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Jiří Močkoř

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COMPLETE SUBOBJECTS OF FUZZY SETS OVER MV-ALGEBRAS

Jiří Močkoř, Ostrava

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Abstract. A subobjects structure of the category Ω-**FSet** of Ω-fuzzy sets over a complete MV-algebra $\Omega = (L, \wedge, \vee, \otimes, \rightarrow)$ is investigated, where an Ω-fuzzy set is a pair $\mathbf{A} = (A, \delta)$ such that A is a set and $\delta\colon A \times A \rightarrow \Omega$ is a special map. Special subobjects (called *complete*) of an Ω-fuzzy set $\mathbf{A}$ which can be identified with some *characteristic morphisms* $\mathbf{A} \rightarrow \Omega^* = (L \times L, \mu)$ are then investigated. It is proved that some truth-valued morphisms $\neg_\Omega\colon \Omega^* \rightarrow \Omega^*, \cap_\Omega, \cup_\Omega\colon \Omega^* \times \Omega^* \rightarrow \Omega^*$ are characteristic morphisms of complete subobjects.

Keywords: fuzzy set over MV-lagebra, complete subobjects, subobjects classification

MSC 2000: 06D15, 18B05, 03E72

1. Introduction

There are various categories which can be used as a natural basis for generalization of classical $[0,1]$-fuzzy sets and their internal logic (see for example [1], [4], [6], [14], [15], [16] and others). A very natural and classical example is the category Ω-**Set** of Ω-sets ([3]), where Ω is a complete Heyting algebra $\Omega = (\wedge, \vee, \rightarrow)$ and an Ω-set is a pair (A, δ) such that A is a set and $\delta\colon A \times A \rightarrow \Omega$ is a map such that

(1) $(\forall x, y \in A)\ \ \delta(x, y) = \delta(y, x)$,

(2) $(\forall x, y, z \in A)\ \ \delta(x, y) \wedge \delta(y, z) \leqslant \delta(x, z)$,

with naturally defined morphisms. It can then be proved that this category Ω-**Set** can represent classical fuzzy sets $A \rightarrow \Omega$ with their morphisms on the one hand, but on the other hand its structure is very different from that of classical fuzzy sets. The principal reason is that the category Ω-**Set** is a topos and, hence, its external (and internal, as well) logic is intuitionistic and based on Heyting algebra structure, while the external logic of classical fuzzy sets is based mostly on Lukasiewicz algebra, i.e. on

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different operations $\otimes$ and $\to$. Hence, the category Ω-**Set** is the best structure for interpretation of some logic based on the Heyting connectives, but it offers quite poor possibilities to interpret other logical connectives, not directly based on the Heyting connective. It seems that the principal reason for the disadvantages of the category Ω-**Set** lies directly in the Heyting algebra Ω. In principle, we are able to interpret only those logical formulas which are constructed over the connectives which can be interpreted in Ω. In case that such connective (contained in a formula) could not be interpreted naturally in Ω, it would be almost impossible to interpret the formula in a reasonable way. Unfortunately, Heyting algebra structure enables the interpretation of classical connectives but not the interpretation of the Lukasiewicz fuzzy conjunction and implication, since these are based on the connectives which are not present in this algebra. Hence, a method improving this situation could be based on some modification of the underlying lattice Ω. This method could make it possible to create a category which could be considered as a generalization of fuzzy sets in a more convenient way.

In this paper we deal with the category Ω-**FSet** of Ω-fuzzy sets $\mathbf{A} = (A, \delta)$, where $\Omega = (L, \wedge, \vee, \otimes, \to, 0_\Omega, 1_\Omega)$ is a complete MV-algebra, A is a set and $\delta\colon\ A \times A \to \Omega$ is a map which satisfies the condition (1) but instead of condition (2) it satisfies some modification of this condition. This category was (in a little more general form) introduced by U. Höhle [5]–[9] and he also investigated a lot of important properties of this category. Höhle also observed that this category is not a topos and that it does not possess a subobject classifier in general. He also investigated some objects in this category which could be used as subobjects classifiers (in some sense). Recall that this classification problem is connected with the bijection relation

$$\mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A}) \cong \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A}, \Omega_0),$$

where Ω_0 is a version of a subobject classifier (see [3], [12]). This bijection exists in any topos, but the category Ω-**FSet** is not a topos in general. Hence, in the category Ω-**FSet** we need to find some analogy of this bijection. In this paper we characterize in a rather simple way subobjects $\mathbf{B} \in \mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})$ (which will be called *complete subobjects*) for which a characteristic morphism $\chi_{\mathbf{A}}(\mathbf{B})\colon\ \mathbf{A} \to \Omega^*$ can be found such that the diagram

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\ f\ } & (A, \delta) \\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \chi_{\mathbf{A}}(\mathbf{B})} \\ \Omega & \xrightarrow{\ \top\ } & \Omega^* \end{array}$$

is a pullback diagram. Moreover, we prove that among these characteristic morphisms of complete subobjects the truth-valued morphisms $\neg_\Omega, \cap_\Omega, \cup_\Omega\colon\ \Omega^* \times \Omega^* \to$

Ω^* are contained which could interprete $\neg, \wedge, \vee$, respectively. The principal advantage of these results is that they provide a rather simple tool for subobjects and characteristic morphisms identification in a category Ω-**FSet**.

2. Subobjects in the category Ω-FSet

Let $\Omega = (L, \wedge, \vee, \otimes, \rightarrow, 1_\Omega, 0_\Omega)$ be a complete MV-algebra, i.e. a complete residuated lattice, where $(a \rightarrow b) \rightarrow b = a \vee b$ holds for every $a, b \in L$. By an *Ω-fuzzy set* we mean (A, δ), where A is a set and $\delta\colon\ A \times A \rightarrow \Omega$ is a map such that

(i) $(\forall x, y \in A) \quad \delta(x, y) \leqslant \delta(x, x) \wedge \delta(y, y)$,

(ii) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,

(iii) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes (\delta(y, y) \rightarrow \delta(y, z)) \leqslant \delta(x, z)$.

Moreover, a Ω-fuzzy set (A, α) is called *separated* if it satisfies the axiom

$$\alpha(x, x) \vee \alpha(y, y) \leqslant \alpha(x, y) \Rightarrow x = y.$$

The category Ω-**FSet** of Ω-fuzzy sets then consists of separated Ω-fuzzy sets as objects and morphisms between objects $(A, \alpha), (B, \beta)$, which are maps $f\colon\ A \rightarrow B$ such that

(1) $(\forall x, y \in A) \quad \beta(f(x), f(y)) \geqslant \alpha(x, y)$,

(2) $(\forall x \in A) \quad \alpha(x, x) = \beta(f(x), f(x))$.

The composition of morphisms is the usual composition of maps. Höhle [5] proved that the category Ω-**FSet** is complete and cocomplete. We mention here only that a terminal object is **To** $= (L, \wedge)$ with the unique morphism $!\colon\ (A, \delta) \rightarrow$ **To** such that $!(a) = \delta(a, a)$. Moreover, a morphism f in this category is a monomorphism if and only if f is injective. In this category a subobject classifier does not exist in general. On the other hand, Höhle proves that there exists an object which has very similar property and which can classify some special subobjects. This subobject is of the form

$$\Omega^* = (\{(\alpha, \beta) \in L \times L \mid \alpha \geqslant \beta\}, \mu),$$
$$\mu((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \alpha_1 \otimes (\beta_1 \rightarrow \beta_2) \wedge \alpha_2 \otimes (\beta_2 \rightarrow \beta_1).$$

The principal aim of this paper is to show that among objects which could be classified by this object Ω^* are the so called complete subobjects of any Ω-fuzzy set.

Recall that a map $s\colon\ A \rightarrow L$ is an *Ω-subset* of Ω-fuzzy set $\mathbf{A} = (A, \delta)$ (abbreviated as $s \subseteq \mathbf{A}$), if

(i) $(\forall x, y \in A) \quad s(x) \otimes (\delta(x, x) \rightarrow \delta(x, y)) \leqslant s(y)$,

(ii) $(\forall x \in A) \quad s(x) \leqslant \delta(x, x)$.

(In [7] this map is called δ-extensional and δ-strict.) Moreover, let $\top\colon (\mathbf{To}, \wedge) \to (\Omega^*, \mu)$ be defined such that $\top(\omega) = (\omega, \omega)$. Then we have $\mu(\top(x), \top(y)) = x \wedge y$ and it follows that $\top$ is a morphism.

The following proposition then extends Theorem 3.7 in [7].

Proposition 2.1. *Let $\mathbf{A} = (A, \delta)$ be an Ω-fuzzy set and let $\mathcal{S}(\mathbf{A}) = \{s \mid s \subseteq \mathbf{A}$ be an Ω-subset$\}$. Then $\mathcal{S}$ is a functor $\Omega\text{-}\mathbf{FSet}^{\mathrm{op}} \to \mathbf{Set}$ and there exists a natural isomorphism*

$$\zeta\colon\ \mathcal{S}(-) \cong \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}}(-, \Omega^*).$$

P r o o f. Let $\mathbf{A} = (A, \delta) \in \Omega\text{-}\mathbf{FSet}$. For a morphism $(A, \delta) \xrightarrow{f} (B, \beta)$ and for $s \in \mathcal{S}(\mathbf{B})$ we set $\mathcal{S}(f)(s) = s \cdot f \in \mathcal{S}(\mathbf{A})$. This definition is correct since

$$\begin{aligned} &\mathcal{S}(f)(s)(a) \otimes (\delta(a,a) \to \delta(a,b)) \\ &\leqslant s(f(a)) \otimes (\beta(f(a), f(a)) \to \beta(f(a), f(b))) \leqslant \mathcal{S}(f)(s)(b). \end{aligned}$$

Hence, $\mathcal{S}$ is a functor. We define a map $\zeta_{\mathbf{A}}$ such that for $s \in \mathcal{S}(\mathbf{A})$ we set

$$(\forall a \in A) \quad \zeta_{\mathbf{A}}(s)(a) = (\delta(a,a), s(a)) \in \Omega^*.$$

Then $\zeta_{\mathbf{A}}(s)\colon\ \mathbf{A} \to \Omega^*$ is a morphism in $\Omega\text{-}\mathbf{FSet}$. In fact, for $a, b \in A$ we have $\delta(a,a) \to \delta(a,b) \leqslant s(a) \to s(b)$ and it follows that

$$\begin{aligned} \mu(\zeta_{\mathbf{A}}(s)(a), \zeta_{\mathbf{A}}(s)(b)) &= (\delta(a,a) \otimes (s(a) \to s(b))) \wedge (\delta(b,b) \otimes (s(b) \to s(a))) \\ &\geqslant \delta(a,a) \otimes (\delta(a,a) \to \delta(a,b)) \wedge \delta(b,b) \otimes (\delta(b,b) \to \delta(a,b))) \\ &= \delta(a,a) \wedge \delta(b,b) \wedge \delta(a,b) = \delta(a,b). \end{aligned}$$

Moreover, we have further $\mu(\zeta_{\mathbf{A}}(s)(a), \zeta_{\mathbf{A}}(s)(a)) = \delta(a,a)$ and it follows that $\zeta_{\mathbf{A}}(s)$ is a morphism.

Conversely, for a morphism $f\colon\ (A, \delta) \to \Omega^*$ we define a map s such that $\zeta_{\mathbf{A}}^{-1}(f) = s = \mathrm{pr}_2 \circ f$, where $\mathrm{pr}_2\colon\ \Omega^* \to L$ is the second projection map. Then $s \subseteq \mathbf{A}$. In fact, let $a \in A$, $f(a) = (f_1, f_2)$. Then we have $\delta(a,a) = f_1$ and $s(a) = f_2 \leqslant f_1 = \delta(a,a)$. Moreover, for $a, b \in A$ we have

$$\delta(a,b) \leqslant \mu(f(a), f(b)) \leqslant \delta(a,a) \otimes (s(a) \to s(b)),$$

and it then follows that

$$\begin{aligned} &s(a) \otimes (\delta(a,a) \to \delta(a,b)) \\ &\leqslant s(a) \otimes (\delta(a,a) \to (\delta(a,a) \otimes (s(a) \to s(b)))) \\ &= s(a) \otimes ((s(a) \to s(b)) \vee \neg\delta(a,a)) \\ &= (s(a) \otimes (s(a) \to s(b))) \vee (s(a) \otimes \neg\delta(a,a)) \\ &\leqslant s(b) \vee (\delta(a,a) \otimes \neg\delta(a,a)) = s(b). \end{aligned}$$

Then $s \in \mathcal{S}$. Let $a \in A$. Since $\delta(a,a) = \mu(f(a), f(a))$, we have

$$\zeta_{\mathbf{A}} \circ \zeta_{\mathbf{A}}^{-1}(f)(a) = (\delta(a,a), \zeta_{\mathbf{A}}^{-1}(f)(a)) = (\delta(a,a), \mathrm{pr}_2 \circ f(a)) = f(a).$$

Analogously, we have $\zeta_{\mathbf{A}}^{-1} \circ \zeta_{\mathbf{A}}(s)(a) = \mathrm{pr}_2(\delta(a,a), s(a)) = s(a)$. Hence, $\zeta_{\mathbf{A}}$, $\zeta_{\mathbf{A}}^{-1}$ are mutually inverse. Finally, for any morphism $(A,\delta) \xrightarrow{f} (B,\beta)$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{S}(\mathbf{B}) & \xrightarrow{\zeta_{\mathbf{B}}} & \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{B}, \Omega^*) \\ {\scriptstyle \mathcal{S}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}}(f,\Omega^*)} \\ \mathcal{S}(\mathbf{A}) & \xrightarrow{\zeta_{\mathbf{A}}} & \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A}, \Omega^*) \end{array}$$

It is simple to prove that the corresponding diagram for $\zeta_{\mathbf{A}}^{-1}$ commutes as well. □

Let $\mathbf{A} = (A,\delta)$ be an Ω-fuzzy set. Then a set $S \subseteq A$ is called *complete* (in $\mathbf{A}$) if

$$S = \left\{ a \in A \colon \bigvee_{x \in S} \delta(a,x) = \delta(a,a) \right\}.$$

Let $\mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})$ be the set of all subobjects of $\mathbf{A}$ which are of the form (S,δ) where $S \subseteq A$ and let $\mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})$ be the set of all complete subobjects, i.e. subobjects (S,δ) such that S is complete in $\mathbf{A}$. Then we obtain two functors $\mathrm{Sub}(-), \mathrm{Sub}^{\mathrm{c}}(-)\colon \Omega\text{-}\mathbf{FSet}^{\mathrm{op}} \to \mathbf{Set}$ such that for a morphism $(A,\delta) \xrightarrow{f} (B,\beta)$ and $(S,\beta) \in \mathrm{Sub}(\mathbf{B})$ we have $\mathrm{Sub}(f)(S,\beta) = (f^{-1}(S),\delta)$ and analogously for $\mathrm{Sub}^{\mathrm{c}}(f)$. This definition is correct since if (S,β) is complete in (B,β) then $(f^{-1}(S),\delta)$ is complete in (A,δ) as well. In fact, let $a \in A$ be such that $\bigvee_{x \in f^{-1}(S)} \delta(a,x) = \delta(a,a)$. Then we have

$$\begin{aligned} \beta(f(a),f(a)) &\geqslant \bigvee_{y \in S} \beta(f(a),y) \geqslant \bigvee_{x \in f^{-1}(S)} \beta(f(a),f(x)) \\ &\geqslant \bigvee_{x \in f^{-1}(S)} \delta(a,x) = \delta(a,a) = \beta(f(a),f(a)), \end{aligned}$$

and it follows that $a \in f^{-1}(S)$.

Complete subsets of $\mathbf{A}$ define a closure system in A. Namely, for any $S \subseteq A$ we set

$$\overline{S} = \left\{ a \in A \colon \bigvee_{x \in S} \delta(a,x) = \delta(a,a) \right\}.$$

Lemma 2.2. *For any subset $S \subseteq A$, $\overline{S}$ is a complete set such that $S \subseteq \overline{S}$ and $\overline{\overline{S}} = \overline{S}$. Moreover, any intersection of complete sets is a complete set.*

Proof. Let $a \in A$ be such that $\bigvee_{x \in \overline{S}} \delta(a,x) = \delta(a,a)$. Then we have

$$\begin{aligned}
\delta(a,a) &= \bigvee_{x\in\overline{S}} \delta(a,x) = \bigvee_{x\in\overline{S}} (\delta(a,x)\wedge\delta(x,x)) \\
&= \bigvee_{x\in\overline{S}} (\delta(a,x)\wedge \bigvee_{y\in S}\delta(x,y)) = \bigvee_{x\in\overline{S}}\bigvee_{y\in S}\delta(a,x)\wedge\delta(x,y) \\
&\leqslant \bigvee_{x\in\overline{S}}\bigvee_{y\in S}\delta(a,x)\otimes(\delta(x,x)\to\delta(x,y)) \\
&\leqslant \bigvee_{y\in S}\delta(a,y)\leqslant\delta(a,a).
\end{aligned}$$

Hence, $a \in \overline{S}$. □

Proposition 2.3. *Let Ω-$\mathbf{FSet}_1$ be a subcategory of the category Ω-$\mathbf{FSet}$ with the same objects and with morphisms $f\colon (A,\delta)\to(B,\beta)$ such that f is surjective and $\beta(f(x),f(y))=\delta(x,y)$ for all $x,y\in A$. Let $\mathcal{S}_1$, $\mathrm{Sub}_1\colon \Omega\text{-}\mathbf{FSet}_1\to\mathbf{Set}$ be the restrictions of functors $\mathcal{S}, \mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}$, respectively.*

(1) *There exists a natural transformation*

$$\sigma\colon\ \mathcal{S}\to\mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}\,.$$

(2) *For any $\mathbf{A}\in\Omega$-$\mathbf{FSet}$ there exists a map*

$$\psi_{\mathbf{A}}\colon\ \mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})\to\mathcal{S}(\mathbf{A}).$$

Moreover, $\psi=\{\psi_{\mathbf{A}}\colon\ \mathbf{A}\in\Omega\text{-}\mathbf{FSet}\}\colon\ \mathrm{Sub}_1\to\mathcal{S}_1$ is a natural transformation.

(3) *For any $\mathbf{S}\in\mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})$, the following diagram commutes.*

$$\begin{array}{ccc}
\mathbf{S} & \longrightarrow & \mathbf{A} \\
{\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_{\mathbf{A}}\psi_{\mathbf{A}}(\mathbf{S})} \\
\mathbf{To} & \xrightarrow{\ \top\ } & (\Omega^*,\mu)
\end{array}$$

This diagram is a pullback if and only if $\mathbf{S}$ is a complete subobject.

(4) *For any $s\in\mathcal{S}(\mathbf{A})$, the following diagram commutes.*

$$\begin{array}{ccc}
\sigma_{\mathbf{A}}(s) & \xrightarrow{\ \hookrightarrow\ } & \mathbf{A} \\
{\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_{\mathbf{A}}(s)} \\
\mathbf{To} & \xrightarrow{\ \top\ } & (\Omega^*,\mu)
\end{array}$$

(5) *For any* $s \in \mathcal{S}(\mathbf{A})$ *we have* $\psi_{\mathbf{A}}\sigma_{\mathbf{A}}(s) \leqslant s$.
(6) *For any* $(S,\delta) \in \mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(A,\delta)$ *we have* $\sigma_{\mathbf{A}}\psi_{\mathbf{A}}(S,\delta) = (\overline{S},\delta)$.

Proof. For $(S,\delta) \in \mathrm{Sub}(\mathbf{A})$ we define $s = \psi_{\mathbf{A}}(S,\delta)$ by

$$(\forall a \in A) \quad s(a) = \bigvee_{x\in S} \delta(a,x).$$

Then $s \subseteq \mathbf{A}$. In fact, we have

$$\begin{aligned} s(a) \otimes (\delta(a,a) \to \delta(a,b)) &= \bigvee_{x\in S} (\delta(a,x) \otimes (\delta(a,a) \to \delta(a,b)) \\ &\leqslant \bigvee_{x\in S} \delta(x,b) = s(b). \end{aligned}$$

The map $\sigma_{\mathbf{A}}$ is defined so that for any $s \in \mathcal{S}(\mathbf{A})$,

$$\sigma_{\mathbf{A}}(s) = (\{a \in A\colon\ s(a) = \delta(a,a)\},\delta) \hookrightarrow \mathbf{A}.$$

Then for a morphism $(A,\delta) \xrightarrow{f} (B\beta)$ in the category Ω-**FSet** the diagram

$$\begin{array}{ccc} \mathcal{S}(\mathbf{A}) & \xrightarrow{\sigma_{\mathbf{A}}} & \mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A}) \\ {\scriptstyle \mathcal{S}(f)}\uparrow & & \uparrow{\scriptstyle \mathrm{Sub}(f)} \\ \mathcal{S}(\mathbf{B}) & \xrightarrow{\sigma_{\mathbf{B}}} & \mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{B}) \end{array}$$

commutes since for any $s \subseteq (B,\beta)$ we have $(\sigma_{\mathbf{B}} \cdot \mathcal{S}(f))(s) = (\{a \in A\colon\ s(f(a)) = 1\},\delta) = (f^{-1}(\{b \in B\colon\ s(b) = 1\}),\delta) = \mathrm{Sub}(f) \cdot \sigma_{\mathbf{B}}(s)$. Moreover, if f is a morphism in the category $\mathrm{Sub}_{\Omega\text{-}\mathbf{FSet}_1}$ then the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Sub}(\mathbf{A}) & \xrightarrow{\psi_{\mathbf{A}}} & \mathcal{S}(\mathbf{A}) \\ {\scriptstyle \mathrm{Sub}(f)}\uparrow & & \uparrow{\scriptstyle \mathcal{S}(f)} \\ \mathrm{Sub}(\mathbf{B}) & \xrightarrow{\psi_{\mathbf{B}}} & \mathcal{S}(\mathbf{B}). \end{array}$$

It is simple to prove that both diagrams from (3) and (4) then commute. Let (S,δ) be a complete subobject of $\mathbf{A}$. We show that the diagram from (3) is then a pullback. In fact, let (B,β) be an Ω-fuzzy set with a morphism u such that the following diagram commutes.

$$\begin{array}{ccc} (B,\beta) & \xrightarrow{u} & (A,\delta) \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \zeta_{\mathbf{A}}\psi_{\mathbf{A}}(S)} \\ \mathbf{To} & \xrightarrow{\top} & (\Omega^*,\mu) \end{array}$$

Then for $b \in B$ we obtain $\delta(u(b), u(b)) = \beta(b, b) = \bigvee_{x \in S} \delta(u(b), x)$ and since S is complete, it follows that $u(b) \in S$. Hence, $u\colon (B, \beta) \to (S, \delta)$ is a morphism and it follows that the diagram is a pullback. Conversely, let the diagram from (3) be a pullback for $\mathbf{X} = (S, \delta)$. Let us assume that there exists $a \in A \setminus S$ such that $\delta(a, a) = \bigvee_{x \in S} \delta(a, x)$. Then the following diagram commutes

$$\begin{array}{ccc} (S \cup \{a\}, \delta) & \xrightarrow{\hookrightarrow} & (A, \delta) \\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_{\mathbf{A}} \psi_{\mathbf{A}}(S)} \\ \mathbf{To} & \xrightarrow{\top} & (\Omega^*, \mu) \end{array}$$

and there exists a morphism $r\colon S \cup \{a\} \to S$ such that $a = r(a) \in S$, a contradiction.

Finally, let $s \in \mathcal{S}(\mathbf{A})$ and $a \in A$. Then we have

$$\begin{aligned} \psi_{\mathbf{A}} \sigma_{\mathbf{A}}(s)(a) &= \bigvee_{\substack{x \in A \\ s(x) = \delta(x,x)}} \delta(a, x) = \bigvee_{\substack{x \in A \\ s(x) = \delta(x,x)}} \delta(a, x) \wedge s(x) \\ &= \bigvee_{\substack{x \in A \\ s(x) = \delta(x,x)}} s(x) \otimes (\delta(x, x) \to \delta(a, x)) \leqslant s(a). \end{aligned}$$

Analogously, for $(S, \delta) \hookrightarrow (A, \delta)$ and for $a \in \sigma_{\mathbf{A}} \psi_{\mathbf{A}}(S)$ we have $\bigvee_{x \in S} \delta(a, x) = \delta(a, a)$ and $a \in \overline{S}$. □

Proposition 2.4.

(1) *For any* $\mathbf{A} \in \Omega\text{-}\mathbf{FSet}$ *we have* $\sigma_{\mathbf{A}}(s) \in \mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})$. *Hence,* $\sigma\colon \mathcal{S} \to \mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}$ *is a natural transformation.*

(2) *For any* $\mathbf{A} \in \Omega\text{-}\mathbf{FSet}$ *we have* $\sigma_{\mathbf{A}} \cdot \psi'_{\mathbf{A}} = \mathbf{id}$, *where* $\psi'_{\mathbf{A}}$ *is a restriction of* $\psi_{\mathbf{A}}$ *onto* $\mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}$.

(3) *For any* $(S, \delta) \in \mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})$, *the diagram*

$$\begin{array}{ccc} (S, \delta) & \xrightarrow{\hookrightarrow} & \mathbf{A} \\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_{\mathbf{A}} \cdot \psi'_{\mathbf{A}}(S)} \\ \mathbf{To} & \xrightarrow{\top} & (\Omega^*, \mu) \end{array}$$

is a pullback.

P r o o f. We show first that for any $s \in \mathcal{S}(\mathbf{A})$, the subobject $\sigma_{\mathbf{A}}(s)$ is complete. From the proof of 2.3, it follows that $\sigma_{\mathbf{A}}(s) = \{a \in A\colon s(a) = \delta(a, a)\}$. Thus we have to prove that

$$\sigma_{\mathbf{A}}(a) = \Big\{a \in A\colon \bigvee_{x\in\sigma_{\mathbf{A}}(s)} \delta(a,x) = \delta(a,a)\Big\}.$$

Let a be an element of the set on the right side. Since $\zeta_{\mathbf{A}}(s)\colon \mathbf{A} \to (\Omega^*, \mu)$ is a morphism in the category Ω-**FSet**, for any $x \in \sigma_{\mathbf{A}}(s)$ we then have

$$\begin{aligned}\delta(a,x) &\leqslant \mu(\zeta_{\mathbf{A}}(s)(x), \zeta_{\mathbf{A}}(s)(a)) = \mu((\delta(a,a), s(a)), (\delta(x,x), s(x)))\\ &= \mu((\delta(a,a), s(a)), (s(x), s(x)))\\ &\leqslant (\delta(a,a) \otimes (s(a) \to s(x))) \wedge s(a) \leqslant s(a).\end{aligned}$$

Hence, we have

$$\delta(a,a) \geqslant s(a) \geqslant \bigvee_{x\in\sigma_{\mathbf{A}}(s)} \delta(a,x) = \delta(a,a),$$

and it follows that $a \in \sigma_{\mathbf{A}}(s)$. Thus, $\sigma_{\mathbf{A}}(s)$ is complete.

Further, for any $(S,\delta) \in \mathrm{Sub}^{\mathrm{c}}(\mathbf{A})$ we have $\sigma_{\mathbf{A}} \cdot \psi'_{\mathbf{A}}(S,\delta) = \{a \in A\colon \delta(a,a) = \bigvee_{x\in S} \delta(a,x)\} = \overline{S} = S$, according to Lemma 2.2. Hence, $\sigma_{\mathbf{A}} \cdot \psi'_{\mathbf{A}} = \mathbf{id}$. The rest follows directly from 2.3. □

The following Theorem summarizes all the previous results. Let $\mathrm{Sub}^{\mathrm{c}}_1$ be the restriction of $\mathrm{Sub}^{\mathrm{c}}$ onto the category Ω-$\mathbf{FSet}_1$.

Theorem 2.5.

(1) *There exists a natural transformation*

$$\chi^{-1}\colon \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}}(-, \Omega^*) \to \mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}(-).$$

(2) *For any* $\mathbf{A} \in \Omega$-**FSet** *there exists a map*

$$\chi_{\mathbf{A}} : \mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A}) \to \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A}, \Omega^*),$$

such that $\chi = \{\chi_{\mathbf{A}}\colon \mathbf{A} \in \Omega\text{-}\mathbf{FSet}_1\}\colon \mathrm{Sub}^{\mathrm{c}}_1(-) \to \mathrm{Hom}_{\Omega\text{-}\mathbf{FSet}_1}(-, \Omega^*)$ *is a natural transformation.*

(3) *For any* $(S,\delta) \in \mathrm{Sub}^{\mathrm{c}}_{\Omega\text{-}\mathbf{FSet}}(\mathbf{A})$, *the following diagram is a pullback:*

$$\begin{array}{ccc} (S,\delta) & \xrightarrow{\ \hookrightarrow\ } & (A,\delta) \\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \chi_{\mathbf{A}}(S,\delta)} \\ \mathbf{To} & \xrightarrow{\ \top\ } & (\Omega,\mu). \end{array}$$

(4) $\chi^{-1} \cdot \chi = \mathbf{id}$.

Proof. Let $\chi_{\mathbf{A}}$, $\chi_{\mathbf{A}}^{-1}$ be the compositions of the following maps from 2.1, 2.4:

$$\chi_{\mathbf{A}}\colon\ \mathrm{Sub}^{\mathrm{c}}(\mathbf{A}) \xrightarrow{\psi_{\mathbf{A}}'} \mathcal{S}(\mathbf{A}) \xrightarrow{\zeta_{\mathbf{A}}} \mathrm{Hom}(\mathbf{A}, \Omega^*),$$
$$\chi_{\mathbf{A}}^{-1}\colon\ \mathrm{Hom}(\mathbf{A}, \Omega^*) \xrightarrow{\zeta_{\mathbf{A}}^{-1}} \mathcal{S}(\mathbf{A}) \xrightarrow{\sigma_{\mathbf{A}}} \mathrm{Sub}^{\mathrm{c}}(\mathbf{A}).$$

Then the proposition follows from 2.1 and 2.4. □

It should be noted that $\chi_{\mathbf{A}} \cdot \chi_{\mathbf{A}}^{-1}$ is not the identity in general. In fact, let L be a complete MV-algebra such that $\otimes \neq \wedge$ in general. Recall that the product $(\Omega^* \times \Omega^*, \gamma)$ in Ω-**FSet** is defined such that

$$\begin{aligned}\Omega^* \times \Omega^* = \{((\alpha_1, \beta_1), (\alpha_2, \beta_2)) \mid (\alpha_i, \beta_i) \in \Omega^*,\ \mu((\alpha_1, \beta_1), (\alpha_1, \beta_1)) \\ = \mu((\alpha_2, \beta_2), (\alpha_2, \beta_2))\} = \{((\alpha, \beta_1), (\alpha, \beta_2)) \mid \alpha \geqslant \beta_1, \beta_2\},\end{aligned}$$

and

$$\gamma(((\alpha, \beta_1), (\alpha, \beta_2)), ((\varrho, \tau_1), (\varrho, \tau_2))) = \mu((\alpha, \beta_1), (\varrho, \tau_1)) \wedge \mu((\alpha, \beta_2), (\varrho, \tau_2)).$$

Let us define a map $\otimes_\Omega\colon\ \Omega^* \times \Omega^* \to \Omega^*$ by

$$\otimes_\Omega((\alpha, \beta_1), (\alpha, \beta_2)) = (\alpha, \beta_1 \otimes \beta_2).$$

Then $\otimes_\Omega$ is a morphism in Ω-**FSet**. Indeed, for $\mathbf{a} = ((\alpha, \beta_1), (\alpha, \beta_2)) \in \Omega^* \times \Omega^*$ we have

$$\mu(\otimes_{\Omega^*}(\mathbf{a}), \otimes_{\Omega^*}(\mathbf{a})) = \alpha = \sigma(\mathbf{a}, \mathbf{a}).$$

Furthermore, for $\mathbf{a} = ((\alpha, \beta_1), (\alpha, \beta_2))$ and $\mathbf{b} = ((\varrho, \tau_1), (\varrho, \tau_2))$ we have

$$\begin{aligned}\gamma(\mathbf{a}, \mathbf{b}) &= (\alpha \otimes (\beta_1 \to \tau_1)) \wedge (\alpha \otimes (\beta_2 \to \tau_2)) \wedge (\varrho \otimes (\tau_1 \to \beta_1)) \wedge (\varrho \otimes (\tau_2 \to \beta_2)) \\ &= \alpha \otimes ((\beta_1 \to \tau_1) \wedge (\beta_2 \to \tau_2)) \wedge \varrho \otimes ((\tau_1 \to \beta_1) \wedge (\tau_2 \to \beta_2)) \\ &\leqslant \alpha \otimes (\beta_1 \otimes \beta_2 \to \tau_1 \wedge \tau_2) \wedge \varrho \otimes (\tau_1 \otimes \tau_2 \to \beta_1 \wedge \beta_2) \\ &\leqslant \alpha \otimes (\beta_1 \otimes \beta_2 \to \tau_1 \otimes \tau_2) \wedge \varrho \otimes (\tau_1 \otimes \tau_2 \to \beta_1 \otimes \beta_2) \\ &= \mu(\otimes_{\Omega^*}(\mathbf{a}), \otimes_{\Omega^*}(\mathbf{b})).\end{aligned}$$

We show that $\chi_\Omega \cdot \chi_\Omega^{-1}(\otimes_\Omega) \neq \otimes_\Omega$ in general. In fact, we have

$$\begin{aligned}S = \chi_\Omega^{-1}(\otimes_\Omega) &= \sigma_{\Omega^* \times \Omega^*} \cdot \zeta_{\Omega^* \times \Omega^*}^{-1}(\otimes_\Omega) \\ &= \{((\alpha, \beta_1), (\alpha, \beta_2))\colon\ \alpha, \beta_i \in L,\ \beta_1 \otimes \beta_2 = \alpha, \beta_i \leqslant \alpha\} \\ &= \{((\alpha, \alpha), (\alpha, \alpha))\colon\ \alpha \otimes \alpha = \alpha\}.\end{aligned}$$

Further, we have $\chi_{\Omega^*\times\Omega^*}(S) = \zeta_{\Omega^*\times\Omega^*}\cdot\psi'_{\Omega^*\times\Omega^*}(S)$, where $s = \psi'_{\Omega^*\times\Omega^*}(S)\colon \Omega^*\times\Omega^*\to L$ is such that for $\mathbf{a} = ((\alpha,\beta_1),(\alpha,\beta_2))$ we have

$$\begin{aligned} s(\mathbf{a}) &= \bigvee_{\mathbf{x}\in S}\gamma(\mathbf{a},\mathbf{x}) = \bigvee_{\varrho\in L}\gamma((\alpha,\beta_1),(\alpha,\beta_2),(\varrho,\varrho),(\varrho,\varrho)) \\ &= \alpha\wedge\beta_1\wedge\beta_2 = \beta_1\wedge\beta_2. \end{aligned}$$

Hence, for β_1,β_2 such that $\beta_1\otimes\beta_2 < \beta_1\wedge\beta_2$ we have

$$\chi_\Omega\cdot\chi_\Omega^{-1}(\otimes_\Omega)(\mathbf{a}) = (\gamma(\mathbf{a},\mathbf{a}), s(\mathbf{a})) \neq (\alpha,\beta_1\otimes\beta_2) = \otimes_\Omega(\mathbf{a}).$$

3. Examples of complete subobjects in Ω-**FSet**

Since the category Ω-**FSet** seems to be a well-defined basis for investigation of fuzzy sets, it could also be used for interpretation of formulas. The interpretation of a formula Φ of any logic in a category $\mathcal{K}$, where Φ has its free variables contained in a set X of free variables, is based on a construction of a characteristic morphism $\|\Phi\|\colon M(X)\to\Omega$, where Ω is an analogy of a subobject classifier and $M(X) = \prod_{x\in X} M(\iota_x)$ is the product in $\mathcal{K}$ of interpretations of the types ι_x corresponding to the variables $x\in X$. If a binary logical connective $\triangledown$ appears in a formula Φ, then some interpretation of $\triangledown$ has to be defined first, which is a morphism $\Omega\times\Omega \xrightarrow{\triangle} \Omega$.

In this part we want to show how the logical connectives $\wedge,\vee,\neg$ can be interpreted in the category Ω-**FSet** by using results from the previous sections.

Example [construction of $\cap_\Omega$]**.** Recall that the interpretation $\cap_\Omega$ of $\wedge$ is (classically) the characteristic morphism of the subobject $\top\times\top\colon \mathbf{To}\to\Omega^*\times\Omega^*$. In the category Ω-**FSet** this construction can be used if this subobject is complete.

Hence, we have $S_\wedge = \top\times\top(\mathbf{To}) = \{((\omega,\omega),(\omega,\omega))\colon \omega\in\Omega\}$ and let $\mathbf{a} = ((\alpha,\beta_1),(\alpha,\beta_2))\in\Omega^*\times\Omega^*$ be such that

$$\gamma(\mathbf{a},\mathbf{a}) = \bigvee_{\mathbf{x}\in S_\wedge}\gamma(\mathbf{a},\mathbf{x}).$$

Then we have

$$\begin{aligned} \gamma(\mathbf{a},\mathbf{a}) &= \bigvee_{\omega\in\Omega}(\alpha\otimes(\beta_1\to\omega)\wedge\omega\otimes(\omega\to\beta_1)\wedge\alpha\otimes(\beta_2\to\omega)\wedge\omega\otimes(\omega\to\beta_2)) \\ &= \beta_1\wedge\beta_2\wedge\bigvee_{\omega\in\Omega}(\alpha\otimes(\beta_1\to\omega)\wedge\otimes(\beta_2\to\omega)\wedge\omega) \\ &= \beta_1\wedge\beta_2\wedge\alpha = \beta_1\wedge\beta_2, \\ \alpha &= \gamma(\mathbf{a},\mathbf{a}). \end{aligned}$$

Hence, we have $\alpha = \beta_1 = \beta_2$ and it follows that $\mathbf{a} \in S_\wedge$. Then according to 2.5, the interpretation $\cap_\Omega$ of $\wedge$ is the characteristic morphism $\Omega^* \times \Omega^* \xrightarrow{\cap_\Omega} \Omega^*$ such that $\cap_\Omega = \chi(S_\wedge)$, i.e. for $\mathbf{a} = ((\alpha, \beta_1), (\alpha, \beta_2)) \in \Omega^* \times \Omega^*$ we have

$$\cap_\Omega(\mathbf{a}) = \zeta_{\mathbf{A}} \cdot \psi_{\mathbf{A}}(S_\wedge)(\mathbf{a}) = \left(\gamma(\mathbf{a}, \mathbf{a}), \bigvee_{\mathbf{x} \in S_\wedge} \gamma(\mathbf{a}, \mathbf{x})\right) = (\alpha, \beta_1 \wedge \beta_2),$$

which can be shown easily.

Example [construction of $\cup_\Omega$]**.** Let us consider the following subobject $S_\vee$ of the object $\Omega^* \times \Omega^*$,

$$S_\vee = \{((\beta_1 \vee \beta_2, \beta_1), (\beta_1 \vee \beta_2, \beta_2)) \colon \ \beta_1, \beta_2 \in \Omega\}.$$

Then $S_\vee$ is complete. In fact, let $\mathbf{a} = ((\alpha, \beta_1), (\alpha, \beta_2)) \in \Omega^* \times \Omega^*$ be such that

$$\beta_1 \vee \beta_2 \leqslant \alpha = \gamma(\mathbf{a}, \mathbf{a}) = \bigvee_{\mathbf{x} \in S_\vee} \gamma(\mathbf{a}, \mathbf{x}).$$

Then we have

$$\begin{aligned}
\beta_1 \vee \beta_2 &\leqslant \bigvee_{\tau_1, \tau_2 \in \Omega} \mu((\alpha, \beta_1), (\tau_1 \vee \tau_2, \tau_1)) \wedge \mu((\alpha, \beta_2), (\tau_1 \vee \tau_2, \tau_2)) \\
&= \bigvee_{\tau_1, \tau_2 \in \Omega} \alpha \otimes ((\beta_1 \to \tau_1) \wedge (\beta_2 \to \tau_2)) \wedge (\tau_1 \vee \tau_2) \\
&\quad \otimes ((\tau_1 \to \beta_1) \wedge (\tau_2 \to \tau_2)) \\
&\leqslant \bigvee_{\tau_1, \tau_1 \in \Omega} \alpha \otimes ((\beta_1 \vee \beta_2) \to (\tau_1 \vee \tau_2)) \wedge (\tau_1 \vee \tau_2) \\
&\quad \otimes ((\tau_1 \vee \tau_2) \to (\beta_1 \vee \beta_2)) \\
&= \alpha \otimes (\beta_1 \vee \beta_2 \to 1) \wedge (\beta_1 \vee \beta_2) = \beta_1 \vee \beta_2.
\end{aligned}$$

Hence, $\alpha = \beta_1 \vee \beta_2$ and $\mathbf{a} \in S_\vee$. Then the characteristic morphism $\cup_\Omega$ of a complete object $S_\vee$ is $\chi(S_\vee)$ and we have

$$\cup_\Omega(\mathbf{a}) = \left(\gamma(\mathbf{a}, \mathbf{a}), \bigvee_{\mathbf{x} \in S_\vee} \gamma(\mathbf{a}, \mathbf{x})\right) = (\alpha, \beta_1 \vee \beta_2).$$

Example [construction of $\neg_\Omega$]**.** Recall that the interpretation $\neg_\Omega$ of $\neg$ is defined in a category as a characteristic morphism of $\bot \colon \mathbf{To} \to \Omega^*$, where $\bot$ is defined by $\bot(\alpha) = (\alpha, 0)$. This construction can be used in the category Ω-**FSet** if the subobject corresponding to $\perp$ is complete, i.e. in case that the set $S = \{(\alpha, 0)\colon \alpha \in L\}$ is complete. But we have

$$\begin{aligned}\overline{S} &= \left\{(\alpha,\beta) \in \Omega^*\colon \bigvee_{\varrho \in L} \alpha \otimes (\beta \to 0) \wedge \varrho \otimes (0 \to \beta) = \alpha\right\}\\ &= \{(\alpha,\beta) \in \Omega^*\colon \alpha = \alpha \otimes (\beta \to 0)\}\end{aligned}$$

and it follows that $S \neq \overline{S}$, in general. The interpretation $\neg_\Omega$ can then be defined as the characteristic morphism of a complete subobject $\overline{S}$. Hence, we have $\neg_\Omega(\alpha,\beta) = \zeta_{\Omega^*} \cdot \psi'_{\Omega^*}(\overline{S})(\alpha,\beta) = (\alpha, \alpha \otimes (\beta \to 0))$.

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Author's address: Department of Mathematics, Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, CZ-702 00 Ostrava, e-mail: `Jiri.Mockor@osu.cz`.