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ON SEMICONVEXITY PROPERTIES OF ROTATIONALLY INVARIANT FUNCTIONS IN TWO DIMENSIONS

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Abstract. Let f be a function defined on the set $\mathbf{M}^{2\times 2}$ of all 2 by 2 matrices that is invariant with respect to left and right multiplications of its argument by proper orthogonal matrices. The function f can be represented as a function $\tilde{f}$ of the signed singular values of its matrix argument. The paper expresses the ordinary convexity, polyconvexity, and rank 1 convexity of f in terms of its representation $\tilde{f}$.

Keywords: semiconvexity, rank 1 convexity, polyconvexity, convexity, rotational invariance

MSC 2000: 49J45, 74B20

1. Introduction

In the two-dimensional nonlinear elasticity and in the theory of phase transitions in solids, one deals with the energy functional

$$I(\mathbf{u}) = \int_\Omega f(D\mathbf{u})\,\mathrm{d}\mathbf{x}$$

where $\Omega \subset \mathbb{R}^2$, $\mathbf{u}\colon \Omega \to \mathbb{R}^2$ is a deformation with the gradient $D\mathbf{u}$ and $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ is the stored energy defined on the set $\mathbf{M}^{2\times 2}$ of all real 2×2 matrices. The elastic equilibrium, if it exists, corresponds to the minimum of I on an appropriate function space. The existence/nonexistence of the minimizer, the formation/absence of microstructure and other important properties of I are related to the semiconvexity properties of f, i.e., the rank 1 convexity, quasiconvexity, polyconvexity, and

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convexity, [11], [6], [7]. For an isotropic body the stored energy is rotationally invariant. A function $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ is said to be *rotationally invariant* (briefly, invariant) if $f(\mathbf{A}) = f(\mathbf{QAR})$ for all $\mathbf{A} \in \mathbf{M}^{2\times 2}$ and all $\mathbf{Q}$, $\mathbf{R}$ proper orthogonal (i.e., with $\det \mathbf{Q} = \det \mathbf{R} = 1$); if the same holds for all $\mathbf{Q}$, $\mathbf{R}$ orthogonal then f is called *fully rotationally invariant* (briefly, fully invariant). A combination of the polar and spectral decomposition theorems implies that an invariant function is expressible as

$$f(\mathbf{A}) = \tilde{f}(\tau), \quad \mathbf{A} \in \mathbf{M}^{2\times 2}, \tag{1}$$

where $\tilde{f}$ and τ are as follows. The function $\tilde{f}\colon \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$, called the *representation* of f, is symmetric and even:

$$\tilde{f}(\alpha_1, \alpha_2) = \tilde{f}(\alpha_2, \alpha_1), \quad \tilde{f}(\alpha) = \tilde{f}(-\alpha), \quad \alpha \in \mathbb{R}^2. \tag{2}$$

The pair $\tau = (\tau_1, \tau_2)$ is called the (pair of) *signed singular values* of $\mathbf{A}$, defined, [12], [15], as the unique pair such that $\tau_1 \geqslant |\tau_2|$ are ordered eigenvalues of $\sqrt{\mathbf{AA}^{\mathrm{T}}}$ and $\operatorname{sgn} \tau_2 = \operatorname{sgn} \det \mathbf{A}$. We write $\tau(\mathbf{A}) = (\tau_1(\mathbf{A}), \tau_2(\mathbf{A}))$. Note that

$$\tau_1(\mathbf{A}) \pm \tau_2(\mathbf{A}) = \frac{1}{\sqrt{2}} |\mathbf{A} \pm \operatorname{cof} \mathbf{A}|$$

and hence

$$\tau_{1,2}(\mathbf{A}) = \frac{1}{2\sqrt{2}} \big(|\mathbf{A} + \operatorname{cof} \mathbf{A}| \pm |\mathbf{A} - \operatorname{cof} \mathbf{A}| \big).$$

Indeed, in view of the invariance of the above expressions, it suffices to verify them on diagonal matrices, which is trivial. Furthermore, since the norm is convex and the cofactor is linear in dimension 2, one sees that the functions $\tau_1, \tau_1 + \tau_2, \tau_1 - \tau_2$ are convex on $\mathbf{M}^{2\times 2}$.

It is immediate that if f is invariant with the representation $\tilde{f}$ then

$$\tilde{f}(\alpha) = f(\operatorname{diag}(\alpha)), \quad \alpha \in \mathbb{R}^2,$$

and that f is fully invariant if and only if $\tilde{f}$ is *fully even,* i.e., in addition to (2) also

$$\tilde{f}(\alpha_1, \alpha_2) = \tilde{f}(-\alpha_1, \alpha_2), \quad \alpha \in \mathbb{R}^2.$$

Recall that $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ is said to be *convex* if

$$f((1-t)\mathbf{A} + t\mathbf{B}) \leqslant (1-t) f(\mathbf{A}) + t f(\mathbf{B}) \tag{3}$$

for every $\mathbf{A}, \mathbf{B} \in \mathbf{M}^{2\times 2}$ and every $t \in [0,1]$, while f is said to be *rank* 1 *convex* if (3) holds only provided additionally $\operatorname{rank}(\mathbf{A} - \mathbf{B}) \leqslant 1$. The function f is said to be *polyconvex* if there exists a convex function $g\colon \mathbf{M}^{2\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$f(\mathbf{A}) = g(\mathbf{A}, \det \mathbf{A}), \quad \mathbf{A} \in \mathbf{M}^{2\times 2}.$$

The paper continues the line [13–18] which seeks the understanding of the semiconvexity properties in terms of the shortened language of $\tilde{f}$. Specifically, reference [17] shows that invariant convex and rank 1 convex functions have certain monotonicity properties; see (5) $\Rightarrow$ (6), (27) $\Rightarrow$ (28) below for the two-dimensional case. (The fully invariant case has been treated in [10, Section 7.3].) The monotonicities are closely related to the ordered forces inequalities and to the Baker-Ericksen inequalities, [13]. On the other hand, recent literature on invariant functions contains global (as opposed to differential) conditions for convexity, [13], [12], [15], rank 1 convexity, [3], [15], [14], [16], and polyconvexity, [12], [15]. The paper integrates conditions of this type with the monotonicity. The results reveal the importance of the convex functions $\tau_1 \pm \tau_2$ and the representation of invariant functions in terms of them.

Theorems 2.1, 3.1, 4.1 and 5.3 (below) take natural forms in as much as they provide necessary and sufficient conditions, do not involve differentiability hypotheses, and apply to functions ranging in $\mathbb{R} \cup \{\infty\}$. They are suitable for the calculation of semiconvex hulls of functions and sets. In addition, Proposition 4.2 and Theorem 5.3 provide an interesting 'estimation' of the difference between the rank 1 convexity and polyconvexity.

2. Symmetric even convex functions

In view of the representation theorem (1) it is convenient to begin with the examination of the convexity of symmetric even functions. The results will be applied to invariant convex functions on $\mathbf{M}^{2\times 2}$ in Section 3 and to invariant polyconvex functions in Section 4.

Let $\mathbb{R}^2_+ := [0,\infty)^2$, $\mathbf{G}^2 := \{x \in \mathbb{R}^2 \colon x_1 \geqslant |x_2|\}$, and note that $\tau(\mathbf{A}) \in \mathbf{G}^2$ for each $\mathbf{A} \in \mathbf{M}^{2\times 2}$. A function $h\colon D \to \mathbb{R} \cup \{\infty\}$, where $D \subset \mathbb{R}^2$, is said to be non-decreasing if $h(x) \leqslant h(y)$ for every $x, y \in D$ such that $x_i \leqslant y_i$, $i = 1, 2$.

Theorem 2.1. *Let $g\colon \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ be a symmetric even function. The following conditions are equivalent:*

(i) *g is convex;*

(ii) *if $\alpha, \beta, \gamma \in \mathbf{G}^2$ and $t \in [0,1]$ satisfy*

$$\gamma_1 + \varepsilon\gamma_2 \leqslant (1-t)(\alpha_1 + \varepsilon\alpha_2) + t(\beta_1 + \varepsilon\beta_2) \quad \textit{for each } \varepsilon \in \{-1, 1\}$$

then

$$g(\gamma) \leqslant (1-t)g(\alpha) + tg(\beta);$$

(iii) *there exists a convex nondecreasing function* $h\colon \mathbb{R}_+^2 \to \mathbb{R} \cup \{\infty\}$ *such that*

$$g(\alpha) = h(\alpha_1 + \alpha_2, \alpha_1 - \alpha_2) \tag{4}$$

for every $\alpha \in \mathbf{G}^2$.

Proof. (i) $\Rightarrow$ (ii): Let g be convex on $\mathbb{R}^2$. If $\alpha, \beta \in \mathbf{G}^2$ satisfy

$$\alpha_1 + \varepsilon\alpha_2 \leqslant \beta_1 + \varepsilon\beta_2 \quad \text{for each } \varepsilon \in \{-1, 1\} \tag{5}$$

then

$$g(\alpha) \leqslant g(\beta). \tag{6}$$

This is just a reformulation of [8, Lemma 1.2]. Alternatively, this is proved by specializing the proof of [17, Lemma 4.2] to dimension 2. Combining the monotonicity (5) $\Rightarrow$ (6) with the convexity, we obtain (ii).

(ii) $\Rightarrow$ (iii): Since the mapping $\alpha \mapsto (\alpha_1 + \alpha_2, \alpha_1 - \alpha_2)$ maps $\mathbf{G}^2$ bijectively onto $\mathbb{R}_+^2$, the implication is immediate.

(iii) $\Rightarrow$ (i): Since we have (4) for each $\alpha \in \mathbf{G}^2$ and g is symmetric even, one deduces that

$$g(\alpha) = h(|\alpha_1 + \alpha_2|, |\alpha_1 - \alpha_2|)$$

for each $\alpha \in \mathbb{R}^2$. Thus g is a composition of a convex nondecreasing function with two convex functions. $\square$

Recall that fully invariant functions are represented by symmetric fully even functions. For these, Theorem 2.1 remains valid, but the following is also true.

Theorem 2.2. *Let* $g\colon \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ *be a symmetric fully even function. The following conditions are equivalent:*

(i) g *is convex;*

(ii) *if* $\alpha, \beta, \gamma \in \mathbf{G}^2 \cap \mathbb{R}_+^2$ *and* $t \in [0,1]$ *satisfy*

$$\gamma_1 \leqslant (1-t)\alpha_1 + t\beta_1, \quad \gamma_1 + \gamma_2 \leqslant (1-t)(\alpha_1 + \alpha_2) + t(\beta_1 + \beta_2)$$

then

$$g(\gamma) \leqslant (1-t)g(\alpha) + tg(\beta);$$

(iii) *there exists a convex nondecreasing function* $h\colon \mathbf{P}^2 \to \mathbb{R} \cup \{\infty\}$, *defined on*

$$\mathbf{P}^2 := \{x \in \mathbb{R}^2 \colon\ x_1 \geqslant x_2 \geqslant \tfrac{1}{2}x_1\} \subset \mathbf{G}^2 \cap \mathbb{R}^2_+, \tag{7}$$

such that

$$g(\alpha) = h(\alpha_1 + \alpha_2, \alpha_1) \tag{8}$$

for each $\alpha \in \mathbf{G}^2 \cap \mathbb{R}^2_+$.

Proof. (i) $\Rightarrow$ (ii): Let g be convex on $\mathbb{R}^2$. Prove first that if $\alpha, \beta \in \mathbf{G}^2 \cap \mathbb{R}^2_+$ satisfy

$$\alpha_1 \leqslant \beta_1, \quad \alpha_1 + \alpha_2 \leqslant \beta_1 + \beta_2, \tag{9}$$

then

$$g(\alpha) \leqslant g(\beta). \tag{10}$$

If in addition to (9) also $\alpha_1 - \alpha_2 \leqslant \beta_1 - \beta_2$, then (10) follows from Theorem 2.1. Suppose now that $\alpha_1 - \alpha_2 > \beta_1 - \beta_2$. Let $\gamma = (\gamma_1, \gamma_2)$ be determined from

$$\gamma_1 = \beta_1, \quad \alpha_1 - \alpha_2 = \gamma_1 - \gamma_2,$$

i.e., $\gamma = (\beta_1, \gamma_2), \gamma_2 = \beta_1 - \alpha_1 + \alpha_2$. Then

$$\alpha_1 + \varepsilon\alpha_2 \leqslant \gamma_1 + \varepsilon\gamma_2 \quad \text{for each } \varepsilon \in \{-1, 1\}; \quad \text{moreover} \quad \gamma_2 \leqslant \beta_2. \tag{11}$$

From $(11)_1$ and Theorem 2.1, $g(\alpha) \leqslant g(\gamma)$. Furthermore, $g(\beta_1, \cdot)$ is convex and even; hence nondecreasing on $[0, \infty)$ and thus $g(\gamma) \leqslant g(\beta)$ by $(11)_2$. To summarize, $g(\alpha) \leqslant g(\gamma) \leqslant g(\beta)$ which completes the proof of the monotonicity (9) $\Rightarrow$ (10). A combination with the convexity of g provides (ii).

(ii) $\Rightarrow$ (iii): The mapping $\alpha \mapsto (\alpha_1 + \alpha_2, \alpha_1)$ maps $\mathbf{G}^2 \cap \mathbb{R}^2_+$ onto $\mathbf{P}^2$; the rest is immediate.

(iii) $\Rightarrow$ (i): Prove first that (iii) implies that

$$g(\beta) = h\bigl(|\beta_1| + |\beta_2|, \max\{|\beta_1|, |\beta_2|\}\bigr) \tag{12}$$

for every $\beta \in \mathbb{R}^2$. Indeed, the argument of h is in $\mathbf{P}^2$ for each $\beta \in \mathbb{R}^2$. Next, the function defined by the right-hand side of (12) is symmetric and fully even. Moreover, for $\beta \in \mathbf{G}^2 \cap \mathbb{R}^2_+$ the argument on the right-hand side of (12) reduces to $(\beta_1 + \beta_2, \beta_1)$ and thus the equality in (12) holds by (8) in this case. That (12) holds generally is then deduced by noting that both sides of this equality are symmetric and fully even functions. Finally, the functions $\beta \mapsto |\beta_1| + |\beta_2|, \max\{|\beta_1|, |\beta_2|\}$ are convex and h is nondecreasing and convex. Thus g is convex. $\square$

3. Invariant convex functions

Recall the mapping $\mathbf{A} \mapsto \tau(\mathbf{A}) = (\tau_1(\mathbf{A}), \tau_2(\mathbf{A}))$ which associates the pair of signed singular values with a matrix $\mathbf{A} \in \mathbf{M}^{2\times 2}$ (see the introduction). The functions

$$\tau_1 \pm \tau_2, \quad \tau_1 + |\tau_2|, \quad \tau_1$$

are convex on $\mathbf{M}^{2\times 2}$. Indeed, the convexity of $\tau_1 \pm \tau_2, \tau_1$ has been proved in Introduction and the convexity of $\tau_1 + |\tau_2|$ follows from $\tau_1 + |\tau_2| = \max\{\tau_1 + \tau_2, \tau_1 - \tau_2\}$. Alternatively, this follows from the $SO(n)$-invariant version of the von Neumann's trace inequality, [13, Proposition 18.3.2(2)]; see also [12] and [15].

Theorem 3.1. *Let $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ be invariant and let $\tilde f$ be its representation. The following conditions are equivalent:*

(i) *f is convex;*

(ii) *$\tilde f$ is convex;*

(iii) *if $\alpha, \beta, \gamma \in \mathbf{G}^2$ and $t \in [0,1]$ satisfy*

$$\gamma_1 + \varepsilon\gamma_2 \leqslant (1-t)(\alpha_1 + \varepsilon\alpha_2) + t(\beta_1 + \varepsilon\beta_2) \quad \textit{for each } \varepsilon \in \{-1, 1\}$$

then

$$\tilde f(\gamma) \leqslant (1-t)\tilde f(\alpha) + t\tilde f(\beta);$$

(iv) *there exists a convex nondecreasing function $h\colon \mathbb{R}^2_+ \to \mathbb{R} \cup \{\infty\}$ such that*

$$\tilde f(\alpha) = h(\alpha_1 + \alpha_2, \alpha_1 - \alpha_2)$$

for every $\alpha \in \mathbf{G}^2$.

In particular, the representation $g := \tilde f$ of a convex invariant function has monotonicity: if $\alpha, \beta \in \mathbf{G}^2$ satisfy (5) then (6) holds. The equivalence (i) $\Leftrightarrow$ (ii) is due to Dacorogna & Koshigoe [8, Theorem 1.1].

Proof. (i) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (iii) follows from Theorem 2.1. (iii) $\Rightarrow$ (iv) is immediate.

(iv) $\Rightarrow$ (i): Condition (iv) implies that

$$f(\mathbf{A}) = h(\tau_1(\mathbf{A}) + \tau_2(\mathbf{A}), \tau_1(\mathbf{A}) - \tau_2(\mathbf{A}))$$

for every $A \in \mathbf{M}^{2\times 2}$ and it suffices to recall that $\tau_1 \pm \tau_2$ are convex. $\square$

For fully invariant functions we have additionally the following result.

Theorem 3.2. *Let $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R}\cup\{\infty\}$ be fully invariant and let $\tilde f$ be its representation. The following conditions are equivalent:*

(i) *f is convex;*

(ii) *$\tilde f$ is convex;*

iii) *if $\alpha,\beta,\gamma \in \mathbf{G}^2\cap\mathbb{R}^2_+$ and $t\in[0,1]$ satisfy*

$$\gamma_1 \leqslant (1-t)\alpha_1 + t\beta_1, \quad \gamma_1+\gamma_2 \leqslant (1-t)(\alpha_1+\alpha_2) + t(\beta_1+\beta_2)$$

then

$$\tilde f(\gamma) \leqslant (1-t)\tilde f(\alpha) + t\tilde f(\beta);$$

(iv) *there exists a convex nondecreasing function $h\colon \mathbf{P}^2 \to \mathbb{R}\cup\{\infty\}$, where $\mathbf{P}^2$ is defined in (7), such that*

$$g(\alpha) = h(\alpha_1+\alpha_2, \alpha_1)$$

for each $\alpha\in\mathbf{G}^2\cap\mathbb{R}^2_+$.

In particular, the representation $g := \tilde f$ of a convex invariant function has monotonicity: if $\alpha,\beta\in\mathbf{G}^2\cap\mathbb{R}^2_+$ satisfy (9) then (10) holds.

Proof. (i) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (iii) follows from Theorem 2.2. (iii) $\Rightarrow$ (iv) is immediate.

(iv) $\Rightarrow$ (i): Condition (iv) implies that

$$f(\mathbf{A}) = h(\tau_1(\mathbf{A}) + |\tau_2(\mathbf{A})|, \tau_1(\mathbf{A}))$$

for every $\mathbf{A}\in\mathbf{M}^{2\times 2}$ and it suffices to recall that $\tau_1, \tau_1+|\tau_2|$ are convex. $\square$

4. Invariant polyconvex functions

The treatment of polyconvex invariant functions is based on the monotonicity of symmetric even convex functions.

Theorem 4.1. *Let $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R}\cup\{\infty\}$ be an invariant function with the representation $\tilde f$. The following conditions are equivalent:*

(i) *f is polyconvex;*

(ii) *there exists a convex $h\colon \mathbb{R}^3 \to \mathbb{R}\cup\{\infty\}$, symmetric and even in the first two variables, such that*

$$\tilde f(\alpha) = h(\alpha_1, \alpha_2, \alpha_1\alpha_2)$$

for every $\alpha\in\mathbf{G}^2$;

(iii) *there exists a convex function* $k\colon\ \mathbb{R}^2_+\times\mathbb{R}\to\mathbb{R}\cup\{\infty\}$, *nondecreasing in the first two variables, such that*

$$\tilde f(\alpha)=k(\alpha_1+\alpha_2,\alpha_1-\alpha_2,\alpha_1\alpha_2)$$

for every $\alpha\in\mathbf{G}^2$;

(iv) *if* $\beta,\alpha^i\in\mathbf{G}^2$, $t^i\geqslant 0$, $i=1,\dots,p$, *satisfy*

$$\sum_{i=1}^p t^i=1, \tag{13}$$

$$\beta_1\beta_2=\sum_{i=1}^p t^i\alpha_1^i\alpha_2^i,\quad \beta_1+\varepsilon\beta_2\leqslant\sum_{i=1}^p t^i\alpha_1^i+\varepsilon\alpha_2^i\quad \textit{for each } \varepsilon\in\{-1,1\} \tag{14}$$

then

$$\tilde f(\beta)\leqslant\sum_{i=1}^p t^i\tilde f(\alpha^i). \tag{15}$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is essentially contained in [8], [12], [15] and the proof is therefore omitted. (ii) $\Rightarrow$ (iii) follows from (ii) $\Rightarrow$ (iii) in Theorem 2.1. (iii) $\Rightarrow$ (iv) is immediate.

(iv) $\Rightarrow$ (i): Let (iv) hold, and let $\mathbf{B},\mathbf{A}^i\in\mathbf{M}^{2\times2}$, $t^i\geqslant 0$, $i=1,\dots,p$, be such that (13) holds and

$$\det\mathbf{B}=\sum_{i=1}^p t^i\det\mathbf{A}^i,\quad \mathbf{B}=\sum_{i=1}^p t^i\mathbf{A}^i. \tag{16}$$

Denote by β, α^i the signed singular values of $\mathbf{B}$, $\mathbf{A}^i$, respectively. Then (16) reads as (14) and the convexity of $\tau_1\pm\tau_2$ and $(16)_2$ implies $(14)_2$. Hence (15) holds which reads

$$f(\mathbf{B})\leqslant\sum_{i=1}^p t^i f(\mathbf{A}^i). \tag{17}$$

The implication (16) $\Rightarrow$ (17) gives the polyconvexity [7, Subsection 4.1.1.2]. $\square$

This section is concluded with the following necessary condition, stated here for future reference (Section 5).

Proposition 4.2. *If* $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ *is polyconvex then*

$$\tilde{f}(\gamma) \leqslant (1-t)\tilde{f}(\alpha) + t\tilde{f}(\beta)$$

for every $\alpha, \beta, \gamma \in \mathbf{G}^2$ *and* $t \in [0,1]$ *satisfying*

$$\begin{cases} \gamma_1\gamma_2 = (1-t)\alpha_1\alpha_2 + t\beta_1\beta_2, \\ \gamma_1 + \varepsilon\gamma_2 \leqslant (1-t)(\alpha_1 + \varepsilon\alpha_2) + t(\beta_1 + \varepsilon\beta_2), \end{cases} \tag{18}$$

where

$$\varepsilon = \begin{cases} +1 & \textit{if } (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \geqslant 0, \\ -1 & \textit{if } (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) < 0. \end{cases} \tag{19}$$

P r o o f. If

$$\bar{\gamma} := (1-t)\alpha + t\beta \tag{20}$$

then from (18),

$$\gamma_1 + \varepsilon\gamma_2 \leqslant \bar{\gamma}_1 + \varepsilon\bar{\gamma}_2. \tag{21}$$

The function $\varphi(t) := \varepsilon\bar{\gamma}_1\bar{\gamma}_2$, $0 \leqslant t \leqslant 1$, where $\bar{\gamma}$ is given by (20), is quadratic and its second derivative is $2\varepsilon(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \geqslant 0$. Therefore its convexity implies

$$\varepsilon\bar{\gamma}_1\bar{\gamma}_2 \leqslant (1-t)\varphi(0) + t\varphi(1) = \varepsilon\big((1-t)\alpha_1\alpha_2 + t\beta_1\beta_2\big) = \varepsilon\gamma_1\gamma_2. \tag{22}$$

Combining the square of (21) with (22) and taking the square root, we obtain

$$\gamma_1 - \varepsilon\gamma_2 \leqslant \bar{\gamma}_1 - \varepsilon\bar{\gamma}_2.$$

Hence

$$\gamma_1 - \varepsilon\gamma_2 \leqslant (1-t)(\alpha_1 - \varepsilon\alpha_2) + t(\beta_1 - \varepsilon\beta_2)$$

and the conclusion follows from (iv) of Theorem 4.1. $\square$

5. Invariant rank 1 convex functions

Recall that the rank 1 convexity is the convexity restricted to line segments with endpoints $\mathbf{A}$, $\mathbf{B}$ satisfying $\operatorname{rank}(\mathbf{A}-\mathbf{B}) \leqslant 1$. A matrix $\mathbf{B} \in \mathbf{M}^{2\times 2}$ is said to be a rank 1 perturbation of $\mathbf{A} \in \mathbf{M}^{2\times 2}$ if $\operatorname{rank}(\mathbf{A}-\mathbf{B}) \leqslant 1$. The following remark shows that the signed singular values of a rank 1 perturbation are restricted by definite inequalities which will occur in the subsequent results.

Remark 5.1. ([4], [5], [15]) Let $\mathbf{A} \in \mathbf{M}^{2\times 2}$ have signed singular values $\alpha \in \mathbf{G}^2$. Then $\beta \in \mathbf{G}^2$ are the signed singular values of some rank 1 perturbation of $\mathbf{A}$ if and only if

$$|\alpha_2| \leqslant \beta_1, \quad |\beta_2| \leqslant \alpha_1. \tag{23}$$

The following proposition is a special case of a general result; cf. [15], [14].

Proposition 5.2. *Let $\mathbf{A} \in \mathbf{M}^{2\times 2}$ have signed singular values α, let $\beta \in \mathbf{G}^2$ satisfy* (23), *and let ε be given by* (19). *Then there exists a* rank 1 *perturbation* $\mathbf{B}$ *of $\mathbf{A}$ such that if $t \in [0,1]$ and $C := (1-t)\mathbf{A} + t\mathbf{B}$, $\gamma := \tau(\mathbf{C})$ then*

$$\begin{cases} \gamma_1\gamma_2 = (1-t)\alpha_1\alpha_2 + t\beta_1\beta_2, \\ \gamma_1 + \varepsilon\gamma_2 = (1-t)(\alpha_1+\varepsilon\alpha_2) + t(\beta_1+\varepsilon\beta_2). \end{cases} \tag{24}$$

The rank 1 perturbation $\mathbf{B}$ is given explicitly in the cited papers but their form is irrelevant for our purposes. What is more important for understanding the results to follow is the occurrence of ε. This is not an artifact of the proof nor a matter of convenience; rather, ε is inherently connected with the structure of the set of all rank 1 perturbations of $\mathbf{A}$. Thus, e.g., if α, β are as in Proposition 5.2 such that $\varepsilon = 1$, there is no rank 1 perturbation $\mathbf{B}$ of $\mathbf{A}$ such that we would have (24) with $\varepsilon = -1$.

Theorem 5.3. *An invariant function $f\colon \mathbf{M}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ is* rank 1 *convex if and only if*

$$\tilde{f}(\gamma) \leqslant (1-t)\tilde{f}(\alpha) + t\tilde{f}(\beta) \tag{25}$$

for every $\alpha, \beta, \gamma \in \mathbf{G}^2$ and $t \in [0,1]$ that satisfy (23) *and*

$$\begin{cases} \gamma_1\gamma_2 = (1-t)\alpha_1\alpha_2 + t\beta_1\beta_2, \\ \gamma_1 + \varepsilon\gamma_2 \leqslant (1-t)(\alpha_1+\varepsilon\alpha_2) + t(\beta_1+\varepsilon\beta_2) \end{cases} \tag{26}$$

where ε is given by (19).

P r o o f. Let f be rank 1 convex, and prove first that if $\alpha, \beta \in \mathbf{G}^2$ satisfy

$$\alpha_1 \leqslant \beta_1, \quad \alpha_1\alpha_2 = \beta_1\beta_2 \tag{27}$$

then

$$\tilde{f}(\alpha) \leqslant \tilde{f}(\beta). \tag{28}$$

Indeed, [17, Theorem 5.4] implies that (28) holds if

$$\alpha_1 + \alpha_2 \leqslant \beta_1 + \beta_2, \quad \alpha_1\alpha_2 = \beta_1\beta_2. \tag{29}$$

It now suffices to note that for $\alpha, \beta \in \mathbf{G}^2$, (29) and (27) are equivalent, since $\alpha_1 \mapsto \alpha_1 + \alpha_2$ is an increasing function of α_1 on the hyperbola $\alpha_1\alpha_2 = \beta_1\beta_2$. Let now α, β, γ, t be as in the statement of the theorem. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be as in Proposition 5.2 and denote the signed singular values of $\mathbf{C}$ by $\overline{\gamma}$ so that

$$\begin{cases} \overline{\gamma}_1\overline{\gamma}_2 = (1-t)\alpha_1\alpha_2 + t\beta_1\beta_2, \\ \overline{\gamma}_1 + \varepsilon\overline{\gamma}_2 = (1-t)(\alpha_1 + \varepsilon\alpha_2) + t(\beta_1 + \varepsilon\beta_2), \end{cases}$$

and (26) implies

$$\gamma_1\gamma_2 = \overline{\gamma}_1\overline{\gamma}_2, \quad \gamma_1 + \varepsilon\gamma_2 \leqslant \overline{\gamma}_1 + \varepsilon\overline{\gamma}_2. \tag{30}$$

Combining the square of $(30)_2$ with $(30)_1$ we obtain $\gamma_1 - \varepsilon\gamma_2 \leqslant \overline{\gamma}_1 - \varepsilon\overline{\gamma}_2$ and hence $\gamma_1 \leqslant \overline{\gamma}_1$. Combining the last inequality and $(30)_1$ with the monotonicity of $\tilde{f}$ we obtain

$$\tilde{f}(\gamma) \leqslant \tilde{f}(\overline{\gamma}). \tag{31}$$

On the other hand, applying the rank 1 convexity inequality to the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ from Proposition 5.2 we obtain

$$\tilde{f}(\overline{\gamma}) \leqslant (1-t)\tilde{f}(\alpha) + t\tilde{f}(\beta) \tag{32}$$

and thus (31) and (32) imply (25). Conversely, let $\tilde{f}$ satisfy the condition stated in the theorem, let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{M}^{2\times 2}$, $t \in [0,1]$ satisfy

$$\mathbf{C} = (1-t)\mathbf{A} + t\mathbf{B}, \quad \operatorname{rank}(\mathbf{A} - \mathbf{B}) = 1$$

and denote by α, β, γ the signed singular values of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Then (23) hold, $(26)_1$ holds as a consequence of the rank 1 convexity of the determinant and $(26)_2$ holds as a consequence of the convexity of $\tau_1 + \varepsilon\tau_2$. Hence (25) holds which reads

$$f(\mathbf{C}) \leqslant (1-t)f(\mathbf{A}) + tf(\mathbf{B}).$$

$\square$

Remark 5.4. It is interesting to compare optically the necessary condition for the polyconvexity in Proposition 4.2 with the equivalent condition for the rank 1 convexity in Theorem 5.3. The only difference is the requirement (23), which makes the class of representations $\tilde{f}$ satisfying Proposition 4.2 a subset of those $\tilde{f}$ which satisfy Theorem 5.3. It is well-known that the set of invariant rank 1 convex functions is really wider than that of invariant polyconvex functions [2], [9], [1]. A recent example [19] also shows that the rank 1 convexity and polyconvexity are also different in the narrower class of fully invariant functions.

References

[1] *J. J. Alibert and B. Dacorogna*: An example of a quasiconvex function that is not polyconvex in two dimensions. Arch. Rational Mech. Anal. *117* (1992), 155–166.

[2] *G. Aubert*: On a counterexample of a rank 1 convex function which is not polyconvex in the case $N = 2$. Proc. Roy. Soc. Edinburgh *106A* (1987), 237–240.

[3] *G. Aubert*: Necessary and sufficient conditions for isotropic rank-one convex functions in dimension 2. J. Elasticity *39* (1995), 31–46.

[4] *G. Aubert and R. Tahraoui*: Sur la faible fermeture de certains ensembles de contrainte en élasticité nonlinéaire plane. C. R. Acad. Sci. Paris *290* (1980), 537–540.

[5] *G. Aubert and R. Tahraoui*: Sur la faible fermeture de certains ensembles de contrainte en élasticite nonlinéaire plane. Arch. Rational Mech. Anal. *97* (1987), 33–58.

[6] *J. M. Ball*: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. *63* (1977), 337–403.

[7] *B. Dacorogna*: Direct Methods in the Calculus of Variations. Springer, Berlin, 1989.

[8] *B. Dacorogna and H. Koshigoe*: On the different notions of convexity for rotationally invariant functions. Ann. Fac. Sci. Toulouse *II* (1993), 163–184.

[9] *B. Dacorogna and P. Marcellini*: A counterexample in the vectorial calculus of variations. In: Material Instabilities in Continuum Mechanics (J. M. Ball, ed.). Clarendon Press, Oxford, 1985/1986, pp. 77–83.

[10] *B. Dacorogna and P. Marcellini*: Implicit Partial Differential Equations. Birkhäuser, Basel, 1999.

[11] *C. B. Morrey, Jr.*: Multiple Integrals in the Calculus of Variations. Springer, New York, 1966.

[12] *P. Rosakis*: Characterization of convex isotropic functions. J. Elasticity *49* (1998), 257–267.

[13] *M. Šilhavý*: The Mechanics and Thermodynamics of Continuous Media. Springer, Berlin, 1997.

[14] *M. Šilhavý*: On isotropic rank 1 convex functions. Proc. Roy. Soc. Edinburgh *129A* (1999), 1081–1105.

[15] *M. Šilhavý*: Convexity conditions for rotationally invariant functions in two dimensions. In: Applied Nonlinear Analysis (A. Sequeira et al., ed.). Kluwer Academic, New York, 1999, pp. 513–530; preprint. Mathematical Institute, Prague, 1997.

[16] *M. Šilhavý*: Rotationally invariant rank 1 convex functions. Appl. Math. Optim. *44* (2001), 1–15.

[17] *M. Šilhavý*: Monotonicity of rotationally invariant convex and rank 1 convex functions. Proc. Royal Soc. Edinburgh *132A* (2002), 419–435.

[18] *M. Šilhavý*: Rank 1 Convex hulls of isotropic functions in dimension 2 by 2. Math. Bohem. *126* (2001), 521–529.

[19] *M. Šilhavý*: An $O(n)$ invariant rank 1 convex function that is not polyconvex. Theor. Appl. Mech. *28–29* (2002), 325–336.

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