{h=1}^{p} f(x_h)|A_h| - \int_a^b f \right| \leqslant \sum_{x_h \in \{a_0, \ldots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right|$$

$$+ \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \sum_{x_h \in (\alpha_{i,j}, \alpha_{i,j+1})} \left| f(x_h)|A_h| - \int_{A_h} f \right|$$

$$\leqslant \frac{\varepsilon}{2} + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \frac{\varepsilon}{k 2^{|j|+3}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore condition (LI) is satisfied.

Vice versa, assume that there exist $a = a_0 < a_1 \ldots < a_k = b$, and a real number I satisfying condition (LI). Then f is Kurzweil-Henstock integrable on $[a, b]$ with $I = \int_a^b f$. Moreover, by an easy adaptation of Henstock's lemma (see [4, Lemma 9.11]) we have

$$\sum_{h=1}^{p} \left| f(x_h)|A_h| - \int_{A_h} f \right| < 2\varepsilon,$$

for each δ-fine McShane's partition $\{(A_h, x_h)\}_{h=1}^{p}$ of $[a, b]$ satisfying the K-H condition on E. This implies that f is McShane integrable (hence Lebesgue integrable) on each compact subinterval of (a_i, a_{i+1}), $i = 0, 1, \ldots, k-1$ and, by the continuity

of the Kurzweil-Henstock integral, the limits

$$\lim_{\substack{\alpha \searrow a_i \\ \beta \nearrow a_{i+1}}} \int_\alpha^\beta f$$

exist finite, for $i = 0, 1, \ldots, k - 1$.

In conclusion f is Lebesgue improper integrable on $[a, b]$. $\qquad\square$

4. The Riemann improper integral

In this section we prove

Theorem 2. *A function $f \colon [a, b] \to \mathbb{R}$ is Riemann improper integrable on $[a, b]$ if and only if there exists a finite set $E \subset [a, b]$ and a real number I such that*

(RI) *for each $\varepsilon > 0$ there exists a gauge δ so that*
   (1) *δ is continuous on $[a, b] \setminus E$;*
   (2) *$\left| \sum_P f - I \right| < \varepsilon$, for each δ-fine McShane's partition P of $[a, b]$ satisfying the K-H condition on E.*

The proof is based on the following lemma:

Lemma 1. *If $f \colon [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then for each $\varepsilon > 0$ there exists a constant $\eta > 0$ such that*

$$\sum_{h=1}^p \left| f(x_h)|A_h| - \int_{A_h} f \right| < \varepsilon,$$

for each η-fine McShane's partition $\{(A_h, x_h)\}_{h=1}^p$ of $[a, b]$.

P r o o f.   By the definition of Riemann integral, to each $\varepsilon > 0$ there exists a constant $\eta_1 > 0$ such that

$$\left| \sum_{h=1}^p f(y_h)|A_h| - \int_a^b f \right| < \frac{\varepsilon}{8},$$

for each η_1-fine Kurzweil-Henstock's partition $\{(A_h, y_h)\}_{h=1}^p$ of $[a, b]$. Then, by an easy adaptation of Henstock's lemma, we have

(6) $$\sum_{h=1}^p \left| f(y_h)|A_h| - \int_{A_h} f \right| < \frac{\varepsilon}{4},$$

for each η_1-fine Kurzweil-Henstock's partition $\{(A_h, y_h)\}_{h=1}^p$ of $[a, b]$. Now, for each natural n, let

$$E_n = \left\{ x \in [a, b] \colon\ \omega\left(f, \left(x - \frac{1}{n}, x + \frac{1}{n}\right)\right) \geqslant \frac{\varepsilon}{4(b-a)} \right\},$$

where $\omega(f, A)$ stands for the oscillation of f on the set A.

It is easy to see that $\overline{E}_n \subset E_{n+1}$. Moreover, since the set of points of discontinuity of a Riemann integrable function is a Lebesgue null set, we have $\left| \bigcap\limits_{n=1}^{\infty} E_n \right| = 0$. Hence $|\overline{E}_n| \to 0$. Let $M = \sup\{|f(x)| \colon\ x \in [a, b]\}$, and let n_0 be such that $|\overline{E}_{n_0}| < \varepsilon/(4M)$.

Now, let $\{(\alpha_i, \beta_i)\}_{i=1}^{\infty}$ be a covering of $\overline{E}_{n_0}$ with

$$(7) \qquad\qquad \sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \frac{\varepsilon}{4M}.$$

Then, there is i_0 such that $\overline{E}_{n_0} \subset \bigcup\limits_{i=1}^{i_0} (\alpha_i, \beta_i)$. Set

$$\eta_2 = \inf\left\{ |x - y| \colon\ x \in \overline{E}_{n_0}\ \text{and}\ y \notin \bigcup_{i=1}^{i_0}(\alpha_i, \beta_i) \right\},$$

and

$$\eta = \min\{\eta_1, \eta_2, 1/n_0\}.$$

Let $\{(A_h, x_h)\}_{h=1}^p$ be an η-fine McShane's partition of $[a, b]$, and let $y_h \in A_h$ for $h = 1, \ldots, p$. Then (6) holds. Moreover, by definitions of E_{n_0} and η, for $h = 1, \ldots, p$ we have

$$|f(x_h) - f(y_h)| < \frac{\varepsilon}{4(b-a)}, \quad \text{if } x_h \notin E_{n_0},$$

and

$$A_n \subset \bigcup_{i=1}^{i_0}(\alpha_i, \beta_i), \quad \text{if } x_h \in E_{n_0}.$$

Thus, by (7) we get

$$\sum_{x_h \in E_{n_0}} |A_h| < \frac{\varepsilon}{4M}.$$

In conclusion

$$\sum_{h=1}^{p}\left|f(x_h)|A_h|-\int_{A_h}f\right|$$

$$\leqslant \sum_{h=1}^{p}|f(x_h)-f(y_h)|\,|A_h|+\sum_{h=1}^{p}\left|f(y_h)|A_h|-\int_{A_h}f\right|$$

$$\leqslant \sum_{h=1}^{p}|f(x_h)-f(y_h)|\,|A_h|+\frac{\varepsilon}{4}$$

$$=\sum_{x_h\in E_{n_0}}|f(x_h)-f(y_h)|\,|A_h|+\sum_{x_h\notin E_{n_0}}|f(x_h)-f(y_h)|\,|A_h|+\frac{\varepsilon}{4}$$

$$<2M\sum_{x_h\in E_{n_0}}|A_h|+\frac{\varepsilon}{4(b-a)}\sum_{x_h\notin E_{n_0}}|A_h|+\frac{\varepsilon}{4}$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.$$

$\square$

Corollary 1. *A function $f\colon [a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, if and only if there exists a real number I satisfying the following condition: for each $\varepsilon>0$ there exists a positive constant η such that $\left|\sum_{P}f-I\right|<\varepsilon$, for each η-fine McShane's partition $P=\{(A_h,x_h)\}_{h=1}^{p}$ of $[a,b]$.*

P r o o f of Theorem 2.   Assume that f is Riemann improper integrable on $[a,b]$. Then there exist $a=a_0<a_1<\ldots<a_k=b$ such that f is Riemann integrable on each interval $[\alpha,\beta]\subset(a_i,a_{i+1})$, and the limits (2) exist finite, for $i=0,1,\ldots,k-1$.

Given $i\in\{0,1,\ldots,k-1\}$, let $\{\alpha_{i,j}\}_{j\in\mathbb{Z}}\subset(a_i,b_i)$ be an increasing sequence such that $\lim_{j\to-\infty}\alpha_{i,j}=a_i$ and $\lim_{j\to\infty}\alpha_{i,j}=a_{i+1}$.

For $i\in\{0,1,\ldots,k-1\}$, $j\in\mathbb{Z}$, and $\varepsilon>0$, by Lemma 1 there exist $0<\tau_{i,j}<\min\{\alpha_{i,j+1}-\alpha_{i,j},\alpha_{i,j+2}-\alpha_{i,j+1},\alpha_{i,j+3}-\alpha_{i,j+2}\}$ such that condition (3) is satisfied for each $\tau_{i,j}$-fine McShane's partition $P_{i,j}$ of $[\alpha_{i,j},\alpha_{i,j+3}]$. Define $\delta_{i,j}\leqslant\min\{\tau_{i,j-2},\tau_{i,j-1},\tau_{i,j}\}$ such that $\delta_{i,j}\leqslant\delta_{i,j+1}$ for $j<0$, $\delta_{i,-1}=\delta_{i,0}$ and $\delta_{i,j}\geqslant\delta_{i,j+1}$, for $j\geqslant 0$. Moreover, for $i=0,1,\ldots,k$ take $\delta(a_i)>0$ such that conditions (4) and (5) are satisfied. Finally, for $i\in\{0,1,\ldots,k-1\}$ and $x\in(a_i,a_{i+1})$ define

$$\delta(x)=\begin{cases}\delta_{i,j-1}+\dfrac{x-\alpha_{i,j}}{\alpha_{i,j+1}-\alpha_{i,j}}(\delta_{i,j}-\delta_{i,j-1}), & \text{if }x\in[\alpha_{i,j},\alpha_{i,j+1})\text{ and }j<0;\\[3mm]\delta_{i,j}+\dfrac{x-\alpha_{i,j}}{\alpha_{i,j+1}-\alpha_{i,j}}(\delta_{i,j+1}-\delta_{i,j}), & \text{if }x\in[\alpha_{i,j},\alpha_{i,j+1})\text{ and }j\geqslant 0.\end{cases}$$

Note that δ is continuous on $[a,b] \setminus \bigcup_{i=1}^{p} \{a_i\}$ and $\delta(x) \leqslant \delta_{i,j}$ for $x \in [\alpha_{i,j}, \alpha_{i,j+1})$, $i = 0, 1, \ldots, k-1$, $j \in \mathbb{Z}$.

To verify condition $(\mathrm{RI})_2$, let $\{(A_h, x_h)\}_{h=1}^{p}$ be a δ-fine McShane's partition of $[a,b]$ satisfying the K-H condition on $\{a_0, \ldots, a_k\}$. Then, by (4) and (5) we have

$$(8) \qquad \sum_{x_h \in \{a_0, \ldots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| \leqslant \frac{\varepsilon}{2}.$$

Now, whenever $x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]$ with $j \geqslant 0$, then

$$A_h \subset (x_h - \delta(x_h), x_h + \delta(x_h)) \subset (\alpha_{i,j} - \delta_{i,j}, \alpha_{i,j+1} + \delta_{i,j})$$
$$\subset (\alpha_{i,j} - \tau_{i,j-1}, \alpha_{i,j+1} + \tau_{i,j-1}) \subset [\alpha_{i,j-1}, \alpha_{i,j+2}],$$

and whenever $x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]$ with $j < 0$, then

$$A_h \subset (x_h - \delta(x_h), x_h + \delta(x_h)) \subset (\alpha_{i,j} - \delta_{i,j+1}, \alpha_{i,j+1} + \delta_{i,j+1})$$
$$\subset (\alpha_{i,j} - \tau_{i,j-1}, \alpha_{i,j+1} + \tau_{i,j-1}) \subset [\alpha_{i,j-1}, \alpha_{i,j+2}].$$

Therefore the family $\{(A_h, x_h) \colon x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]\}$ is a $\tau_{i,j-1}$-fine McShane's partial partition in $[\alpha_{i,j-1}, \alpha_{i,j+2}]$. Thus, by (3) and (8) we have

$$\left| \sum_{h=1}^{p} f(x_h)|A_h| - \int_a^b f \right| \leqslant \sum_{x_h \in \{a_0, \ldots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right|$$

$$+ \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \sum_{x_h \in (\alpha_{i,j}, \alpha_{i,j+1})} \left| f(x_h)|A_h| - \int_{A_h} f \right|$$

$$\leqslant \frac{\varepsilon}{2} + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \frac{\varepsilon}{k 2^{|j|+3}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof of condition $(\mathrm{RI})_2$.

Vice versa, assume that there exist $a = a_0 < a_1 \ldots < a_k = b$, and a real number I satisfying condition (RI). Since condition (RI) implies condition (LI), then f is Lebesgue improper integrable on $[a,b]$, by Theorem 1. Thus we have only to prove that f is Riemann integrable on each compact subinterval of (a_i, a_{i+1}), for $i = 0, 1, \ldots, k-1$. Let $[\alpha, \beta] \subset (a_i, a_{i+1})$, $i = 0, 1, \ldots, k-1$. By $(\mathrm{RI})_2$ and by an easy adaptation of Henstock's lemma, we have $\left| \sum_P f - I \right| < 2\varepsilon$, for each δ-fine McShane's partition P of $[\alpha, \beta]$. Now, by continuity of δ on $[\alpha, \beta]$, we have $\eta = \min\{\delta(x) \colon x \in [\alpha, \beta]\} > 0$. Therefore $\left| \sum_P f - I \right| < 2\varepsilon$, for each η-fine McShane's (hence Kurzweil-Henstock's) partition P of $[\alpha, \beta]$. Thus f is Riemann integrable on $[\alpha, \beta]$, and the proof is complete. $\qquad \square$

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