M. Abel; J. Arhippainen

Locally m-pseudoconvex topologies on locally A-pseudoconvex algebras

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 3, 675–680

Persistent URL: `http://dml.cz/dmlcz/127919`

Czechoslovak Mathematical Journal, 54 (129) (2004), 675–680

LOCALLY M-PSEUDOCONVEX TOPOLOGIES ON LOCALLY A-PSEUDOCONVEX ALGEBRAS

M. Abel, Tartu, and J. Arhippainen, Oulu

(Received December 3, 2001)

Abstract. Let (A, T) be a locally A-pseudoconvex algebra over $\mathbb{R}$ or $\mathbb{C}$. We define a new topology $m(T)$ on A which is the weakest among all m-pseudoconvex topologies on A stronger than T. We describe a family of non-homogeneous seminorms on A which defines the topology $m(T)$.

Keywords: locally A-pseudoconvex algebra, locally m-pseudoconvex algebra

MSC 2000: 46H05, 46H20

1. Introduction

Let A be *a locally* A-*pseudoconvex algebra*. This means that A is an associative algebra over $\mathbb{K}$ (where $\mathbb{K}$ is either the field $\mathbb{C}$ of complex numbers or $\mathbb{R}$ of real numbers) equipped with a topology T given by a base $\{U_\lambda\colon \lambda \in \Lambda\}$ of neighbourhoods of zero in which each U_λ is A-pseudoconvex. So for all $\lambda \in \Lambda$, U_λ is balanced, pseudoconvex (i.e. for each $\lambda \in \Lambda$ there is a number $r_\lambda \in (0,1]$ such that $U_\lambda + U_\lambda \subset 2^{1/r_\lambda} U_\lambda$) and absorbs the set $xU_\lambda \cup U_\lambda x$ for all $x \in A$. For each $\lambda \in \Lambda$ let p_λ be a mapping (the r_λ-homogeneous gauge of U_λ) defined by

$$p_\lambda(x) = \inf\{|\mu|^{r_\lambda}\colon\ x \in \mu \operatorname{conv}_{r_\lambda} U_\lambda\}$$

for all $x \in A$ (here $\operatorname{conv}_{r_\lambda} U_\lambda$ means the absolutely r_λ-convex hull of U_λ). Now p_λ is an r_λ-homogeneous A-pseudoconvex seminorm on A (here the numbers r_λ may vary). The A-pseudoconvexity of a seminorm p_λ means that for each $x \in A$ and $\lambda \in \Lambda$ there exist positive numbers $L(x,\lambda)$ and $R(x,\lambda)$ (depending on x and λ) such that $p_\lambda(xy) \leqslant L(x,\lambda)p_\lambda(y)$ and $p_\lambda(yx) \leqslant R(x,\lambda)p_\lambda(y)$ for all $y \in A$. Denote this family of seminorms on A by $\mathscr{P}$ and the corresponding topology on A by $T(\mathscr{P})$.

Now we clearly have $T(\mathscr{P}) = T$. If every $p_\lambda \in \mathscr{P}$ is m-pseudoconvex (i.e. if each p_λ is submultiplicative) then $(A, T(\mathscr{P}))$ is called a locally m-pseudoconvex algebra. In this case (A, T) has a base of neighbourhoods of zero, each element of which is m-pseudoconvex (i.e. is idempotent, balanced and pseudoconvex). If $r_\lambda = 1$ for all $\lambda \in \Lambda$, then each p_λ is A-convex and $(A, T(\mathscr{P}))$ is called a locally A-convex algebra, and if moreover each p_λ is submultiplicative, then $(A, T(\mathscr{P}))$ is called a locally m-convex algebra.

For locally pseudoconvex algebras see [1], [2] or [13] and for locally A-convex algebras see e.g. [3], [4], [5], [6], [7], [9], [10] or [11].

2. Main results

It was shown in [9] that for each locally A-convex topology T on A there exists on A the weakest locally m-convex topology, say $m(T)$, which is stronger than T. We shall give a detailed proof of this fact for the locally A-pseudoconvex case.

Theorem 1. *Let (A, T) be a locally* A-*pseudoconvex algebra, $\mathscr{B}$ the set of all* A-*pseudoconvex neighbourhoods of zero on A and $\mathscr{B}' = \{\varepsilon U' \colon \varepsilon \in (0,1], U \in \mathscr{B}\}$ where $U' = \{x \in U \colon xU \cup Ux \subset U\}$. Then U' is r_U-convex if U is and $\mathscr{B}'$ forms a subbase of the neighbourhoods of zero for a locally* m-*pseudoconvex topology $m(T)$ on A which is stronger than T. In particular, if (A, T) is a locally* m-*pseudoconvex algebra, then $m(T) = T$.*

P r o o f. Let $\mathscr{E}$ be the family of all finite intersections of elements of $\mathscr{B}'$. Clearly $\mathscr{E}$ is a basis for a filter on A. It is easy to see that every $E \in \mathscr{E}$ is balanced and absorbent and U' is r_U-convex if U is. Let now $E = \bigcap\limits_{k=1}^{n} \varepsilon_k U'_k u$, where $\gamma_k = \varepsilon_k 2^{-1/r_{U_k}}$, $\varepsilon_k \in (0,1]$ and $U_k \in \mathscr{B}$ for each $k \in \{1, 2, \ldots, n\}$; then $F \in \mathscr{E}$. If x_1 and $x_2 \in F$, then for each $k \in \{1, 2, \ldots, n\}$ there exist elements $y_{(1,k)}, y_{(2,k)} \in U'_k$ for which $x_1 = \gamma_k y_{(1,k)}$ and $x_2 = \gamma_k y_{(2,k)}$ and

$$(x_1 + x_2)U_k \cup U_k(x_1 + x_2) \subset \gamma_k(U_k + U_k) \subset \gamma_k 2^{1/r_{U_k}} U_k \subset \varepsilon_k U_k.$$

Hence every E defines a F such that $F + F \subset E$. Therefore by Theorem 2.1 of [8], p. 13, there exists a topology $m(T)$ on A for which $(A, m(T))$ is a topological vector space and $\mathscr{E}$ is a base of neighbourhoods of zero for the topology $m(T)$. To show that every $E \in \mathscr{E}$ is m-pseudoconvex let x_1 and $x_2 \in E$. Since $E = \bigcap\limits_{k=1}^{n} \varepsilon_k U'_k$ for some $\varepsilon_k \in (0,1]$ and $U_k \in \mathscr{B}$, we have

$$(x_1x_2)U_k \cup U_k(x_1x_2) \subset \varepsilon_k(x_1U_k \cup U_kx_2) \subset \varepsilon_k^2 U_k \subset \varepsilon_k U_k$$

and

$$(x_1+x_2)U_k \cup U_k(x_1+x_2) \subset \varepsilon_k(U_k+U_k) \subset 2^{1/r_{U_k}}\varepsilon_k U_k \subset 2^{1/r}\varepsilon_k U_k$$

for all $k \in \{1,2,\dots,n\}$, where $r = \min\{r_{U_1}, r_{U_2}, \dots, r_{U_n}\}$. Thus each $E \in \mathscr{E}$ is idempotent and pseudoconvex. This shows that $m(T)$ is a locally m-pseudoconvex topology on A which is stronger than T, since $U' \subset U$ for each $U \in \mathscr{B}$. In particular, if each $U \in \mathscr{B}$ is idempotent (which means that T is locally m-pseudoconvex), then $U' = U$ and thus $T = m(T)$. □

Theorem 2. *Let (A_1, T_1) and (A_2, T_2) be two locally* A-*pseudoconvex algebras and φ a continuous isomorphism from (A_1, T_1) onto (A_2, T_2). Then φ is a continuous isomorphism from $(A_1, m(T_1))$ onto $(A_2, m(T_2))$.*

Proof. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the sets of all A-pseudoconvex neighbourhoods of zero of the algebras (A_1, T_1) and (A_2, T_2), respectively. Let $\mathscr{B}_1'$ and $\mathscr{B}_2'$ be the subbases of neighbourhoods of zero for the algebras $(A_1, m(T_1))$ and $(A_2, m(T_2))$, respectively, defined in the proof of Theorem 1. If $E \in \mathscr{B}_2'$ is arbitrary, then there exist a set $V \in \mathscr{B}_2$ and $\varepsilon \in (0,1]$ such that $E = \varepsilon V'$ where $V' = \{x \in V \colon xV \cup Vx \subset V\}$. Since φ is a continuous surjection, $U = \varphi^{-1}(V)$ is a neighbourhood of zero in (A_1, T_1) and $\varphi(U) = V$. Clearly U is an A-pseudoconvex subset of A_1, since φ is an isomorphism, which implies that $U \in \mathscr{B}_1$. Let now $x \in U'$ be given. Then

$$\varphi(x)V \cup V\varphi(x) = \varphi(x)\varphi(U) \cup \varphi(U)\varphi(x) = \varphi(xU \cup Ux) \subset \varphi(U) = V.$$

This shows that $\varphi(U') \subset V'$ implies $\varphi(\varepsilon U') \subset \varepsilon V' = E$. As $\varepsilon U' \in \mathscr{B}_1$, it follows that φ is a continuous map from $(A_1, m(T_1))$ onto $(A_2, m(T_2))$.

Corollary 1. *Let T be a locally* A-*pseudoconvex topology on A and let T_1 be an arbitrary locally* m-*pseudoconvex topology on A which is stronger than T. Then $m(T)$ is weaker than T_1.*

Proof. Let I be the identity map on A. Then I is a continuous isomorphism from (A, T_1) onto (A, T). Therefore I is also continuous as a map from $(A, m(T_1))$ onto $(A, m(T))$ by Theorem 2. Since T_1 is locally m-pseudoconvex, we have $m(T_1) = T_1$ by Theorem 1. Hence $m(T)$ is weaker than T_1. □

Corollary 2. *Let T_1 and T_2 be two locally* A*-pseudoconvex topologies on A. If T_1 is weaker than T_2, then $m(T_1)$ is weaker than $m(T_2)$.*

Proof. Let I be the identity map on A. If T_1 is weaker than T_2, then I is a continuous isomorphism from (A,T_2) onto (A,T_1). Therefore, I is also continuous as a map from $(A,m(T_2))$ onto $(A,m(T_1))$ by Theorem 2. Hence $m(T_1)$ is weaker than $m(T_2)$. $\square$

3. Seminorms defining $m(T)$

Let $(A,T(\mathscr{P}))$ be a locally A-pseudoconvex algebra, where $\mathscr{P}$ is a family of all continuous r_λ-homogeneous A-pseudoconvex seminorms on (A,T) with $r_\lambda \in (0,1]$, defining the topology $T(\mathscr{P})$. We shall now give a description of seminorms which define the topology $m(T(\mathscr{P}))$. To this end it let

$$\widetilde{p}_\lambda(x) = \sup_{p_\lambda(y)\leqslant 1} \max\{p_\lambda(xy), p_\lambda(yx)\}$$

for each $x \in A$ and $\lambda \in \Lambda$ (see [12], p. 19). Then $\widetilde{p}_\lambda$ is an r_λ-homogeneous submultiplicative seminorm on A for each $\lambda \in \Lambda$ and the family $\widetilde{\mathscr{P}} = \{\widetilde{p}_\lambda\colon \lambda \in \Lambda\}$ defines on A a topology $T(\widetilde{\mathscr{P}})$ which is not necessarily a Hausdorff topology even though $T(\mathscr{P})$ is. Let now

$$q_\lambda(x) = \max\{p_\lambda(x), \widetilde{p}_\lambda(x)\}$$

for each $x \in A$ and $\lambda \in \Lambda$. Then q_λ is an r_λ-homogeneous and submultiplicative seminorm on A for each $\lambda \in \Lambda$. Let $\mathscr{Q} = \{q_\lambda\colon \lambda \in \Lambda\}$. Then $T(\mathscr{Q})$ is a locally m-pseudoconvex topology on A which is stronger than $T(\mathscr{P})$.

In [9] the case has been considered when $(A,T(\mathscr{P}))$ is a locally A-convex algebra and it was stated without proof that $m((T(\mathscr{P})) = T(\mathscr{Q})$ where the seminorms q_λ are defined by $q_\lambda(x) = \max\{p_\lambda(x), \widetilde{p}_\lambda(x)\}$ with $\widetilde{p}_\lambda(x) = \sup\limits_{p_\lambda(y)\leqslant 1} p_\lambda(xy)$ for each $x \in A$.

We will show that the results of Oubbi and Oudadess in [9] and [11] are in fact valid not only for the locally A-convex case, but also for the locally A-pseudoconvex case.

Theorem 3. *Let (A,T) be a locally* A*-pseudoconvex algebra and let $\mathscr{P}$ be the family of all continuous r_λ-homogeneous* A*-pseudoconvex seminorms on (A,T) defining the topology T. Then $m(T(\mathscr{P})) = T(\mathscr{Q})$. Furthermore, $m(T(\mathscr{P}))$ is separated if and only if $T(\mathscr{P})$ is separated.*

P r o o f. By Corollary 1, $m(T(\mathscr{P}))$ is weaker than $T(\mathscr{Q})$. To show that $T(\mathscr{Q})$ coincides with $m(T(\mathscr{P}))$ let O be an arbitrary element in the base of the neighbourhoods of zero for the topology $T(\mathscr{Q})$. Then there exist $\varepsilon > 0$, $n \in \mathbb{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$ such that

$$O = \bigcap_{k=1}^{n} \{x \in A\colon\ q_{\lambda_k}(x) < \varepsilon\}.$$

Furthermore, for each $\lambda \in \Lambda$ let U_λ be the A-pseudoconvex neighbourhood of zero which defines on A the r_λ-homogeneous seminorm p_λ. Let U'_λ be the element of the subbase of the neighbourhoods of zero for the topology $m(T(\mathscr{P}))$ defined in Theorem 1 and let p'_λ be the r_λ-homogeneous gauge of U'_λ. Suppose that $x \in A$ is an element for which $p'_\lambda(x) \leqslant 1$. For $n \in \mathbb{N}$ let $x_n = (1 - 1/n)x$. Then $p'_\lambda(x_n) = |1 - 1/n|^{r_\lambda} p'_\lambda(x) < 1$ for each $n \in \mathbb{N}$. Furthermore, let $y \in A$ be an element for which $p_\lambda(y) \leqslant 1$ and let $y_n = (1 - 1/n)y$ for each $n \in \mathbb{N}$. As above we have $p_\lambda(y_n) < 1$ for all $n \in \mathbb{N}$. Thus $x_n \in \operatorname{conv}_{r_\lambda} U'_\lambda$ and $y_n \in \operatorname{conv}_{r_\lambda} U_\lambda$ for each $n \in \mathbb{N}$. Since $U'_\lambda U_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$, it follows that $\operatorname{conv}_{r_\lambda} U'_\lambda \operatorname{conv}_{r_\lambda} U_\lambda \subset \operatorname{conv}_{r_\lambda} U_\lambda$. This implies that $p_\lambda(x_n y_n) \leqslant 1$ for each $n \in \mathbb{N}$. Since

$$\begin{aligned}\lim_{n\to\infty} p_\lambda(x_n y_n - xy) &= \lim_{n\to\infty} p_\lambda\Big(\Big(1 - \frac{1}{n}\Big)^2 xy - xy\Big)\\ &= p_\lambda(xy) \lim_{n\to\infty}\Big|\Big(1 - \frac{1}{n}\Big)^2 - 1\Big|^{r_\lambda} = 0,\end{aligned}$$

we have $p_\lambda(xy) \leqslant 1$. So we have shown that $p_\lambda(xy) \leqslant 1$ if $p'_\lambda(x) \leqslant 1$ and $p_\lambda(y) \leqslant 1$. In the same way we have also $p_\lambda(yx) \leqslant 1$ if $p'_\lambda(x) \leqslant 1$ and $p_\lambda(y) \leqslant 1$. This implies that the condition $p'_\lambda(x) \leqslant 1$ yields that $\widetilde{p}_\lambda(x) \leqslant 1$. Let now $r = \max\{r_{\lambda_1}, r_{\lambda_2}, \ldots, r_{\lambda_n}\}$, $\delta \in (0, \varepsilon^{1/r})$ and

$$U = \delta \bigcap_{k=1}^{n} U'_{\lambda_k}.$$

Then U is a neighbourhood of zero in A in the topology $m(T(\mathscr{P}))$. Thus there exists an element, say V, of the base of the neighbourhoods of zero of A in the topology $m(T(\mathscr{P}))$ such that $V \subset U$. To show that $V \subset O$ let $x \in V$ be given. Since $V \subset \delta U'_{\lambda_k} \subset \delta \operatorname{conv}_{\lambda_k} U'_{\lambda_k}$ for each k, we have $x = \delta u_k$ for some $u_k \in \operatorname{conv}_{r_{\lambda_k}} U'_{\lambda_k}$. Therefore it follows from $p'_{\lambda_k}(x) \leqslant \delta^{r_{\lambda_k}}$ that $\widetilde{p}_{\lambda_k}(x) \leqslant \delta^{r_{\lambda_k}}$ for all k. As $U'_{\lambda_k} \subset U_{\lambda_k}$ we have $\operatorname{conv}_{r_{\lambda_k}} U'_{\lambda_k} \subset \operatorname{conv}_{r_{\lambda_k}} U_{\lambda_k}$ for each k. Hence $p_{\lambda_k}(x) \leqslant \delta^{r_{\lambda_k}}$ for each k. Consequently, it follows from $x \in V$ that $q_{\lambda_k}(x) \leqslant \delta^{r_{\lambda_k}}$ for all $k = 1, 2, \ldots, n$. But this means that $V \subset O$ and we have shown that $T(\mathscr{Q}) = m(T(\mathscr{P}))$.

To show that $T(\mathscr{Q})$ is separated if and only if $T(\mathscr{P})$ is separated it suffices to show that $\ker q_\lambda = \ker p_\lambda$ for each $\lambda \in \Lambda$. Let $\lambda \in \Lambda$ be given. Clearly $\ker q_\lambda \subset \ker p_\lambda$. On the other hand if $x \in \ker p_\lambda$, then $p_\lambda(xy) = p_\lambda(yx) = 0$ for all $y \in A$. This implies that $\widetilde{p}_\lambda(x) = 0$ and thus also $q_\lambda(x) = 0$. So $\ker q_\lambda = \ker p_\lambda$, which completes the proof. □

Let now $(A, T(\mathscr{P}))$ be a locally A-pseudoconvex algebra. We say that $T(\mathscr{P})$ is *weakly regular* if for each $\lambda \in \Lambda$ there is a constant $m_\lambda > 0$ such that $p_\lambda(x) \leqslant m_\lambda \widetilde{p}_\lambda(x)$ for all $x \in A$. Note that if A has a unit element (denoted by e), then $(A, T(\mathscr{P}))$ is weakly regular (we can take $m_\lambda = p_\lambda(e)$ for each $\lambda \in \Lambda$, see [4]).

Corollary 3. *Let $(A, T(\mathscr{P}))$ be as in Theorem* 3. *If $T(\mathscr{P})$ is weakly regular (in particular if A has a unit), then $m(T(\mathscr{P}))$ is equivalent to $T(\widetilde{\mathscr{P}})$.*

References

[1] *M. Abel*: Gelfand-Mazur Algebras, Topological Vector Spaces, Algebras and Related Areas (Hamilton, ON). Pitman Research Notes in Math. Series 316. Longman Scientific & Technical, London, 1994, pp. 116–129.

[2] *M. Abel and A. Kokk*: Locally pseudoconvex Gelfand-Mazur algebras. Eesti Tead. Akad. Toimetised Füüs.-Mat. *37*, 377–386. (In Russian.)

[3] *J. Arhippainen*: On functional representation of uniformly A-convex algebras. Math. Japon. *46* (1997), 509–515.

[4] *J. Arhippainen*: On functional representation of commutative locally A-convex algebras. Rocky Mountain J. Math. *30* (2000), 777–794.

[5] *A. C. Cochran*: Topological algebras and Mackey topologies. Proc. Amer. Math. Soc. *30* (1971), 115–119.

[6] *A. C. Cochran*: Representation of A-convex algebras. Proc. Amer. Math. Soc. *41* (1973), 473–479.

[7] *A. C. Cochran, R. Keown and C. R. Williams*: On class of topological algebras. Pacific J. Math. *34* (1970), 17–25.

[8] *T. Husain*: The Open Mapping and Closed Graph Theorems in Topological Vector Spaces. Clarendon Press, Oxford, 1965.

[9] *L. Oubbi*: Topologies m-convexes dans les algebres A-convexes. Rend. Circ. Mat. Palermo *XLI* (1992), 397–406.

[10] *L. Oubbi*: Representation of locally convex algebras. Rev. Mat. Univ. Complut. Madrid *35* (1994), 233–244.

[11] *M. Oudadess*: Unité et semi-normes dans les algèbres localement convexes. Rev. Colombiana Mat. *16* (1982), 141–150.

[12] *T. W. Palmer*: Banach Algebras and the General Theory of *-Algebras. Cambridge Univ. Press, New York, 1994.

[13] *L. Waelbroeck*: Topological Vector Spaces and Algebras. Lecture Notes in Math. 230. Springer-Verlag, Berlin-New York, 1971.

Author's address: M. Abel, Institute of Pure Mathematics, Univ. of Tartu, Estonia, e-mail: `mati.abel@math.ut.ee`; J. Arhippainen, Department of Math. Sciences, Univ. of Oulu, Finland, e-mail: `jarhippa@sun3.oulu.fi`.