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DETERMINANTS OF MATRICES ASSOCIATED WITH INCIDENCE FUNCTIONS ON POSETS

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Abstract. Let $S = \{x_1, \ldots, x_n\}$ be a finite subset of a partially ordered set P. Let f be an incidence function of P. Let $[f(x_i \wedge x_j)]$ denote the $n \times n$ matrix having f evaluated at the meet $x_i \wedge x_j$ of x_i and x_j as its i, j-entry and $[f(x_i \vee x_j)]$ denote the $n \times n$ matrix having f evaluated at the join $x_i \vee x_j$ of x_i and x_j as its i, j-entry. The set S is said to be meet-closed if $x_i \wedge x_j \in S$ for all $1 \leqslant i, j \leqslant n$. In this paper we get explicit combinatorial formulas for the determinants of matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S. We also obtain necessary and sufficient conditions for the matrices $f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S to be nonsingular. Finally, we give some number-theoretic applications.

Keywords: meet-closed set, greatest-type lower, incidence function, determinant, nonsingularity

MSC 2000: 11C20, 15A57

1. Introduction

Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. The matrix having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry is called the *greatest common divisor* (GCD) *matrix*, denoted by $[(x_i, x_j)]$. The matrix having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry is called the *least common multiple* (LCM) *matrix*, denoted by $([x_i, x_j])$. The set S is said to be *factor-closed* if it contains every divisor of x for any $x \in S$. H. J. S. Smith [10] showed that the determinant of the GCD matrix $[(x_i, x_j)]$ on a factor-closed set S is the product $\prod_{i=1}^{n} \varphi(x_i)$, where φ is Euler's totient function. The set S is said to be *gcd-closed* if

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$(x_i, x_j) \in S$ for all $1 \leqslant i, j \leqslant n$. It is clear that a factor-closed set is a gcd-closed set but not conversely.

Let f be an arithmetical function. Let $[f(x_i, x_j)]$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry. In [10], Smith also considered the determinant of the matrix $[f(x_i, x_j)]$ on a factor-closed set S. It was shown to be the product $\prod_{k=1}^{n} (f * \mu)(x_k)$, where $f * \mu$ is the Dirichlet product of f and μ. In [4], Bourque and Ligh obtained a generalization of Smith's result. Haukkanen [5] gave an abstract generalization of Bourque and Ligh's result.

Now let f be an incidence function and $S = \{x_1, \ldots, x_n\}$ a meet-closed set of a finite partially ordered set (poset) P (for related definitions, see the next section). Let $[f(x_i \wedge x_j)]$ denote the $n \times n$ matrix having f evaluated at the meet $x_i \wedge x_j$ of x_i and x_j as its i, j-entry, and let $[f(x_i \vee x_j)]$ denote the $n \times n$ matrix having f evaluated at the join $x_i \vee x_j$ of x_i and x_j as its i, j-entry. In this paper we will obtain explicit combinatorial formulas for the determinants of the matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S. We will also get necessary and sufficient conditions for the matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S to be nonsingular. In the last section we give some number-theoretic applications.

2. Preliminaries and definitions

Let $(P, \leqslant)$ be a poset. We say that P is a *meet semilattice* if for any $x, y \in P$ there exists a unique $z \in P$ such that

(i) $z \leqslant x$ and $z \leqslant y$, and

(ii) if $w \leqslant x$ and $w \leqslant y$ for some $w \in P$, then $w \leqslant z$.

In such a case z is called the *meet* of x and y and is denoted by $x \wedge y$. Let S be a subset of P. We call S *lower-closed* if for every $x, y \in P$ with $x \in S$ and $y \leqslant x$ we have $y \in S$. We call S *meet-closed* if for every $x, y \in S$ we have $x \wedge y \in S$. It is clear that a lower-closed set is always meet-closed but not conversely. The concepts of "lower-closed" and "meet closed" are generalizations of "factor-closed" and "gcd-closed" [2], [3], respectively.

Let f be a complex-valued function on $P \times P$ such that $f(x, y) = 0$ whenever $x \nleqslant y$. Then we say that f is an *incidence function* of P. If f and g are incidence functions of P, their *sum* $f + g$ is defined by $(f + g)(x, y) = f(x, y) + g(x, y)$ and their *convolution* $f * g$ is defined by$(f * g)(x, y) = \sum_{x \leqslant z \leqslant y} f(x, z)g(z, y)$. The set of all incidence functions of P under addition and convolution forms a ring with unity, where the *unity* δ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The incidence function ζ is defined by $\zeta(x,y)=1$ if $x\leqslant y$, and $\zeta(x,y)=0$ otherwise. The Möbius function μ of P is the inverse of ζ.

In what follows, let $(P,\leqslant)=(P,\wedge,\vee)$ be a finite meet semilattice. Let S be a subset of P and denote $S=\{x_1,x_2,\dots,x_n\}$ with $x_i<x_j\Rightarrow i<j$. For any incidence function f of P we denote $f(0,x)=f(x)$, where $0=\min P$. For example, let $(P,\leqslant)=(\mathbf{Z}^+,|)$. Then $\mu(1,n)$ is the usual number-theoretic function $\mu(n)$.

Proposition 2.1 ([5])**.** *Let $S=\{x_1,\dots,x_n\}$ be a meet-closed set. Then the determinant of the matrix $[f(x_i\wedge x_j)]$ defined on $S=\{x_1,\dots,x_n\}$ is equal to the product $\prod_{k=1}^{n}\psi_f(x_k)$, where*

$$\psi_f(x_k)=\sum_{\substack{d\leqslant x_k\\ d\nleqslant x_t,\, t<k}}(f*\mu)(d). \tag{1}$$

Note that Haukkanen [5] writes this formula without using convolution of incidence functions.

Definition 2.2. Let T be a given subset of P. For any $a,b\in T$ and $a<b$, we say that a is a *greatest-type lower* of b in T, if $a\leqslant c$, $c<b$ and $c\in T$ implies $c=a$.

If $(P,\leqslant)=(\mathbf{Z}^+,|)$, then the concept of greatest-type lower reduces to that of *greatest-type divisor* introduced in [7].

Definition 2.3. Let f be a complex-valued function on P. Then f is said to be *semi-multiplicative* if for any $x,y\in P$, one has $f(x)f(y)=f(x\wedge y)f(x\vee y)$.

The above concept of a semi-multiplicative function on P is a generalization of the known concept of a *semi-multiplicative arithmetical function* [9, p. 49].

Definition 2.4. For any incidence function f, we define for any $x\in P$ the function $1/f$ to be 0 if $f(x)=0$; $1/f(x)$ if $f(x)\neq 0$.

It is easy to check that the following is true.

Proposition 2.5. *Let f be an incidence function. Then f is semi-multiplicative if and only if $1/f$ is semi multiplicative.*

3. Combinatorial formulas for $\det[f(x_i \wedge x_j)]$ and $\det(f[x_i \vee x_j])$

Throughout this paper, denote by $|A|$ the cardinality of any finite set A. In the present section we give reductions for $\psi_f(x_k)$ using the ideas in [6], [7]. First one needs a generalization of the principle of cross-classification in [6] to give a preliminary reduction for the formula of $\psi_f(x_k)$. For an alternative proof using induction, see [8].

Lemma 3.1 ([6, Lemma 1])**.** *Let R be a given finite set and f any complex-valued function defined on R. For a subset T of R, we denote by $\overline{T}$ the set of those elements of R which are not in T, i.e., $\overline{T} = R \setminus T$. If $R_1, \dots, R_m$ are given m distinct subsets of R, then*

$$\sum_{x \in \bigcap_{i=1}^{m} \overline{R}_i} f(x) = \sum_{x \in R} f(x) + \sum_{t=1}^{m} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant m} \sum_{x \in \bigcap_{j=1}^{t} R_{i_j}} f(x).$$

Lemma 3.2. *Let f be an incidence function of P. Then*

$$\sum_{x \leqslant z \leqslant y} (f * \mu)(x, z) = f(x, y)$$

for all $x, y \in P$. In particular, one has

$$\sum_{z \leqslant y} (f * \mu)(z) = f(y)$$

for all $y \in P$.

Proof. Let $x, y \in P$ be given. Note that $f * \delta = f$ and $\mu * \zeta = \delta$. Then

$$\begin{aligned} f(x, y) &= (f * \delta)(x, y) = (f * (\mu * \zeta))(x, y) = ((f * \mu) * \zeta)(x, y) \\ &= \sum_{x \leqslant z \leqslant y} (f * \mu)(x, z)\zeta(z, y) = \sum_{x \leqslant z \leqslant y} (f * \mu)(x, z). \end{aligned}$$

The first assertion is proved. For the other assertion, one needs only to pick $x = \min P$. The proof is complete. □

Lemma 3.3. *Let n be an integer. Let $S=\{x_1,\ldots,x_n\}$ be a meet-closed set with $x_i<x_j \Rightarrow i<j$. If $\psi_f(x_k)$ is defined as in* (1), *then*

$$\psi_f(x_k)=f(x_k)+\sum_{t=1}^{k-1}(-1)^t\sum_{1\leqslant i_1<\ldots<i_t\leqslant k-1} f(x_k\wedge x_{i_1}\wedge\ldots\wedge x_{i_t}), \tag{2}$$

where $f(x_k\wedge x_{i_1}\wedge\ldots\wedge x_{i_t})$ denotes f evaluated at the meet of $x_k,x_{i_1},\ldots,x_{i_t}$.

P r o o f. In Lemma 3.1, let $m=k-1$ and $R=\{d\colon\ d\leqslant x_k,\ x_k\in S\}$. For $1\leqslant i\leqslant k-1$, let $R_i=\{d\in R\colon\ d\leqslant x_i,\ x_i\in S\}$. Then one has $R_i=\{d\colon\ d\leqslant x_k\wedge x_i\}$. By Lemma 3.1, one has

$$\psi_f(x_k)=\sum_{d\leqslant x_k}(f*\mu)(d)+\sum_{t=1}^{k-1}(-1)^t\sum_{1\leqslant i_1<\ldots<i_t\leqslant k-1}\ \sum_{d\leqslant x_k\wedge x_{i_1}\wedge\ldots\wedge x_{i_t}}(f*\mu)(d). \tag{3}$$

By Lemma 3.2, one has $\sum\limits_{d\leqslant x_k}(f*\mu)(d)=f(x_k)$ and for $1\leqslant i_1<\ldots<i_t\leqslant k-1$ $(1\leqslant t\leqslant k-1)$, one has

$$\sum_{d\leqslant x_k\wedge x_{i_1}\wedge\ldots\wedge x_{i_t}}(f*\mu)(d)=f(x_k\wedge x_{i_1}\wedge\ldots\wedge x_{i_t}). \tag{4}$$

It then follows from Equations (3) and (4) that (2) holds. This completes the proof of Lemma 3.3. □

Now, we give further reduction for the formula of $\psi_f(x_k)$. The ideas of the proofs of the following two lemmas are due to our article [7].

Lemma 3.4. *Let $S=\{x_1,\ldots,x_n\}$ be a meet-closed set with $x_i<x_j\Rightarrow i<j$. For $1\leqslant k\leqslant n$, let $I_k=\{i\colon\ 1\leqslant i\leqslant k-1$ and $x_i\nleqslant x_k\}$ and $J_k=\{1,2,\ldots,k-1\}\setminus I_k$. Then*

$$\psi_f(x_k)=f(x_k)+\sum_{r=1}^{|J_k|}(-1)^r\sum_{\substack{i_1<\ldots<i_r\\ i_j\in J_k}} f(x_k\wedge x_{i_1}\wedge\ldots\wedge x_{i_r}). \tag{5}$$

P r o o f. If $|I_k|=0$, then it follows from Lemma 3.3 that Lemma 3.4 holds. In what follows let $|I_k|\geqslant 1$. Note that for $i\in J_k$ one has $x_i\leqslant x_k$. Since S is meet-closed, $x_1\leqslant x_k$. Thus one has $|J_k|\geqslant 1$. Note also that $|I_k|+|J_k|=k-1$. By Lemma 3.3, one has

$$\psi_f(x_k)=f(x_k)+\Delta'+\Delta, \tag{6}$$

where

$$\Delta' = \sum_{r=1}^{|J_k|}(-1)^r \sum_{\substack{i_1<\ldots<i_r\\ i_j\in J_k}} f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r})$$

and

$$(7)\qquad \Delta = \sum_{r=1}^{|J_k|} \sum_{\substack{i_1<\ldots<i_r\\ i_j\in J_k}} \sum_{s=1}^{|I_k|}(-1)^{r+s} \sum_{\substack{t_1<\ldots<t_s\\ t_u\in I_k}} f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s}).$$

For any given $t_1 < \ldots < t_s$, $t_u \in I_k$ $(1 \leqslant u \leqslant s)$, it follows from the fact that S is meet-closed that $x_k \wedge x_{t_1} \wedge \ldots \wedge x_{t_s} \in S$. Let $x_l = x_k \wedge x_{t_1} \wedge \ldots \wedge x_{t_s}$. Then $x_l \leqslant x_k$ and $x_l \leqslant x_{t_u}$ for $1 \leqslant u \leqslant s$. So one has $l \in J_k$. Then by (7), one has

$$\begin{aligned}
(8)\ \Delta &= \sum_{s=1}^{|I_k|} \sum_{\substack{t_1<\ldots<t_s\\ t_u\in I_k}} \sum_{r=1}^{|J_k|}(-1)^{r+s} \sum_{\substack{i_1<\ldots<i_r\\ i_j\in J_k}} f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s})\\
&= \sum_{s=1}^{|I_k|} \sum_{\substack{t_1<\ldots<t_s\\ t_u\in I_k}} \sum_{r=0}^{|J_k|-1} \sum_{\substack{i_1<\ldots<i_r\\ i_j\in J_k,\, i_j\neq l}} \big((-1)^{r+s} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s})\\
&\qquad + (-1)^{r+s+1} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_l \wedge x_{t_1} \wedge \ldots \wedge x_{t_s})\big)\\
&= \sum_{s=1}^{|I_k|} \sum_{\substack{t_1<\ldots<t_s\\ t_u\in I_k}} \sum_{r=0}^{|J_k|-1} \sum_{\substack{i_1<\ldots<i_r\\ i_j\in J_k,\, i_j\neq l}} \big((-1)^{r+s} \cdot f(x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_l)\\
&\qquad + (-1)^{r+s+1} \cdot f(x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_l)\big) = 0.
\end{aligned}$$

Therefore it follows from Equations (6) and (8) that (5) holds. The proof of Lemma 3.4 is complete. $\square$

Now we can use the concept of greatest-type lower to give a further reduction for $\psi_f(x_k)$.

Lemma 3.5. *Let $S = \{x_1, \ldots, x_n\}$ be a meet-closed set. For $1 \leqslant k \leqslant n$, let $R_k = \{i\colon 1 \leqslant i \leqslant k-1,\ x_i$ is the greatest-type lower of x_k in $S\}$. Then*

$$\psi_f(x_k) = f(x_k) + \sum_{r=1}^{|R_k|}(1)^r \sum_{\substack{i_1<\ldots<i_r\\ i_j\in R_k}} f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r}).$$

P r o o f. For the case $k \leqslant 2$, the lemma is clearly true. In what follows let $k \geqslant 3$. Let $J_k = \{i\colon 1 \leqslant i \leqslant k-1$ and $x_i \leqslant x_k\}$. Then $|J_k| \geqslant 1$. It is clear that $R_k \subseteq J_k$.

If $|J_k| = 1$, then $J_k = \{1\}$. Note that $|R_k| \geqslant 1$. So one has $R_k = \{1\} = J_k$. Thus by Lemma 3.4, the result is true. In the following let $|J_k| \geqslant 2$. Let $L_k = J_k \setminus R_k$. We claim that $L_k \neq \emptyset$. Assuming otherwise implies that $R_k = J_k$. But $1 \in J_k$, hence $1 \in R_k$. From $|J_k| \geqslant 2$ one deduces that there is an $i \in J_k$, $i \neq 1$, such that $i \in J_k = R_k$. Since S is meet-closed, one has $x_1 < x_i$. This is impossible since x_1 and x_i cannot both be greatest-type lowers of x_k in S. Therefore the claim is true. In a similar way to that in (6), one has by Lemma 3.4 that

$$\psi_f(x_k) = f(x_k) + \overline{\Delta}' + \overline{\Delta},$$

where

$$\overline{\Delta}' = \sum_{r=1}^{|R_k|} (-1)^r \sum_{\substack{i_1 < \ldots < i_r \\ i_j \in R_k}} f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r})$$

and

$$(9)\quad \Delta = \sum_{r=0}^{|R_k|} \sum_{\substack{i_1 < \ldots < i_r \\ i_j \in R_k}} \sum_{s=1}^{|L_k|} \sum_{\substack{t_1 < \ldots < t_s \\ t_u \in L_k}} (-1)^{r+s} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s})$$

$$= \sum_{s=1}^{|L_k|} \sum_{\substack{t_1 < \ldots < t_s \\ t_u \in L_k}} (-1)^s \sum_{r=0}^{|R_k|} \sum_{\substack{i_1 < \ldots < i_r \\ i_j \in R_k}} (-1)^r \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s}).$$

To prove the lemma, one needs only to show that $\overline{\Delta} = 0$, which we will do in the following.

For any given $t_1 < \ldots < t_s$ ($1 \leqslant s \leqslant |L_k|$), $t_u \in L_k$, $1 \leqslant u \leqslant s$, let $T = \{i\colon\ i \in R_k,\ \text{and}\ x_{t_u} \leqslant x_i \text{ for some } t_u,\ 1 \leqslant u \leqslant s\}$ and $Q = R_k \setminus T$. Let $|T| = h$ and $|Q| = h'$. Clearly one has that $1 \leqslant h \leqslant |R_k|$ and $0 \leqslant h' \leqslant |R_k| - 1$. Then one has

$$(10)\quad \sum_{r=0}^{|R_k|} \sum_{\substack{i_1 < \ldots < i_r \\ i_j \in R_k}} (-1)^r \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s})$$

$$= \sum_{r'=0}^{h'} \sum_{\substack{i_1 < \ldots < i_{r'} \\ i_u \in Q}} \sum_{r=0}^{h} \sum_{\substack{j_1 < \ldots < j_r \\ j_v \in T}} (-1)^{r+r'} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_{r'}} \wedge x_{j_1} \wedge \ldots \wedge x_{j_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s})$$

$$= \sum_{r'=0}^{h'} \sum_{\substack{i_1 < \ldots < i_{r'} \\ i_u \in Q}} \sum_{r=0}^{h} \sum_{\substack{j_1 < \ldots < j_r \\ j_v \in T}} (-1)^{r+r'} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s})$$

(since by the definition of T one has $x_{j_1} \wedge \ldots \wedge x_{j_r} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s} = x_{t_1} \wedge \ldots \wedge x_{t_s}$ for any $j_1 < \ldots < j_r,\ j_v \in T$)

$$= \sum_{r'=0}^{h'} \sum_{\substack{i_1<\ldots<i_{r'} \\ i_u\in Q}} (-1)^{r'} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s}) \cdot \Biggl(1 + \sum_{r=1}^{h} (-1)^r \sum_{\substack{j_1<\ldots<j_r \\ j_v\in T}} 1\Biggr)$$

$$= \sum_{r'=0}^{h'} \sum_{\substack{i_1<\ldots<i_{r'} \\ i_u\in Q}} (-1)^{r'} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s}) \cdot \Biggl(1 + \sum_{r=1}^{h} (-1)^r \cdot \binom{h}{r}\Biggr)$$

$$= \sum_{r'=0}^{h'} \sum_{\substack{i_1<\ldots<i_{r'} \\ i_u\in Q}} (-1)^{r'} \cdot f(x_k \wedge x_{i_1} \wedge \ldots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \ldots \wedge x_{t_s}) \cdot (1-1)^h = 0.$$

It now follows from Equations (9) and (10) that $\overline{\Delta} = 0$. This completes the proof of Lemma 3.5. □

Theorem 3.6. *Let $S = \{x_1, \ldots, x_n\}$ be a meet-closed set and f an incidence function. Then*

$$\det[f(x_i \wedge x_j)] = \prod_{i=1}^{n} \Biggl(f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1\leqslant i_1<\ldots<i_t\leqslant n(x_i)} f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t}) \Biggr),$$

where $f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S.

Proof. This theorem follows from Proposition 2.1 and Lemma 3.5. □

Lemma 3.7. *Let f be a semi-multiplicative function and $S = \{x_1, \ldots, x_n\}$ a meet-closed set. If $f(x_i) \neq 0$ for all $1 \leqslant i \leqslant n$, then*

$$[f(x_i \vee x_j)] = \operatorname{diag}\{f(x_1), \ldots, f(x_n)\} \cdot \left[\frac{1}{f}(x_i \wedge x_j)\right] \cdot \operatorname{diag}\{f(x_1), \ldots, f(x_n)\}.$$

Proof. It follows from definition of a semi-multiplicative function that this lemma is true. □

Theorem 3.8. *Let $S = \{x_1, \ldots, x_n\}$ be a meet-closed set. If f is a semi-multiplicative function satisfying $f(x_i) \neq 0$ for all $1 \leqslant i \leqslant n$, then*

$$\det [f(x_i \vee x_j)] = \prod_{i=1}^{n} [f(x_i)]^2 \Bigg(\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} \frac{1}{f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t})} \Bigg),$$

where $f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S.

Proof. This theorem follows from Lemma 3.7 and Theorem 3.6 applied to the function $1/f$. The proof is complete. □

It follows from Theorems 3.6 and 3.8 that the following two corollaries are true.

Corollary 3.9. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be lower-closed and let f be an incidence function. Then each of the following is true:*

(i) *One has* $\det[f(x_i \wedge x_j)] = \prod_{i=1}^{n} (f * \mu)(x_i)$;

(ii) *If f is semi-multiplicative and $f(x_i) \neq 0$ for all $1 \leqslant i \leqslant n$, then* $\det[f(x_i \vee x_j)] = \prod_{i=1}^{n} [f(x_i)]^2 \big((1/f) * \mu\big)(x)$.

Corollary 3.10. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a chain with $x_1 < x_2 < \ldots < x_{n-1} < x_n$ and f an incidence function. Then each of the following is true:*

(i) *One has* $\det[f(x_i \wedge x_j)] = f(x_1) \prod_{i=2}^{n} [f(x_i) - f(x_{i-1})]$;

(ii) *If f is semi-multiplicative and $f(x_i) \neq 0$ for all $1 \leqslant i \leqslant n$, then* $\det[f(x_i \vee x_j)] = f(x_n) \prod_{i=2}^{n} [f(x_{i-1}) - f(x_i)]$.

Proof. For k, $1 \leqslant k \leqslant n$, since $x_1 < x_2 < \ldots < x_n$, one has that x_{k-1} is the only greatest-type lower of x_k in S. It then follows from Theorems 3.6 and 3.8 that this corollary is true. □

4. Nonsingularity of matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$

We can now use the results of the preceding section to give a characterization for nonsingularity of matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ as follows.

Theorem 4.1. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a meet-closed set and let f be an incidence function. Then the matrix $[f(x_i \wedge x_j)]$ defined on S is nonsingular if and only if for all $1 \leqslant i \leqslant n$, one has*

$$f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t}) \neq 0,$$

where $f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S.

Proof. First, one has that the matrix $[f(x_i \wedge x_j)]$ on S is nonsingular if and only if $\det([f(x_i \wedge x_j)]) \neq 0$. From Theorem 3.6 one knows that

$$\det[f(x_i \wedge x_j)] = \prod_{i=1}^{n} \Bigg(f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t}) \Bigg),$$

where $f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S. So $[f(x_i \wedge x_j)]$ is nonsingular if and only if for all $1 \leqslant i \leqslant n$, one has

$$f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t}) \neq 0,$$

as desired. □

Theorem 4.2. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a meet-closed set and let f be a semi-multiplicative function. Then the matrix $[f(x_i \vee x_j)]$ defined on S is nonsingular if and only if for all $1 \leqslant i \leqslant n$ one has $f(x_i) \neq 0$ and*

$$\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} \frac{1}{f(x \wedge y_{i_1} \wedge \ldots \wedge y_{i_t})} \neq 0,$$

where $f(x_i \wedge y_{i_1} \wedge \ldots \wedge y_{i_t})$ denotes f evaluated at the meet of $x, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S.

Proof. This theorem follows immediately from Theorem 3.8. □

Corollary 4.3. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be lower-closed. Then each of the following is true:*

(i) *The matrix $[f(x_i \wedge x_j)]$ defined on S is nonsingular if and only if $(f * \mu)(x_i) \neq 0$ for all $1 \leqslant i \leqslant n$;*

(ii) *The matrix $(f(x_i \vee x_j))$ defined on S is nonsingular if and only if $f(x_i) \neq 0$ and $((1/f) * \mu)(x_i) \neq 0$ for all $1 \leqslant i \leqslant n$.*

Corollary 4.4. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a chain with $x_1 < x_2 < \ldots < x_{n-1} < x_n$. Then each of the following is true:*

(i) *The matrix $[f(x_i \wedge x_j)]$ defined on S is nonsingular if and only if $f(x_1) \neq 0$ and for all k, $2 \leqslant k \leqslant n$, one has $f(x_{k-1}) \neq f(x_k)$;*

(ii) *The matrix $[f(x_i \vee x_j)]$ defined on S is nonsingular if and only if $f(x_1) \neq 0$, and for all k, $2 \leqslant k \leqslant n$, one has $f(x_k) \neq 0$ and $f(x_{k-1}) \neq f(x_k)$.*

5. Applications to matrices $[f(x_i, x_j)]$ and $(f[x_i, x_j])$

In the present section, we give number-theoretic applications of the results presented in Sections 3 and 4.

Theorem 5.1. *Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set and let f be an arithmetical function. Then*

$$\det[f(x_i, x_j)] = \prod_{i=1}^{n} \Bigg(f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} f(x_i, y_{i_1}, \ldots, y_{i_t}) \Bigg),$$

where $f(x_i, y_{i_1}, \ldots, y_{i_t})$ denotes f evaluated at the greatest common divisor $(x_i, y_{i_1}, \ldots, y_{i_t})$ of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S.

Proof. Let $(P, \leqslant) = (\mathbb{Z}^+, |)$. Then this theorem follows from Theorem 3.6. $\square$

Theorem 5.2. *Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set. If f is a semi-multiplicative arithmetical function satisfying $f(x_i) \neq 0$ for all $1 \leqslant i \leqslant n$, then*

$$\det(f[x_i, x_j]) = \prod_{i=1}^{n} [f(x_i)]^2 \Bigg(\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} \frac{1}{f(x_i, y_{i_1}, \ldots, y_{i_t})} \Bigg),$$

where $f(x_i, y_{i_1}, \ldots, y_{i_t})$ denotes f evaluated at the greatest common divisor $(x_i, y_{i_1}, \ldots, y_{i_t})$ of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S.

Proof. Let $(P, \leqslant) = (\mathbb{Z}^+, |)$. Then this theorem follows from Theorem 3.8. □

Theorem 5.3. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers and f an arithmetical function. If S is gcd-closed, then the matrix $[f(x_i, x_j)]$ defined on S is nonsingular if and only if for all $1 \leqslant i \leqslant n$ one has*

$$f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} f(x_i, y_{i_1}, \ldots, y_{i_t}) \neq 0,$$

where $f(x_i, y_{i_1}, \ldots, y_{i_t})$ denotes f evaluated at the greatest common divisor of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S.

Proof. Let $(P, \leqslant) = (\mathbb{Z}^+, |)$. Then this theorem follows immediately from Theorem 4.1. □

Note that Theorem 5.3 gives an answer to the problem raised by Bourque and Ligh in [4].

Theorem 5.4. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers and f a semi-multiplicative arithmetical function. If S is gcd-closed, then the matrix $(f[x_i, x_j])$ defined on S is nonsingular if and only if for all $1 \leqslant i \leqslant n$ one has $f(x_i) \neq 0$ and*

$$\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant n(x_i)} \frac{1}{f(x_i, y_{i_1}, \ldots, y_{i_t})} \neq 0,$$

where $f(x_i, y_{i_1}, \ldots, y_{i_t})$ denotes f evaluated at the greatest common divisor of $x_i, y_{i_1}, \ldots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S, and $\{y_1, y_2, \ldots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S.

Proof. Let $(P, \leqslant) = (\mathbb{Z}^+, |)$. Then this theorem follows immediately from Theorem 4.2. □

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