Sangwon Park; Eunha Cho

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INJECTIVE AND PROJECTIVE PROPERTIES OF $R[x]$-MODULES

Sangwon Park and Eunha Cho, Pusan

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Abstract. We study whether the projective and injective properties of left R-modules can be implied to the special kind of left $R[x]$-modules, especially to the case of inverse polynomial modules and Laurent polynomial modules.

Keywords: module, inverse polynomial module, injective module, projective modules

MSC 2000: 16E30, 13C11, 16D80

1. Introduction

Northcott [3] and McKerrow in [1] proved that if R is a left Noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. In [5] Park showed that $P[x^{-1}]$ is not a projective left $R[x]$-module while $P[x]$ is a projective left $R[x]$-module for a projective left R-module P. In this paper we study whether the projective and injective properties of left R-modules can be implied to the special kind of left $R[x]$-modules. We prove that for any non zero left R-module E, that the Laurent polynomial module $E[x,x^{-1}]$ is not an injective left $R[x]$-module and $E[x_1^{-1},x_2^{-2},\ldots]$ is not an injective left $R[x_1,x_2,\ldots]$-module, in general. We also give another proof of Northcott's and McKerrow's result by using locally nilpotent. And then we prove that for a projective left R-module P, the inverse power series module $P[[x^{-1}]]$ and the Laurent polynomial module $P[x,x^{-1}]$ are not projective left $R[x]$-module. Inverse polynomial modules were studied in [2], [4], [5] and recently in [6], [7], [8].

Definition 1.1. Let R be a ring and M a left R-module, then $M[x^{-1}]$ is the left $R[x]$-module such that

$$x(m_0 + m_1x^{-1} + \ldots + m_nx^{-n}) = m_1 + m_2x^{-1} + \ldots + m_nx^{-n+1}$$

and

$$r(m_0 + m_1x^{-1} + \ldots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \ldots + rm_nx^{-n}$$

where $r \in R$.

Similarly, we can also define $M[[x^{-1}]], M[x, x^{-1}], M[x, x^{-1}]]$, and also $M[[x, x^{-1}]$ as left $R[x]$-modules where, for example, $M[[x, x^{-1}]$ is the set of Laurent series in x with coefficients in M, i.e. the set of all formal sums $\sum\limits_{k \geqslant n_0} m_kx^k$ with n_0 any element of $\mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers).

Lemma 1.2. *Let M be a left R-module. Then*

$$\operatorname{Hom}_R(R[x], M) \cong M[[x^{-1}]]$$

as left $R[x]$-modules.

Proof. Define $\varphi\colon \operatorname{Hom}_R(R[x], M) \to M[[x^{-1}]]$ by

$$\varphi(f) = f(1) + f(x)x^{-1} + f(x^2)x^{-2} + \ldots.$$

Then φ is an isomorphism. □

We note that if E is an injective left R-module, then $\operatorname{Hom}_R(R[x], E)$ is an injective left $R[x]$-module so by the above Lemma 1.2, $E[[x^{-1}]]$ is an injective left $R[x]$-module.

2. Injective properties of $R[x]$-modules

Definition 2.1. Given any module M and $f \in \operatorname{End}(M)$ we say that f is locally nilpotent on M if for every $x \in M$, there exist $n \geqslant 1$ such that $f^n(x) = 0$.

The following Lemma 2.2 is originally due to Matlis and Gabriel.

Lemma 2.2. *If R is a left Noetherian ring, E is an injective left R-module, and $f \in \operatorname{End}(E)$ is such that E is an essential extension of* $\operatorname{Ker}(f)$, *then f is locally nilpotent on E.*

Proof. Let K be the kernel of f and E an essential extension of K. Consider the direct sum $K \oplus K \oplus \ldots$ of countable number of K's. Choose $(a_1, a_2, \ldots) \in E \oplus E \oplus \ldots$, then $a_i = 0$ for all $i \geqslant n$ for some n. Since E is an essential extension of K, choose $r_1 \in R$ such that $r_1 a_1 \in K$. Then choose $r_2 \in R$ such that $r_2(r_1 a_2) \in K$ and so on. Finally, choose $r_k \in R$ such that $r_k(r_{k-1} \ldots r_2 r_1 a_k) \in K$. Then

$$(r_n r_{n-1} \ldots r_2 r_1)(a_1, a_2, \ldots, a_n, 0, 0, \ldots) \in K \oplus K \oplus \ldots.$$

Thus $E \oplus E \oplus \ldots$ is an essential extension of $K \oplus K \oplus \ldots$. Since R is left Noetherian, $E \oplus E \oplus \ldots$ is injective, so it is an injective envelope of $K \oplus K \oplus \ldots$. If $M \subset E_1$, $M \subset E_2$ are injective envelopes of M and $\varphi\colon E_1 \to E_2$ is the identity on M then φ is an isomorphism. So define the map

$$\begin{aligned} \varphi\colon E \oplus E \oplus \ldots &\longrightarrow E \oplus E \oplus \ldots \\ (x_1, x_2, \ldots) &\longmapsto (x_1, x_2 - f(x_1), x_3 - f(x_2), \ldots). \end{aligned}$$

Then φ is a homomorphism, and $\varphi|_{K\oplus K\oplus\ldots} = \mathrm{id}_{K\oplus K\oplus\ldots}$. So φ is an automorphism of $E \oplus E \oplus \ldots$ and in particular φ is onto. Let $x \in E$ and consider $(x, 0, 0, \ldots)$. Then $\varphi(x_1, x_2, x_3, \ldots) = (x, 0, 0, \ldots)$ for some $(x_1, x_2, x_3, \ldots) \in E \oplus E \oplus \ldots$. Then

$$\begin{aligned} x_1 &= x, \\ x_2 - f(x_1) &= 0, \\ x_3 - f(x_2) &= 0, \end{aligned}$$

and so on. So $x_n = f^{n-1}(x)$ for all $n \geqslant 2$. But for some n, $x_{n+1} = 0$, i.e., $f^n(x) = 0$. Therefore, f is locally nilpotent on E. □

The following Theorem 2.3 is originally due to Northcott and McKerrow. We give another proof by using locally nilpotent.

Theorem 2.3. *Let R be a commutative Noetherian ring and E an injective left R-module. Then $E[x^{-1}]$ is an injective left $R[x]$-module.*

Proof. Let E be an injective left R-module. Then

$$\operatorname{Hom}_R(R[x], E) \cong E[[x^{-1}]]$$

is an injective left $R[x]$-module. Define $\varphi\colon\ E[[x^{-1}]] \to E[[x^{-1}]]$ by

$$\varphi(f) = xf$$

for $f \in E[[x^{-1}]]$, then φ is not locally nilpotent on $E[[x^{-1}]]$. So $E[[x^{-1}]]$ is not an essential extension of $\mathrm{Ker}(\varphi)$. Let $\overline{E}$ be an injective envelope of $\mathrm{Ker}(\varphi)$, then $\mathrm{Ker}(\varphi) \subset \overline{E} \subset E[[x^{-1}]]$. Then $\varphi\colon\ \overline{E} \to \overline{E}$ defined by

$$\varphi(f) = xf$$

for $f \in \overline{E}$ is locally nilpotent on $\overline{E}$. So $\overline{E} \subset E[x^{-1}]$. But $E[x^{-1}]$ is an essential extention of $\mathrm{Ker}(\varphi)$, so that $E[x^{-1}]$ is an esssestial extention of $\overline{E}$. Therefore, $\overline{E} = E[x^{-1}]$. Hence, $E[x^{-1}]$ is an injective left $R[x]$-module. □

We note that $E[x]$ is not an injective left $R[x]$-module if $E \neq 0$.

Theorem 2.4. *For any non zero left R-module E, $E[x, x^{-1}]$ is not an injective left $R[x]$-module.*

Proof. Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & (1+x) & \xrightarrow{\ i\ } & R[x] \\ & & \Big\downarrow{\scriptstyle h} & & \\ & & E[x, x^{-1}] & & \end{array}$$

defined by $h(1+x) = e$, $e \in E$; here i is the inclusion map. Then we can not complete the above diagram as a commutative diagram. □

Theorem 2.5. *Let E be an injective left R-module. Then $E[x_1^{-1}, x_2^{-2}, \ldots]$ is not an injective left $R[x_1, x_2, \ldots]$-module, in general.*

Proof. We give a counterexample for the case of $E = \mathbb{Q}$ (the set of all rational numbers), and $R = \mathbb{Z}$ (the set of all integers). Let $I = (x_1, x_2, x_3 \ldots)$ and J be an ideal generated by $x_i x_j$, for $i \neq j$, and x_i^3, for all i. Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{\ i\ } & Q[x_1, x_2, \ldots] \\ & & \Big\downarrow & & \\ & & I/J & & \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ & & Q[x^{-1}, x^{-2}, \ldots] & & \end{array}$$

defined by $\varphi\colon I/J \longrightarrow \mathbb{Q}[x_1^{-1}, x_2^{-2}, \ldots]$, $\varphi(x_i^2 + J) = 1$ and $\varphi(x_i + J) = x_i^{-1}$, and $i\colon I \longrightarrow \mathbb{Q}[x_1, x_2, \ldots]$ the inclusion map. Then we can not complete the above diagram to a commutative diagram.

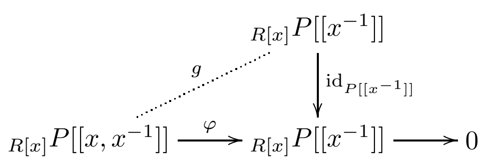
3. Projective properties of $R[x]$-modules

Theorem 3.1. *$P[[x^{-1}]]$ is not a projective left $R[x]$-module for P a projective left R-module.*

Proof. Let P be a left R-module and $P[[x, x^{-1}]], P[[x^{-1}]]$ be $R[x]$-modules, then $f\colon P[[x, x^{-1}]] \to P[[x^{-1}]]$ defined by

$$\varphi(\ldots + a_3x^3 + a_2x^2 + a_1x + a_0 + b_1x^{-1} + b_2x^{-2} + b_3x^{-3} + \ldots)$$
$$= a_0 + b_1x^{-1} + b_2x^{-2} + b_3x^{-3} + \ldots$$

is a surjective $R[x]$-linear map. If $P[[x^{-1}]]$ is an projective left $R[x]$-module, then we should be able to complete the following diagram as a commutative diagram by an $R[x]$-linear map g.



Let $a_0 \in P[[x^{-1}]]$ and $a_0 \neq 0$. Then $g(a_0) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$. But $xg(a_0) = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \ldots \neq 0$ and $g(xa_0) = g(0) = 0$. So, $g(xa_0) \neq xg(a_0)$. Therefore, g is not an $R[x]$-linear map. Hence, $P[[x^{-1}]]$ is not a projective left $R[x]$-module. □

Theorem 3.2. *$P[x, x^{-1}]$ is not a projective left $R[x]$-module for P a projective left R-module.*

Proof. We show that $R[x, x^{-1}]$ is not a projective left $R[x]$-module. Let $R[x]$ be considered as a left $R[x]$-module over itself. Consider the subsets $x^nR[x]$, for $n \geqslant 1$, then clearly the intersection of these sets is 0. We can argue the same for any free left $R[x]$-module F (so F is a direct sum of copies of $R[x]$). Now recalling that any projective left $R[x]$-module is direct summand of a free left $R[x]$-module, we see that the intersection of all the x^nP for P a projective left $R[x]$-module and $n \geqslant 1$ is also 0. But $x^nR[x, x^{-1}] = R[x, x^{-1}]$ for any $n \geqslant 1$. So $R[x, x^{-1}]$ is not a projective left $R[x]$-module. □

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Authors' address: Department of Mathematices, Dong-A University, Pusan, Korea 604-714, e-mails: `swpark@donga.ac.kr`, `choeh@donga.ac.kr`.