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# EXISTENCE OF SOLUTIONS AND OF MULTIPLE SOLUTIONS FOR NONLINEAR NONSMOOTH PERIODIC SYSTEMS

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*Abstract.* In this paper we examine nonlinear periodic systems driven by the vectorial $p$-Laplacian and with a nondifferentiable, locally Lipschitz nonlinearity. Our approach is based on the nonsmooth critical point theory and uses the subdifferential theory for locally Lipschitz functions. We prove existence and multiplicity results for the "sublinear" problem. For the semilinear problem (i.e. $p=2$) using a nonsmooth multidimensional version of the Ambrosetti-Rabinowitz condition, we prove an existence theorem for the "superlinear" problem. Our work generalizes some recent results of Tang (PAMS 126(1998)).

*Keywords*: $p$-Laplacian, nonsmooth critical point theory, Clarke subdifferential, saddle point theorem, periodic solution, Poincare-Wirtinger inequality, Sobolev inequality, nonsmooth Palais-Smale condition

*MSC 2000*: 34C25

## 1. Introduction

In two recent papers Tang [12], [13] obtained existence and multiplicity results for semilinear nonautonomous periodic systems with a continuously differentiable sublinear nonlinearity. More precisely, he studied the following problem:

$$(1)\qquad \begin{cases} x''(t) = \nabla\varphi(t, x(t)) \quad \text{a.e. on } T=[0,b], \\ x(0)=x(b),\ x'(0)=x'(b). \end{cases}$$

Assuming that $\varphi(t,x)$ is measurable in $t\in T$, continuously differentiable in $x\in\mathbb{R}^N$, $\|\varphi(t,x)\|, \|\nabla\varphi(t,x)\| \leqslant \alpha(\|x\|)c(t)$ a.e. on $T$ with $\alpha\in C(\mathbb{R}_+)$, $c\in L^1(T)_+$, $\|\nabla\varphi(t,x)\| \leqslant g(t)+f(t)\|x\|^\theta$ a.e. on $T$, with $f,g\in L^1(T)_+$, $0\leqslant\theta<1$, and $\lim\limits_{\|x\|\to\infty}(1/\|x\|^{2\theta})\int_0^b\varphi(t,x)\,\mathrm{d}t = \pm\infty$, Tang proved that problem (1) has a solution (see [12, Theorems 1 and 2]). Moreover, by imposing additional growth conditions on $\varphi(t,x)$, he also proved multiplicity results for problem (1) (see [12, Theorems 3 and 4]). Similar results can be found in [13] but under more restrictive hypotheses on $\varphi(t,x)$. The results of Tang extend earlier ones obtained by Mawhin-Willem (see [9, Section 4.3, pp. 85–87]).

The goal of this paper is to obtain extensions of the results of Tang to quasilinear periodic systems driven by the vectorial $p$-Laplacian and having a nondifferentiable potential. Such problems are also known as "hemivariational inequalities" and have applications in mechanics and engineering (see [10]).

The problem under consideration is the following:

$$\begin{cases} (\|x'(t)\|^{p-2}x'(t))' \in \partial\varphi(t,x(t)) \quad \text{a.e. on } T=[0,b], \\ x(0)=x(b),\ x'(0)=x'(b),\ 2\leqslant p<\infty. \end{cases} \tag{2}$$

Here the differential operator of the left-hand side is the vectorial $p$-Laplacian $\Delta_p x(t) = (\|x'(t)\|^{p-2}\cdot x'(t))'$. The norm inside the parenthesis is the usual Euclidean norm. This is the "ordinary" version of the partial differential operator $\Delta_p x = \operatorname{div}(\|Dx\|^{p-2}Dx)$ when $x$ defined on a bounded domain $Z \subseteq \mathbb{R}^N$ with $N \geqslant 2$. Also $\varphi(t,\cdot)$ is locally Lipschitz on $\mathbb{R}^N$ into $\mathbb{R}$ and $\partial\varphi(t,x)$ denotes the generalized subdifferential in the sense of Clarke [3] (see also Section 2). Our approach is based on the nonsmooth critical point theory of Chang [2] (for extensions see [6]). For the convenience of the reader in the next section we recall the main aspects of this theory.

## 2. Mathematical background

The nonsmooth critical point theory of Chang [2] uses the subdifferential theory of Clarke [3]. So let us start by briefly presenting the main aspects of this theory. More details can be found in [3].

Let $X$ be a Banach space and $X^*$ its topological dual. By $\|\cdot\|$ we denote the norm of $X$, by $\|\cdot\|_*$ the norm of $X^*$ and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $(X,X^*)$. A function $\varphi\colon X\to\mathbb{R}$ is said to be locally Lipschitz if for every bounded open set $U\subseteq X$, there exists $k_U>0$ such that $|\varphi(x)-\varphi(y)|\leqslant k_U\|x-y\|$ for all $x,y\in U$. Recall that if $\psi\colon X\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is proper (i.e. not identically $+\infty$), convex and lower semicontinuous (i.e. $\psi\in\Gamma_0(X)$, see [4, p. 341]), then $\psi$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom}\psi=\{x\in X\colon\ \psi(x)<+\infty\}$. Hence a convex $\mathbb{R}$-valued function on $X$ is locally Lipschitz.

Now given a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, in analogy to the directional derivative of a convex function, we define

$$\varphi^0(x;h) = \limsup_{\substack{y \longrightarrow x \\ \lambda \downarrow 0}} \frac{\varphi(y+\lambda h) - \varphi(y)}{\lambda} \quad \text{for all } y, h \in X.$$

This quantity is called the "generalized directional derivative" of $\varphi$ at $x \in X$ in the direction $h \in X$. For every $x \in \mathbb{R}$, the function $h \to \varphi^0(x;h)$ is sublinear continuous and so by the Hahn-Banach theorem it is the support function of a nonempty, $w^*$-compact and convex set given by

$$\partial\varphi(x) = \{x^* \in X^*\colon \langle x^*, h\rangle \leqslant \varphi^0(x;h) \quad \text{for all } h \in X\}.$$

So $\varphi^0(x;h) = \sup\{\langle x^*, h\rangle : x^* \in \partial\varphi(x)\}$ (i.e. $\varphi^0(x;\cdot)$ is the support function of the set $\partial\varphi(x)$) and the multifunction $\partial\varphi\colon X \to 2^{X^*} \setminus \{\emptyset\}$ is known as the (Clarke or generalized) subdifferential of $\varphi$. This multifunction has a graph $\operatorname{Gr}\partial\varphi = \{(x,x^*) \in X \times X^*\colon x^* \in \partial\varphi(x)\}$ which is sequentially closed in $X \times X^*_{w^*}$. Here by $X^*_{w^*}$ we denote the Banach space $X^*$ furnished with the $w^*$-topology. So if $x_n \to x$ in $X$, $x_n^* \overset{w^*}{\to} x^*$ in $X^*$ and $x_n^* \in \partial\varphi(x_n)$ for all $n \geqslant 1$, then $x^* \in \partial\varphi(x)$. If $\varphi, \psi\colon X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(\varphi+\psi)(x) \subseteq \partial\varphi(x)+\partial\psi(x)$ and $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$. If $\varphi$ is in addition convex, then the subdifferential $\partial\varphi$ coincides with the subdifferential in the sence of convex analysis (see for example [5, p. 267]). Finally, if $\varphi$ is continuously Gateaux differentiable at $x \in X$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

Let $\varphi\colon X \to \mathbb{R}$ be a locally Lipschitz function. In analogy with the smooth case, we say that $x \in X$ is a "critical point" of $\varphi$ if $0 \in \partial\varphi(x)$. Then $c = \varphi(x)$ is a "critical value" of $\varphi$. It is easy to check that if $x \in X$ is a local extremum (i.e. a local minimum or maximum), then $x$ is a critical point, i.e. $0 \in \partial\varphi(x)$. It is well-known that the smooth critical point theory uses a compactness condition, known as the Palais-Smale condition (PS-condition for short). In the present nonsmooth setting, this condition takes the following form:

> "A locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ satisfies the nonsmooth Palais-Smale condition at level $c \in \mathbb{R}$ (nonsmooth $\mathrm{PS}_c$-condition for short) if any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $m(x_n) = \inf\{\|x^*\|_*\colon x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. If $\varphi$ satisfies the nonsmooth $\mathrm{PS}_c$-condition at every level $c \in \mathbb{R}$, we simply say $\varphi$ satisfies the nonsmooth PS-condition."

If $\varphi \in C^1(X)$, then because $\partial\varphi(x) = \{\varphi'(x)\}$ for all $x \in X$, we see that the notion of the nonsmooth PS-condition coincides with the classical one (see [4]). Using this compactness-type property of $\varphi$, we can state the following nonsmooth extension of the classical "Saddle Point Theorem" (see [2]):

**Theorem 1.** *If $X$ is a reflexive Banach space, $X = Y \oplus V$ with $\dim Y < +\infty$, $\varphi\colon X \to \mathbb{R}$ is locally Lipschitz, there exists $R > 0$ such that $\max\{\varphi(y)\colon y \in Y, \|y\| = R\} < \inf\{\varphi(v)\colon v \in V\}$, and $\varphi$ satisfies the nonsmooth* $\mathrm{PS}_{c_0}$*-condition where $c_0 = \inf\limits_{\gamma\in\Gamma}\max\limits_{y\in D}\varphi(\gamma(y))$ with $D = \{y \in Y\colon \|y\| \leqslant R\}$ and $\Gamma = \{\gamma \in C(D, X)\colon \gamma(y) = y$ whenever $\|y\| = R\}$, then $c_0 \geqslant \inf\limits_V \varphi$ and $c_0$ is a critical value of $\varphi$ with corresponding critical point $x \in X$. Moreover, if $c_0 = \inf\limits_V \varphi$, then there exists a critical point $x \in V$ of $\varphi$ with $c_0 = \varphi(x)$.*

**Remark.** A more general version of this theorem can be found in [6].

Another theorem that we shall need in the sequel is the following (see [2]):

**Theorem 2.** *If $X$ is a reflexive Banach space and $\varphi\colon X \to \mathbb{R}$ is locally Lipschitz function which is bounded below and satisfies the nonsmooth* PS*-condition, then $\inf\limits_X \varphi$ is attained at $x \in X$ and $x$ is a critical point of $\varphi$.*

## 3. Existence theorems

In this section we prove three existence theorems for problem (1), thus generalizing the work of Tang to the present quasilinear and nonsmooth setting.

For the first two theorems our hypotheses on the nonsmooth potential function $\varphi(t,x)$ are the following:

$\mathrm{H}(\varphi)_1$ $\varphi\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $\varphi(\cdot,0) \in L^1(T)$ and

(i) for every $x \in \mathbb{R}^N$, $t \to \varphi(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to \varphi(t,x)$ is locally Lipschitz;

(iii) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial\varphi(t,x)$ we have

$$\|u\| \leqslant \alpha(t) + c(t)\|x\|^\theta$$

with $\alpha, c \in L^q(T)_+$, $1/p + 1/q = 1$ and $0 \leqslant \theta < p - 1$;

(iv) one of the following two conditions holds:

$(\mathrm{iv})_1$ $(1/\|x\|^{\theta q})\int_0^b \varphi(t,x)\,\mathrm{d}t \to -\infty$, as $\|x\| \to \infty$, or

$(\mathrm{iv})_2$ $(1/\|x\|^{\theta q})\int_0^b \varphi(t,x)\,\mathrm{d}t \to +\infty$, as $\|x\| \to \infty$.

**Remark.** Our hypotheses are similar to those employed by Tang [12] in the context of semilinear (i.e. $p = 2$) systems with a $C^1$-potential $\varphi(t,\cdot)$. So Theorems 4 and 5 below extend Theorems 1 and 2 of [12]. In what follows by $\mathrm{H}(\varphi)_{1,1}$ we shall denote hypothesis $\mathrm{H}(\varphi)_1$ with $(\mathrm{iv})_1$ in effect and by $\mathrm{H}(\varphi)_{1,2}$, with $(\mathrm{iv})_2$ in effect. When we simply write $\mathrm{H}(\varphi)_1$ then we mean the above hypotheses with either one of $(\mathrm{iv})_1$ or $(\mathrm{iv})_2$ in effect.

Let $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)=\{x\in W^{1,p}(T,\mathbb{R}^N)\colon\ x(0)=x(b)\}$ and let $V\colon\ W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\to\mathbb{R}$ be the energy function defined by

$$V(x)=\frac{1}{p}\|x'\|_p^p+\int_0^b\varphi(t,x(t))\,\mathrm{d}t.$$

We know that $V$ is locally Lipschitz (see [5, p. 313]). Also by $\hat{\Phi}\colon\ L^p(T,\mathbb{R}^N)\to\mathbb{R}$ we denote the integral functional $\hat{\Phi}(x)=\int_0^b\varphi(t,x(t))\,\mathrm{d}t$ and $\Phi=\hat{\Phi}|_{W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)}$. Both are locally Lipschitz.

**Proposition 3.** *If hypotheses* $\mathrm{H}(\varphi)_1$ *hold, then* $V$ *satisfies the nonsmooth* PS-*condition.*

Proof. Let $\{x_n\}\subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ be a sequence such that

$$|V(x_n)|\leqslant M_1\quad\text{for all } n\geqslant 1\quad\text{and}\quad m(x_n)\to 0 \text{ as } n\to\infty.$$

Let $x_n^*\in\partial V(x_n)$ such that $m(x_n)=\|x_n^*\|_*$, $n\geqslant 1$. The existence of such elements follows from the fact that $\partial V(x_n)\subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)^*$ is weakly compact and the norm $\|\cdot\|_*$ is weakly lower semicontinuous. Also let $A\colon\ W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\to W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)^*$ be the nonlinear operator defined by

$$\langle A(x),y\rangle=\int_0^b\|x'(t)\|^{p-2}(x'(t),y'(t))_{\mathbb{R}^N}\,\mathrm{d}t\quad\text{for all } x,y\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N).$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N),W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)^*)$. It is easy to check (see for example [7]) that $A$ is maximal monotone. We have

$$x_n^*=A(x_n)+u_n\quad\text{with } u_n\in\partial\Phi(x_n),\ n\geqslant 1.$$

Since $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is embedded continuously (in fact compactly) and densely in $L^p(T,\mathbb{R}^N)$, from Theorem 2.2 of [2], we have that $\partial\Phi(x_n)\subseteq L^q(T,\mathbb{R}^N)$ and so $u_n\in L^q(T,\mathbb{R}^N)$ for all $n\geqslant 1$. Moreover, we know that $u_n(t)\in\partial\varphi(t,x_n(t))$ a.e. on $T$ (see [3, p. 76]).

We shall show that $\{x_n\}_{n\geqslant 1}\subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded. To this end, consider the direct sum decomposition $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)=\mathbb{R}^N\oplus Y$, where $Y=\{y\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\colon \int_0^b y(t)\,\mathrm{d}t=0\}$. Given $x\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$, we write $x=\overline{x}+\hat{x}$ with $\overline{x}\in\mathbb{R}^N$ and $\hat{x}\in Y$ (of course the decomposition is unique). From the choice of the sequence $\{x_n\}_{n\geqslant 1}\subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ we have

$$|\langle A(x_n),\hat{x}_n\rangle+(u_n,\hat{x}_n)_{pq}|\leqslant\varepsilon_n\|\hat{x}_n\|\quad\text{with }\varepsilon_n\downarrow 0.\tag{3}$$

Here by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(T,\mathbb{R}^N),\, L^q(T,\mathbb{R}^N))$, i.e. $(u_n,\,\hat{x}_n)_{pq}=\int_0^b(u_n(t),\,\hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t$. Using hypothesis $\mathrm{H}(\varphi)_1$(iii) we have that

$$
\begin{aligned}
(4)\qquad (u_n(t),\hat{x}_n(t))_{\mathbb{R}^N} &\leqslant (\alpha(t)+c(t)\|\overline{x}_n+\hat{x}_n(t)\|^\theta)\|\hat{x}_n(t)\|\\
&\leqslant \alpha(t)\|\hat{x}_n(t)\|+2^{\theta-1}c(t)\|\overline{x}_n\|^\theta\|\hat{x}_n(t)\|\\
&\quad +2^{\theta-1}c(t)\|\hat{x}_n(t)\|^{\theta+1}\quad\text{a.e. on } T,\\
\Rightarrow \left|\int_0^b(u_n(t),\hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t\right| &\leqslant \|\alpha\|_1\|\hat{x}_n\|_\infty+2^{\theta-1}\|c\|_1\|\hat{x}_n\|_\infty^{\theta+1}\\
&\quad +2^{\theta-1}\|c\|_1\Big(\frac{\varepsilon}{p}\|\hat{x}_n\|_\infty^p+\frac{1}{\varepsilon q}\|\overline{x}_n\|^{\theta q}\Big)\\
&\leqslant \beta_1\|\hat{x}_n'\|_p+\beta_2\|\hat{x}_n'\|_p^{\theta+1}+\beta_3\frac{\varepsilon}{p}\|\hat{x}_n'\|_p^p+\beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}
\end{aligned}
$$

for $\varepsilon>0$ and for some $\beta_1,\beta_2,\beta_3,\beta_4(\varepsilon)>0$. Here we have used the Poincare-Wirtinger inequality which says that for all $v\in V$, $\|v\|_\infty\leqslant b^{1/q}\|v'\|_p$ (see [9, p. 8]). Also note that $\langle A(x_n),\hat{x}_n\rangle=\|\hat{x}_n'\|_p^p$. So returning to (3) and using these facts, we obtain

$$\|\hat{x}_n'\|_p^p-\beta_1\|\hat{x}_n'\|_p-\beta_2\|\hat{x}_n'\|_p^{\theta+1}-\beta_3\frac{\varepsilon}{p}\|\hat{x}_n'\|_p^p-\beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}\leqslant M_2\|\hat{x}_n\|$$

for some $M_2>0$ and all $n\geqslant 1$, hence

$$(5)\qquad \Big(1-\beta_3\frac{\varepsilon}{p}\Big)\|\hat{x}_n'\|_p^p-(\beta_1+M_3)\|\hat{x}_n'\|_p-\beta_2\|\hat{x}_n'\|_p^{\theta+1}\leqslant\beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}$$

for some $M_3>0$ and all $n\geqslant 1$.

In obtaining the last inequality we have used once more the Poincare-Wirtinger inequality. Choose $\varepsilon>0$ small so that $\beta_3\varepsilon/p<1$. We claim that from (5) we can infer that

$$(6)\qquad \|\hat{x}_n'\|_p^{p-1}\leqslant\beta_5\|\overline{x}_n\|^\theta+\beta_6\quad\text{for some } \beta_5,\beta_6>0\quad\text{and}\quad n\geqslant 1.$$

Indeed, if $\{\hat{x}_n\}_{n\geqslant 1}\subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded, this is clearly the case. Otherwise suppose that $\|\hat{x}_n'\|_p\to\infty$ (by the Poincare-Wirtinger inequality). So from (5) we obtain

$$\beta_7\|\hat{x}_n'\|_p^p-\beta_8\|\hat{x}_n'\|_p^{\theta+1}\leqslant\beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}\quad\text{for some } \beta_7,\beta_8>0\quad\text{and all } n\geqslant 1.$$

Recall that $\theta+1<p$. Using Young's inequality with $\delta>0$ small on the term $\beta_8\|\hat{x}_n'\|_p^{\theta+1}$, we obtain that

$$\beta_9\|\hat{x}_n'\|_p^p\leqslant\beta_4\|\overline{x}_n\|^{\theta q}+\beta_{10}\quad\text{for some } \beta_9,\beta_{10}>0\quad\text{and all } n\geqslant 1,$$

from which (6) follows.

Let $S_n(t) = \{(u,\lambda) \in \mathbb{R}^N \times (0,1)\colon\ u \in \partial\varphi(t, \overline{x}_n + \lambda\hat{x}_n(t)),\ \varphi(t, \overline{x}_n + \hat{x}_n(t)) - \varphi(t,\overline{x}_n) = (u, \hat{x}_n(t))_{\mathbb{R}^N}\}$. From Lebourg's mean value theorem (see [3, p. 41] and [8]), we know that for almost all $t \in T$, $S_n(t) \neq \emptyset$. By redefining $S_n$ on a Lebesgue-null subset of $T$, we may assume without any loss of generality that $S_n(t) \neq \emptyset$ for all $t \in T$. We claim that for every $h \in \mathbb{R}^N$, the function $(t,\lambda) \to \varphi^0(t, \overline{x}_n + \lambda\hat{x}_n(t); h)$ is measurable on $T \times (0,1)$. To this end, from the definition of the generalized directional derivative (see Section 2), we have

$$\begin{aligned}
&\varphi^0(t, \overline{x}_n + \lambda\hat{x}_n(t); h) \\
&\quad = \inf_{m\geqslant 1} \sup\Big\{ \frac{\varphi(t, \overline{x}_n + \lambda\hat{x}_n(t) + r + sh) - \varphi(t, \overline{x}_n + \lambda\hat{x}_n(t) + r)}{s} : \\
&\qquad\qquad\qquad\qquad r \in \mathbb{Q}^N \cap B_{1/m}(0),\ s \in \mathbb{Q} \cap \Big(0, \frac{1}{m}\Big)\Big\}
\end{aligned}$$

where $\mathbb{Q}^N = \mathbb{Q} \times \mathbb{Q} \times \ldots \times \mathbb{Q}$ ($N$times), $B_{1/m}(0) = \{y \in \mathbb{R}^N:\ \|y\| < 1/m\}$. So it follows that $(t,\lambda) \to \varphi^0(t, \overline{x}_n + \lambda\hat{x}_n(t); h)$ is measurable as claimed.

Set $G_n(t,\lambda) = \partial\varphi(t, \overline{x}_n + \lambda\hat{x}_n(t))$ and let $\{h_m\}_{m\geqslant 1} \subseteq \mathbb{R}^N$ be a dense sequence. Exploiting the continuity of $\varphi^0(t, \overline{x}_n + \lambda\hat{x}_n(t); \cdot)$, from the definition of the generalized subdifferential (see Section 2), we have

$$\begin{aligned}
\operatorname{Gr} G_n &= \bigcap_{m\geqslant 1} \{(t,\lambda,u) \in T \times (0,1) \times \mathbb{R}^N:\ (u, h_m)_{\mathbb{R}^N} \leqslant \varphi^0(t, \overline{x}_n + \lambda\hat{x}_n(t); h_m)\} \\
&\Rightarrow \operatorname{Gr} G_n \in \mathcal{L}(T) \times B(I) \times B(\mathbb{R}^N)
\end{aligned}$$

where $\mathcal{L}(T)$ is the Lebesgue $\sigma$-field of $T$, $I = (0,1)$, and $B(I)$, $B(\mathbb{R}^N)$ are the Borel $\sigma$-fields of $I$ and $\mathbb{R}^N$. Then we obtain $\operatorname{Gr} S_n \in \mathcal{L}(T) \times B(I) \times B(\mathbb{R}^N)$. So we can apply the Yankon-von Neumann-Aumann selection theorem (see [4, p. 158] or [14]) to obtain Lebesgue measurable maps $u_n\colon T \to \mathbb{R}^N$ and $\lambda_n\colon T \to I$ such that $(u_n(t), \lambda_n(t)) \in S_n(t)$ for all $t \in T$ and all $n \geqslant 1$. So we have

$$\begin{aligned}
V(x_n) &= \frac{1}{p}\|x_n'\|_p^p + \int_0^b \varphi(t, x_n(t))\,\mathrm{d}t \\
&= \frac{1}{p}\|\hat{x}_n'\|_p^p + \int_0^b \varphi(t, \overline{x}_n + \hat{x}_n(t))\,\mathrm{d}t - \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t + \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t \\
&= \frac{1}{p}\|\hat{x}_n'\|_p^p + \int_0^b (u_n(t), \hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t, \\
&\qquad\qquad\qquad\qquad u_n(t) \in \partial\varphi(t, \overline{x}_n + \lambda_n(t)\hat{x}_n(t)) \text{ a.e. on } T.
\end{aligned}$$

From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$, we have $|V(x_n)| \leqslant M_1$ for all $n \geqslant 1$. First suppose that hypothesis $\mathrm{H}(\varphi)_1(\mathrm{iv})_1$ holds. We have

$$-M_1 \leqslant \frac{1}{p}\|\hat{x}_n'\|_p^p + \int_0^b (u_n(t), \hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t.$$

Using (4) with $\varepsilon = 1$, we obtain

$$-M_1 \leqslant \beta_{11}\|\hat{x}_n'\|_p^p + \beta_1\|\hat{x}_n'\|_p + \beta_2\|\hat{x}_n'\|_p^{\theta+1} + \beta_4\|\overline{x}_n\|^{\theta q} + \int_0^b \varphi(t, \overline{x}_n)\,\mathrm{d}t,$$

with $\beta_{11} = \beta_1 + 1/p$, $\beta_4 = \beta_4(\varepsilon)$. By virtue of (6) and since $p - 1 = p/q$, we have

$$-M_1 \leqslant \beta_{12}\|\overline{x}_n\|^{\theta q} + \beta_{13}\|\overline{x}_n\|^{\theta q/p} + \beta_{14}\|\overline{x}_n\|^{\theta(\theta+1)/(p-1)} + \int_0^b \varphi(t, \overline{x}_n)\,\mathrm{d}t + \beta_{15}$$

for some $\beta_{12}, \beta_{13}, \beta_{14}, \beta_{15} > 0$ and all $n \geqslant 1$.

Note that $\theta q/p, \theta(\theta+1)/(p-1) < \theta q$ (recall that $\theta < p - 1$). Suppose that $\{\overline{x}_n\}_{n\geqslant 1} \subseteq \mathbb{R}^N$ was unbounded. So by passing to a subsequence if necessary, we may assume that $\|\overline{x}_n\| \to \infty$ as $n \to \infty$ and that $\|\overline{x}_n\| \geqslant 1$ for all $n \geqslant 1$. Then we have

$$-M_1 \leqslant \|\overline{x}_n\|^{\theta q}\left(\beta_{16} + \frac{1}{\|\overline{x}_n\|^{\theta q}}\int_0^b \varphi(t, \overline{x}_n)\,\mathrm{d}t + \frac{\beta_{15}}{\|\overline{x}_n\|^{\theta q}}\right)$$

for some $\beta_{16} > 0$ and all $n \geqslant 1$.

Since we have assumed that $\mathrm{H}(\varphi)_1$(iv) holds, from the last inequality we reach a contradiction. This proves the boundedness of $\{\overline{x}_n\}_{n\geqslant 1}$ and then by virtue of (6) we have also the boundedness of $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$.

Next assume that the hypothesis $\mathrm{H}(\varphi)_1(\mathrm{iv})_2$ is in effect. As before, we have

$$\begin{aligned}
&V(x_n) \leqslant M_1 \quad \text{for all } n \geqslant 1\\
&\quad\Rightarrow \frac{1}{p}\|\hat{x}_n'\|_p^p - \beta_1\|\hat{x}_n'\|_p - \beta_2\|\hat{x}_n'\|_p^{\theta+1} - \beta_3\frac{\varepsilon}{p}\|\hat{x}_n'\|_p^p - \beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}\\
&\qquad + \int_0^b \varphi(t, \overline{x}_n)\,\mathrm{d}t \leqslant M_1 \quad \text{(see (4))}\\
&\quad\Rightarrow \left(\frac{1}{p} - \beta_3\frac{\varepsilon}{p}\right)\|\hat{x}_n'\|_p^p - \beta_1\|\hat{x}_n'\|_p - \beta_2\|\hat{x}_n'\|_p^{\theta+1} - \beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}\\
&\qquad + \int_0^b \varphi(t, \overline{x}_n)\,\mathrm{d}t \leqslant M_1.
\end{aligned}$$

Choose $\varepsilon > 0$ so that $\varepsilon < 1/\beta_3$. From the last inequality we have

$$\beta_{17}\|\hat{x}_n'\|_p^p - \beta_1\|\hat{x}_n'\|_p - \beta_2\|\hat{x}_n'\|_p^{\theta+1} + \|\overline{x}_n\|^{\theta q}\left(\frac{1}{\|\overline{x}_n\|^{\theta q}}\int_0^b \varphi(t, \overline{x}_n) - \beta_4(\varepsilon)\right) \leqslant M_1$$

for some $\beta_{17} > 0$ and all $n \geqslant 1$.

If $\{\hat{x}_n'\}_{n\geqslant 1} \subseteq L^p(T, \mathbb{R}^N)$ is unbounded, then we may assume that $\|\hat{x}_n'\| \to \infty$ and so from (6) we also have $\|\overline{x}_n\| \to \infty$. Since $\theta < p - 1$ and hypothesis $\mathrm{H}(\varphi)_1(\mathrm{iv})_2$

is in effect, from the last inequality we have a contradiction. Therefore $\{\hat{x}'_n\}_{n\geqslant 1} \subseteq L^p(T,\mathbb{R}^N)$ is bounded. Suppose that $\|\overline{x}_n\| \to \infty$. Then again from the last inequality and since hypothesis $\mathrm{H}(\varphi)_1(\mathrm{iv})_2$ is in effect, we have a contradiction. Therefore we conclude that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded.

Thus we have proved that in both situations ($\mathrm{H}(\varphi)_1(\mathrm{iv})_1$ and $\mathrm{H}(\varphi)_1(\mathrm{iv})_2$), the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ and $x_n \to x$ in $C(T,\mathbb{R}^N)$ (recall that $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is embedded compactly in $C(T,\mathbb{R}^N)$). We have

$$\langle A(x_n), x_n - x\rangle + (u_n, x_n - x)_{pq} \leqslant \varepsilon_n \|x_n - x\| \leqslant \varepsilon_n \beta_{18}$$

for some $\beta_{18} > 0$ and with $\varepsilon_n \downarrow 0$.

Since $u_n \in \partial\Phi(x_n) \subseteq \partial\hat{\Phi}(x_n)$, $n \geqslant 1$, and $\{x_n\}_{n\geqslant 1} \subseteq L^p(T,\mathbb{R}^N)$ is relatively compact, it follows that $\{u_n\}_{n\geqslant 1} \subseteq L^q(T,\mathbb{R}^N)$ is bounded and so $(u_n, x_n - x)_{pq} \to 0$ as $n \to \infty$.

Thus we have

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leqslant 0.$$

But because $A$ is maximal monotone, it is generalized pseudomonotone (see [4, p. 365]) and so we have

$$\begin{aligned} &\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \\ &\Rightarrow \|x'_n\|_p \to \|x'\|_p. \end{aligned}$$

Recall that $x'_n \xrightarrow{w} x'$ in $L^p(T,\mathbb{R}^N)$ and the space $L^p(T,\mathbb{R}^N)$, being uniformly convex, has the Kadec-Klee property (see [4, p. 28]). So $x'_n \to x'$ in $L^p(T,\mathbb{R}^N)$ and it follows that $x_n \to x$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$. Therefore we have proved that $V$ satisfied the nonsmooth PS-condition. □

Now we can have the first existence theorem for problem (2).

**Theorem 4.** *If hypotheses* $\mathrm{H}(\varphi)_{1,1}$ *hold, then problem* (2) *has a solution* $x \in C^1(T,\mathbb{R}^N)$.

P r o o f. Recall that $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N) = \mathbb{R}^N \oplus Y$ with $Y = \{y \in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\colon \int_0^b y(t)\,\mathrm{d}t = 0\}$.

Let $y \in Y$. As in the proof of Proposition 3, we can find $u \in L^q(T,\mathbb{R}^N)$ and $\lambda\colon T \to (0,1)$ Lebesgue measurable such that $u(t) \in \partial\varphi(t,\lambda(t)y(t))$ and $\varphi(t,y(t)) -$

$\varphi(t,0) = (u(t), y(t))_{\mathbb{R}^N}$ a.e. on $T$. We have

$$\begin{aligned}
V(y) &= \frac{1}{p}\|y'\|_p^p + \int_0^b \varphi(t, y(t))\,\mathrm{d}t \\
&= \frac{1}{p}\|y'\|_p^p + \int_0^b (\varphi(t, y(t)) - \varphi(t,0))\,\mathrm{d}t + \int_0^b \varphi(t,0)\,\mathrm{d}t \\
&= \frac{1}{p}\|y'\|_p^p + \int_0^b (u(t), y(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_0^b \varphi(t,0)\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|y'\|_p^p - \|u\|_1\|y\|_\infty + \int_0^b \varphi(t,0)\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|y'\|_p^p - \beta_{19}\|y'\|_p^{\theta+1} - \beta_{20}
\end{aligned}$$

for some $\beta_{19}, \beta_{20} > 0$.

In the last inequality we have used hypothesis $\mathrm{H}(\varphi)_1$(iii) and the Poincare-Wirtinger inequality. Since $\theta < p-1$, from the last inequality and since $\|y'\|_p$ is an equivalent norm on $Y$ (by the Poincare-Wirtinger inequality), we can conclude that $V$ is coercive on $Y$.

Let $c \in \mathbb{R}^N$. Then from hypothesis $\mathrm{H}(\varphi)_1\mathrm{(iv)}_1$ we have that $V(c) = \int_0^b \varphi(t,c)\,\mathrm{d}t \to -\infty$ as $\|c\| \to \infty$. So $V$ is anti-coercive on $\mathbb{R}^N$. These properties of $V$ together with Proposition 3 permit the use of Theorem 1, which gives us $x \in W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$ such that $0 \in \partial V(x)$. We have $A(u) = -u$ with $u \in L^q(T, \mathbb{R}^N)$ and $u(t) \in \partial\varphi(t, x(t))$ a.e. on $T$ (i.e. $u \in \partial\Phi(x) \subseteq \partial\hat{\Phi}(x) \subseteq L^q(T, \mathbb{R}^N)$). For every $h \in C_0^\infty(T, \mathbb{R}^N)$ we have

$$\begin{gathered}
\langle A(x), h\rangle = (-u, h)_{pq} \\
\Rightarrow \int_0^b \|x'(t)\|^{p-2}(x'(t), h'(t))_{\mathbb{R}^N}\,\mathrm{d}t = \int_0^b (-u(t), h(t))_{\mathbb{R}^N}\,\mathrm{d}t.
\end{gathered}$$

From the definition of the distributional derivative we have

$$(\|x'(t)\|^{p-2}x'(t))' = u(t) \in \partial\varphi(t, x(t)) \quad \text{a.e. on } T, \quad \text{with } x(0) = x(b).$$

From this it follows that $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N) \subseteq C(T, \mathbb{R}^N)$. Since the map $\mu\colon \mathbb{R}^N \to \mathbb{R}^N$ defined by $\mu(x) = \|x\|^{p-2}x$ is a homeomorphism, we have that $x' \in C(T, \mathbb{R}^N)$. Then for every $v \in W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$ from Green's identity (integration by parts), we have that

$$
\begin{aligned}
&\int_0^b ((\|x'(t)\|^{p-2}x'(t))', v(t))_{\mathbb{R}^N}\,\mathrm{d}t \\
&\quad = \|x'(b)\|^{p-2}(x'(b), v(b))_{\mathbb{R}^N} - \|x'(0)\|^{p-2}(x'(0), v(0))_{\mathbb{R}^N} \\
&\qquad - \int_0^b \|x'(t)\|^{p-2}(x'(t), v'(t))_{\mathbb{R}^N}\,\mathrm{d}t \\
\Rightarrow &\int_0^b (u(t), v(t))_{\mathbb{R}^N}\,\mathrm{d}t \\
&\quad = \|x'(b)\|^{p-2}(x'(b), v(b))_{\mathbb{R}^N} - \|x'(0)\|^{p-2}(x'(0), v(0))_{\mathbb{R}^N} - \langle A(x), v\rangle \\
&\quad \Rightarrow \|x'(0)\|^{p-2}(x'(0), v(0))_{\mathbb{R}^N} = \|x'(b)\|^{p-2}(x'(b), v(b))_{\mathbb{R}^N}
\end{aligned}
$$

for all $v \in W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$.

Let $z = (z_m)_{m=1}^N \in \mathbb{R}^N$ be such that $z_i = 1$ for some $i \in \{1, \ldots, N\}$ and $z_m = 0$ for $m \neq i$. Let $v \in W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$ be such that $v(0) = v(b) = z$. We obtain $\|x'(0)\|^{p-2}x_i'(0) = \|x'(b)\|^{p-2}x_i'(b)$. Since $i \in \{1, \ldots, N\}$ was arbitrary we have that $\|x'(0)\|^{p-2}x'(0) = \|x'(b)\|^{p-2}x'(b)$ and so finally $x'(0) = x'(b)$. Therefore we have that $x \in C^1(T, \mathbb{R}^N)$ is a solution of (2). $\square$

The second existence result is the following:

**Theorem 5.** *If hypotheses* $\mathrm{H}(\varphi)_{1,2}$ *hold, then problem* (2) *has a solution* $x \in C^1(T, \mathbb{R}^N)$.

Proof. In this case for every $x \in W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$ we have (see the proof of Proposition 3)

$$
\begin{aligned}
V(x) &= \frac{1}{p}\|x'\|_p^p + \int_0^b \varphi(t, x(t))\,\mathrm{d}t \\
&= \frac{1}{p}\|\hat{x}'\|_p^p + \int_0^b (\varphi(t, \overline{x} + \hat{x}(t)) - \varphi(t, \overline{x}))\,\mathrm{d}t + \int_0^b \varphi(t, \overline{x})\,\mathrm{d}t \\
&= \frac{1}{p}\|\hat{x}'\|_p^p + \int_0^b (u(t), \hat{x}(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_0^b \varphi(t, \overline{x})\,\mathrm{d}t \\
&\qquad\qquad (u(t) \in \partial\varphi(t, \overline{x} + \lambda(t)\hat{x}(t)) \quad \text{a.e. on } T) \\
&\geqslant \frac{1}{p}\|\hat{x}'\|_p^p - \beta_1\|\hat{x}'\|_p - \beta_2\|\hat{x}'\|_p^{\theta+1} - \beta_3\frac{\varepsilon}{p}\|\hat{x}'\|_p^p - \beta_4(\varepsilon)\|\overline{x}\|^{\theta q} \\
&\quad + \int_0^b \varphi(t, \overline{x})\,\mathrm{d}t \quad \text{(see (4))} \\
&= \frac{1}{p}(1 - \beta_3\varepsilon)\|\hat{x}'\|_p^p - \beta_1\|\hat{x}'\|_p - \beta_2\|\hat{x}'\|_p^{\theta+1} \\
&\quad + \|\overline{x}_n\|^{\theta q}\Big(\frac{1}{\|\overline{x}_n\|^{\theta q}} \int_0^b \varphi(t, \overline{x})\,\mathrm{d}t - \beta_4(\varepsilon)\Big).
\end{aligned}
$$

Choose $\varepsilon < 1/\beta_3$ and recall that $\theta < p-1$. It follows that $V$ is coercive (hence bounded below). So we can apply Theorem 2 and obtain $x \in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ such that $0 \in \partial V(x)$. As in the proof of Theorem 4 we can show that $x \in C^1(T,\mathbb{R}^N)$ and that it solves (2). $\square$

Next we present a third existence theorem, in which we assume a stronger growth condition on $\partial\varphi(t,x)$ which implies that $\varphi(t,\cdot)$ is globally Lipschitz (hence it has a sublinear growth), but we weaken the hypothesis $\mathrm{H}(\varphi)_1$(iv).

$\mathrm{H}(\varphi)_2$ $\varphi\colon T\times\mathbb{R}^N \to \mathbb{R}$ is a function such that $\varphi(\cdot,0)\in L^1(T)$ and

(i) for every $x\in\mathbb{R}^N$, $t\to\varphi(t,x)$ is measurable;

(ii) for almost all $t\in T$, $x\to\varphi(t,x)$ is locally Lipschitz;

(iii) there exists $\alpha\in L^q(T)_+$ such that for almost all $t\in T$, all $x\in\mathbb{R}^N$ and all $u\in\partial\varphi(t,x)$ we have

$$\|u\| \leqslant \alpha(t);$$

(iv) $\lim\limits_{||x||\to\infty} \int_0^b \varphi(t,x)\,\mathrm{d}t = -\infty$.

**Remark.** By virtue of the Lebourg mean value theorem and hypothesis $\mathrm{H}(\varphi)_2$(iii) we have that for almost all $t\in T$ and all $x,y\in\mathbb{R}^N$, $|\varphi(t,x)-\varphi(t,y)|\leqslant \alpha(t)\|x-y\|$, i.e. for almost all $t\in T$, $\varphi(t,\cdot)$ is actually globally Lipschitz.

We consider the Lipschitz continuous energy functional $V\colon W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\to\mathbb{R}$ defined by

$$V(x) = \frac{1}{p}\|x'\|_p^p + \int_0^b \varphi(t,x(t))\,\mathrm{d}t.$$

**Proposition 6.** *If hypotheses* $\mathrm{H}(\varphi)_2$ *hold, then* $V$ *satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n\geqslant1}\subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ be a sequence such that

$$|V(x_n)|\leqslant M_1 \quad \text{for all } n\geqslant 1 \quad \text{and} \quad m(x_n)\to 0 \quad \text{as } n\to\infty.$$

Let $x_n^*\in\partial V(x_n)$ be such that $\|x_n^*\|_* = m(x_n)$, $n\geqslant1$, and let $A\colon W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\to W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)^*$ be as in the proof of Proposition 3. We have $x_n^* = A(x_n)+u_n$ with $\{u_n\}_{n\geqslant1}\subseteq L^q(T,\mathbb{R}^N)$ and $u_n(t)\in\partial\varphi(t,x_n(t))$ a.e. on $T$.

We consider the decomposition $x_n = \overline{x}_n + \hat{x}_n$ with $\overline{x}_n\in\mathbb{R}^N$, $\hat{x}_n\in Y$ for all $n\geqslant1$. We have

$$\langle A(x_n),\hat{x}_n\rangle + (u_n,\hat{x}_n)_{p,q} \leqslant \varepsilon_n\|\hat{x}_n\| \quad \text{with } \varepsilon_n\downarrow 0$$

$$\Rightarrow \|\hat{x}_n'\|_p^p + \int_0^b (u_n(t),\hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t \leqslant \varepsilon_n\|\hat{x}_n\| \Rightarrow \|\hat{x}_n'\|_p^p \leqslant \varepsilon_n\|\hat{x}_n\| + \|\alpha\|_1\|\hat{x}_n\|_\infty.$$

Using the Poicare-Wirtinger inequality, we obtain

$$\begin{gathered}\|\hat{x}_n'\|_p^p \leqslant \beta_1 \|\hat{x}_n'\|_p \quad \text{for some } \beta_1 > 0 \text{ and all } n \geqslant 1\\ \Rightarrow \{\hat{x}_n\}_{n\leqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N) \text{ is bounded.}\end{gathered}$$

As in the proof of Proposition 3 we can find $u_n^* \in L^q(T,\mathbb{R}^N)$ and $\lambda_n\colon T \to (0,1)$ Lebesgue measurable such that $u_n^*(t) \in \partial\varphi(t,\overline{x}_n + \lambda_n(t)\hat{x}_n(t))$ a.e. on T and $\varphi(t,\overline{x}_n + \hat{x}_n(t)) - \varphi(t,\overline{x}_n) = (u_n^*(t),\hat{x}_n(t))_{\mathbb{R}^N}$ a.e. on $T$ for all $n \geqslant 1$. Then from the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ we have

$$\begin{aligned}-M_1 \leqslant V(x_n) &= \frac{1}{p}\|x_n'\|_p^p + \int_0^b \varphi(t,x_n(t))\,\mathrm{d}t\\ &= \frac{1}{p}\|\hat{x}_n'\|_p^p + \int_0^b (\varphi(t,x_n(t)) - \varphi(t,\overline{x}_n))\,\mathrm{d}t + \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t\\ &= \frac{1}{p}\|\hat{x}_n'\|_p^p + \int_0^b (u_n^*(t),\hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t\\ &\leqslant \frac{1}{p}\|\hat{x}_n'\|_p^p + \|\alpha\|_1\|\hat{x}_n\|_\infty + \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t.\end{aligned}$$

Since $\{\hat{x}_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded and using the Poincare-Wirtinger inequality, we obtain

$$-M_2 \leqslant \int_0^b \varphi(t,\overline{x}_n)\,\mathrm{d}t \quad \text{for some } M_2 > 0 \text{ and all } n \geqslant 1. \tag{7}$$

Suppose that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is unbounded. Then because $\{\hat{x}_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded we conclude that $\{\overline{x}_n\} \subseteq \mathbb{R}^N$ is unbounded. So we may assume that $\|\overline{x}_n\| \to \infty$. From this, (7) and hypothesis $\mathrm{H}(\varphi)_2$(iv), we reach a contradiction. This shows that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded and so we may say that $x_n \xrightarrow{w} x$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$. The argument in the proof of Theorem 4 applies here too and gives that $x_n \to x$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$. Therefore we conclude that $V$ satisfies the nonsmooth PS-condition. $\square$

Next we show that V is coercive on $Y = \{y \in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\colon \int_0^b y(t)\,\mathrm{d}t = 0\}$.

**Proposition 7.** *If hypotheses* $\mathrm{H}(\varphi)_2$ *hold, then* $V(y) \to +\infty$ *as* $\|y\| \to \infty$, $y \in Y$.

Proof. We proceed by contradiction. Suppose that the proposition was not true. Then we can find $\{y_n\}_{n\geqslant 1} \subseteq Y$ such that $\|y_n\| \to \infty$ and $V(y_n) \leqslant M_3$ for some $M_3 > 0$ and all $n \geqslant 1$. Set $z_n = y_n/\|y_n\|$, $n \geqslant 1$. By passing to a subsequence if necessary, we may assume that $z_n \xrightarrow{w} z$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ and $z_n \to z$ in $C(T,\mathbb{R}^N)$. We have

$$
\begin{aligned}
(8)\qquad & \frac{1}{p}\|y_n'\|_p^p + \int_0^b \varphi(t,y_n(t))\,\mathrm{d}t \leqslant M_3 \quad \text{for all } n \geqslant 1,\\
\Rightarrow\ & \frac{1}{p}\|z_n'\|_p^p + \int_0^b \frac{\varphi(t,y_n(t))}{\|y_n\|^p}\,\mathrm{d}t \leqslant \frac{M_3}{\|y_n\|^p} \quad \text{for all } n \geqslant 1.
\end{aligned}
$$

From hypothesis $\mathrm{H}(\varphi)_2$(iii) and Lebourg's mean value theorem, for almost all $t \in T$ and all $x \in \mathbb{R}^N$ we have

$$|\varphi(t,x)| \leqslant \alpha_1(t) + c_1(t)\|x\| \quad \text{with } \alpha_1 \in L^1(T),\ c_1 \in L^q(T).$$

So we have

$$\frac{|\varphi(t,y_n(t))|}{\|y_n\|^p} \leqslant \frac{\alpha_1(t)}{\|y_n\|^p} + \frac{c_1(t)}{\|y_n\|^{p-1}}\|z_n(t)\| \to 0 \text{ a.e. on } T.$$

Using this in (8), in the limit as $n \to \infty$ we obtain

$$\|z'\|_p = 0, \quad \text{i.e.} \quad z = \xi \in \mathbb{R}^N \quad \text{and} \quad \xi = 0 \quad \text{since } z \in V.$$

Therefore we have that $z_n' \to 0$ in $L^p(T,\mathbb{R}^N)$, hence $z_n \to 0$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$, a contradiction because $\|z_n\| = 1$ for all $n \geqslant 1$. $\square$

**Proposition 8.** *If hypotheses* $\mathrm{H}(\varphi)_2$ *hold, then problem* (2) *has a solution* $x \in C^1(T,\mathbb{R}^N)$.

Proof. By virtue of hypothesis $\mathrm{H}(\varphi)$(iv), we have that $V(c) \to -\infty$ as $\|c\| \to \infty$, $c \in \mathbb{R}^N$. This combined with Propositions 6 and 7 permits the use of Theorem 1 which gives us $x \in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ such that $0 \in \partial V(x)$. Working as in the proof of Theorem 4, we show that $x \in C^1(T,\mathbb{R}^N)$ and that it solves problem (2). $\square$

**Remark.** A simple example of a nonsmooth potential function satisfying hypotheses $\mathrm{H}(\varphi)_2$ is $\varphi(t,x) = \sin\|x\| - h(t)\|x\|$ with $h \in L^1(T)_+$. Then

$$\partial\varphi(t,x) = \begin{cases} (\cos\|x\| - h(t))x/\|x\| & \text{if } x \neq 0,\\ (\cos\|x\| - h(t))\overline{B_1} & \text{if } x = 0, \end{cases}$$

with $\overline{B_1} = \{x \in \mathbb{R}^N : \|x\| \leqslant 1\}$. Also note that $\int_0^b \varphi(t,c)\,\mathrm{d}t \leqslant b - \|c\|\,\|h\|_1 \to -\infty$ as $\|c\| \to \infty$. Thus we satisfy hypotheses $\mathrm{H}(\varphi)_2$.

An example of a function $\varphi(t,x)$ satisfying hypotheses $\mathrm{H}(\varphi)_1$ with $p=3$ is the function $\varphi(t,x)=\frac{1}{2}h(t)(\|x\|^2-d_C^2(x))+\|x\|\ln(\|x\|+2)$, with $d_C$ being the distance function from $C\subseteq\mathbb{R}^N$, a nonempty compact and convex set, and $h\in L^2(T)$. Then

$$\partial\varphi(t,x)=\begin{cases} h(t)p_C(x)+\dfrac{x}{\|x\|}\ln(\|x\|+2)+\dfrac{x}{\|x\|+2} & \text{if } x\neq 0,\\ h(t)p_C(0)+\ln 2\overline{B_1} & \text{if } x=0.\end{cases}$$

Here $p_C$ is the metric projection on $C$. Clearly $\varphi(t,x)$ satisfies $\mathrm{H}(\varphi)_{1,1}$ if $\int_0^b h(t)<0$ and $\mathrm{H}(\varphi)_{1,2}$ if $\int_0^b h(t)>0$.

## 4. Multiplicity results

First we prove a multiplicity result for problems with a smooth potential $\varphi(t,x)$. So the hypotheses on $\varphi(t,x)$ are the following:

$\mathrm{H}(\varphi)_3$ $\varphi\colon T\times\mathbb{R}^N\to\mathbb{R}$ is a function such that

(i) for all $x\in\mathbb{R}^N$, $t\to\varphi(t,x)$ is measurable;

(ii) for almost all $t\in T$, $x\to\varphi(t,x)$ is a $C^1$-function;

(iii) for almost all $t\in T$, and all $x\in\mathbb{R}^N$ we have

$$\|\nabla\varphi(t,x)\|\leqslant\alpha(t)+c(t)\|x\|^\theta$$

with $\alpha,c\in L^q(T)_+$, $0\leqslant\theta<p-1$;

(iv) $(1/\|x\|^{\theta q})\int_0^b\varphi(t,x)\,\mathrm{d}t\to+\infty$ as $\|x\|\to\infty$, $x\in\mathbb{R}^N$;

(v) $\lim\limits_{x\to 0}\varphi(t,x)/\|x\|^p=0$ uniformly for almost all $t\in T$;

(vi) there exists $\hat\delta>0$ such that for almost all $t\in T$ and all $x\in\mathbb{R}^N$ with $\|x\|\leqslant\hat\delta$ we have $\varphi(t,x)\leqslant 0$, and there exists $c\in\mathbb{R}^N$ such that for almost all $t\in T$, $\varphi(t,c)<0$.

**Proposition 9.** *If hypotheses* $\mathrm{H}(\varphi)_3$ *hold, then problem* (2) *has two distinct nonzero solutions in* $C^1(T,\mathbb{R}^N)$.

Proof. Let $V\colon W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)\to\mathbb{R}$ be the smooth (i.e. $C^1$) energy functional defined by

$$V(x)=\frac{1}{p}\|x'\|_p^p+\int_0^b\varphi(t,x(t))\,\mathrm{d}t.$$

We already know from Proposition 3 that $V$ satisfies the PS-condition, while from the proof of Theorem 5 we also know that $V$ is coercive, hence it is bounded below. Moreover, by virtue of hypothesis $\mathrm{H}(\varphi)_3$(vi) we have that $\inf V<0$. Also hypothesis

$\mathrm{H}(\varphi)_3$(v) implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $\|x\| \leqslant \delta$ we have

$$\varphi(t,x) \geqslant -\frac{\varepsilon}{p}\|x\|^p.$$

Let $y \in Y$ with $\|y'\|_p \leqslant \delta/b^{1/q}$. Recall that $\|y\|_\infty \leqslant b^{1/q}\|y'\|_p \leqslant \delta$. So for $y \in Y$ with $\|y\| = (\|y\|_p^p + \|y'\|_p^p)^{1/p} \leqslant \delta/b^{1/q} = \delta_1$, we have $\|y\|_\infty \leqslant \delta$ and so we can write that

$$\begin{aligned} V(y) &= \frac{1}{p}\|y'\|_p^p + \int_0^b \varphi(t,y(t))\,\mathrm{d}t \\ &\geqslant \frac{1}{p}\|y'\|_p^p - \frac{\varepsilon}{p}\|y\|_p^p \\ &\geqslant \frac{1}{p}\Big(1-\frac{\varepsilon}{\beta_1}\Big)\|y'\|_p^p \quad \text{for some } \beta_1 > 0 \end{aligned}$$

by the Poincare-Wirtinger inequality.

So if we choose $0 < \varepsilon < \beta_1$, we have that $V(y) > 0$ for all $y \in Y$ with $\|y\| \leqslant \delta_1$.

In addition by virtue of hypothesis $\mathrm{H}(\varphi)_3$(vi), we can find $\delta_2 > 0$ such that if $x \in \mathbb{R}^N \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ and $\|x\| \leqslant \delta_2$, then we have $V(x) \leqslant 0$. Choose $\delta_3 = \min\{\delta_1,\delta_2\}$. We have

$$V(y) \geqslant 0 \quad \text{for all } y \in Y \text{ with } \|y\| \leqslant \delta_3$$

and

$$V(y) \leqslant 0 \quad \text{for all } y \in \mathbb{R}^N \quad \text{with } \|y\| \leqslant \delta_3.$$

Hence we can apply Theorem 4 of [1] and obtain two distinct nonzero critical points of $V$. We check that these are the two distinct nonzero solutions in $C^1(T,\mathbb{R}^N)$ of problem (2). □

**Remark.** Tang [12] proves multiplicity results for the semilinear (i.e. $p = 2$), smooth problem (see Theorems 3 and 4 in [12]). In Theorem 3 he proves the existence of two distinct solutions, one via the "Saddle Point Theorem" and the second via the "Linking Theorem" (see [11], Theorem 5.29). However, there is no guarantee that the critical points are distinct. In Theorem 4 he proves the existence of three distinct solutions. Two are obtained via the Brezis-Nirenberg Theorem and the third through a minimizer of the energy functional. Again three is no guarantee that the third solution is distinct from the other two. In fact we can also obtain a third solution via the Saddle Point Theorem but there is no guarantee that it is distinct from the other two.

The next multiplicity result concerns problems with nonsmooth potential (hemivariational inequalities). The hypotheses on the potential function are the following:

$\mathrm{H}(\varphi)_4$ $\varphi\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $\varphi(\cdot, 0) \in L^1(T)$ and

(i) for every $x \in \mathbb{R}^N$, $t \to \varphi(t, x)$ is measurable;

(ii) for almost all $t \in T$, $x \to \varphi(t, x)$ is locally Lipschitz;

(iii) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial\varphi(t, x)$ we have

$$\|u\| \leqslant \alpha(t) + c(t)\|x\|^{\theta}$$

with $\alpha, c \in L^q(T)_+$ and $0 \leqslant \theta < p - 1$;

(iv) there exists $0 < \gamma < 2^p 3^{p/2}/(b^p p)$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$, $-\gamma\|x\|^p \leqslant \varphi(t, x)$;

(v) $(1/\|x\|^{\theta q}) \int_0^b \varphi(t, x)\, \mathrm{d}t \to +\infty$ as $\|x\| \to +\infty$, $x \in \mathbb{R}^N$;

(vi) $\limsup\limits_{\|x\| \to 0} \int_0^b \varphi(t, x)\, \mathrm{d}t < 0$.

We have the following multiplicity result for problem (2).

**Theorem 10.** *If hypotheses* $\mathrm{H}(\varphi)_4$ *hold, then problem* (2) *has two distinct solutions in* $C^1(T, \mathbb{R}^N)$.

Proof. Let $V\colon W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N) \to \mathbb{R}$ be the locally Lipschitz energy functional defined by

$$V(x) = \frac{1}{p}\|x'\|_p^p + \int_0^b \varphi(t, x(t))\, \mathrm{d}t.$$

From Proposition 3 we know that $V$ satisfies the nonsmooth PS-condition, while from the proof of Theorem 5 we know that $V$ is bounded below. So by Theorem 2 we can find $x_1 \in W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$ such that $V(x_1) = \inf V$. Then $0 \in \partial V(x_1)$ and from this as before we obtain that $x_1 \in C^1(T, \mathbb{R}^N)$ and that it is a solution of (2). Moreover, by virtue of hypothesis $\mathrm{H}(\varphi)_4$(vi) we have that $V(x_1) < 0$.

Next, again from hypothesis $\mathrm{H}(\varphi)_4$(vi), we can find $\delta > 0$ such that for all $x \in \mathbb{R}^N$ with $\|x\| = \delta$ we have $\int_0^b \varphi(t, x)\, \mathrm{d}t < 0$, hence $V(x) < 0$. On the other hand for every $y \in Y$, by virtue of hypothesis $\mathrm{H}(\varphi)_4$(vi) we have

$$V(y) = \frac{1}{p}\|y'\|_p^p + \int_0^b \varphi(t, y(t))\, \mathrm{d}t \geqslant \frac{1}{p}\|y'\|_p^p - \gamma\|y\|_p^p.$$

Note that $\|y\|_p^p \leqslant b\|y\|_\infty^p$. Moreover, from the Sobolev inequality (see [9, p. 9]) we have

$$\|y\|_\infty \leqslant \frac{b^{1/2}}{2\sqrt{3}}\|y'\|_2 \leqslant \frac{b^{1/2}}{2\sqrt{3}} b^{p-2/2p}\|y'\|_p \Rightarrow \|y\|_\infty^p \leqslant \frac{b^{p-1}}{2^p 3^{p/2}}\|y'\|_p^p.$$

So we have

$$V(y) \geqslant \frac{1}{p}\|y'\|_p^p - \gamma \frac{b^{p-1}}{2^p 3^{p/2}}\|y'\|_p^p \geqslant 0 \quad \text{(see hypothesis } \mathrm{H}(\varphi)_4\text{(iv))}.$$

Therefore we can apply Theorem 1 (the nonsmooth Saddle Point Theorem) and obtain $x_2 \in W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$ such that $0 \in \partial V(x_2)$ and $V(x_2) \geqslant 0$. From the first relation we obtain that $x_2 \in C^1(T, \mathbb{R}^N)$ and that it is a solution of (2), while from the inequality $V(x_2) \geqslant 0$ it follows that $x_2 \neq x_1$. $\square$

## 5. The nonsmooth semilinear problem

In this section we prove an existence theorem for the nonsmooth semilinear (i.e. $p = 2$) problem under a nonsmooth, multidimensional version of the well-known Ambrosetti-Rabinowitz condition (see [11, p. 9]). So in this section the problem under consideration is the following:

$$\text{(9)} \qquad \begin{cases} x''(t) \in \partial\varphi(t, x(t)) & \text{a.e. on } T, \\ x(0) = x(b), \quad x'(0) = x'(b). \end{cases}$$

Our proof is based on the following generalization of the nonsmooth "Mountain Pass Theorem" (see [2] and [6]). The smooth version of this abstract result is due to [11, Theorem 5.3, p. 28].

**Theorem 11.** *If $X$ is a reflexive Banach space, $X = Z \oplus Y$ with $\dim Z < +\infty$, $V\colon X \to \mathbb{R}$ is locally Lipschitz, satisfies the nonsmooth* PS*-condition and the following conditions hold*

(i) *there exist $r, \alpha > 0$ such that for all $y \in Y$ with $\|y\| = r$ we have $V(y) \geqslant \alpha$,*

(ii) *there exist $e \in \partial B_1 \cap Y$ ($B_1 = \{x \in X\colon \|x\| < 1\}$) and $R > r$ such that if $Q = \{z \in Z\colon \|z\| \leqslant R\} \oplus \{\lambda e\colon 0 < \lambda < R\}$, then $V|_{\partial Q} \leqslant 0$ with $\partial Q$ being the boundary of $Q$ in $Z \oplus \mathbb{R}e$,*

*then $c = \inf\limits_{\gamma \in \Gamma} \max\limits_{u \in Q} V(\gamma(u))$ where $\Gamma = \{\gamma \in C(\overline{Q}, X)\colon \gamma|_{\partial Q} = \mathrm{identity}\}$ is a critical value of $V$ with critical point $x \in X$ and $c \geqslant \alpha$. Moreover, if $c = \alpha$, then $x \in V$.*

P r o o f. Using [11, Proposition 5.9] (which is still valid in the present nonsmooth setting), for every $\gamma \in \Gamma$ we have

$$\gamma(Q) \cap \partial B_r \cap Y \neq \emptyset \quad (B_r = \{x \in X\colon \|x\| < r\}) \Rightarrow c \geqslant \alpha.$$

Next we show that $c$ is a critical value of $V$. Suppose that this was not true. Using the nonsmooth deformation theorem [2, Theorem 3.1], we can find $\varepsilon \in (0, \alpha/2)$ and $\xi \colon X \to X$ a homeomorphism such that

$$\xi(x) = x \quad \text{for all} \quad x \notin \left\{u \in X \colon |V(u) - c| < \frac{\alpha}{2}\right\} \quad \text{and} \tag{10}$$

$$V(\xi(x)) \leqslant c - \varepsilon \quad \text{for all } x \in X \text{ with } V(x) \leqslant c + \varepsilon. \tag{11}$$

From the minimax definition of $c$, we can find $\gamma \in \Gamma$ such that

$$\max\{V(\gamma(u)) \colon u \in \overline{Q}\} \leqslant c + \varepsilon. \tag{12}$$

Let $g = \xi \circ \gamma \in C(\overline{Q}, X)$. From hypothesis (ii) and (10), we obtain that $g \in \Gamma$. Then from (11) and (12) we have that $c \leqslant c - \varepsilon$, a contradiction. Finally if $c = \alpha$, then clearly $x \in Y$. $\square$

Using this minimax principle, we can obtain an existence theorem for problem (9) under a generalized Ambrosetti-Rabinowitz condition of $\varphi(t, x)$. More precisely our hypotheses on $\varphi(t, x)$ are the following:

$\underline{\mathrm{H}}(\varphi)_5$ $\varphi \colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $\varphi(\cdot, 0) \in L^2(T)$ and

(i) for every $x \in \mathbb{R}^N$, $t \to \varphi(t, x)$ is measurable;

(ii) for almost all $t \in T$, $x \to \varphi(t, x)$ is locally Lipschitz;

(iii) for every $\varrho > 0$ there exists $\theta_\varrho \in L^2(T)$ such that for almost all $t \in T$, all $x \in \mathbb{R}^N$ with $\|x\| \leqslant \varrho$ and all $u \in \partial\varphi(t, x)$ we have $\|u\| \leqslant \theta_\varrho(t)$;

(iv) there exist $\eta > 2$ and $M > 0$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $\|x\| \geqslant M$ we have
$$\varphi^0(t, x; x) \leqslant \eta\varphi(t, x) < 0;$$

(v) for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $\|x\| \leqslant 1$, we have $-6/b^2 \leqslant \varphi(t, x)$ and for all $x \in \mathbb{R}^N$, $\int_0^b \varphi(t, x)\,\mathrm{d}t < 0$.

**Remark.** Hypothesis $\mathrm{H}(\varphi)_5$(iv) is the nonsmooth multidimensional version of the Ambrosetti-Rabinowitz condition.

**Theorem 12.** *If hypotheses* $\mathrm{H}(\varphi)_5$ *hold, then problem* (9) *has a nonconstant solution* $x \in C^1(T, \mathbb{R}^N)$.

Proof. First we show that for almost all $t \in T$ and all $x \in \mathbb{R}^N$ we have

$$\varphi(t, x) \leqslant \alpha_1(t) - \alpha_2(t)\|x\|^\eta \quad \text{with} \quad \alpha_1, \alpha_2 \in L^2(T)_+. \tag{13}$$

Let $N_0$ be the Lebesgue-null set outside which hypotheses $\mathrm{H}(\varphi)_5$(ii), (iii), (iv) hold and let $t \in T \setminus N_0$, $x \in \mathbb{R}^N$, $\|x\| \geqslant M$. We set $\psi(t, \lambda) = \varphi(t, \lambda x)$, $\lambda \geqslant 1$. Evidently

$\psi(t,\cdot)$ is locally Lipschitz. Moreover, from [3, Theorem 2.3.10, p. 45] (chain rule II), we have that $\partial\psi(t,\lambda) \subseteq (\partial_x\varphi(t,\lambda x), x)_{\mathbb{R}^N}$, hence

$$\begin{aligned}
&\lambda\partial\psi(t,\lambda) \subseteq (\partial_x\varphi(t,\lambda x), \lambda x)_{\mathbb{R}^N} \\
\Rightarrow\ &\lambda\psi'(t,\lambda) \leqslant \eta\psi(t,\lambda) \quad \text{for almost all } \lambda \geqslant 1 \text{ (hypothesis H}(\varphi)_5\text{(iv))} \\
\Rightarrow\ &\frac{\eta}{\lambda} \leqslant \frac{\psi'(t,\lambda)}{\psi(t,\lambda)} \quad \text{for almost all } \lambda \geqslant 1.
\end{aligned}$$

Integrating from 1 to $\lambda_0$ we obtain

$$\ln \lambda_0^\eta \leqslant \ln \frac{\psi(t,\lambda_0)}{\psi(t,1)} \Rightarrow \psi(t,\lambda_0) \leqslant \lambda_0^\eta \psi(t,1).$$

So we have proved that for $t \in T \setminus N_0$, $\|x\| \geqslant M$ and $\lambda \geqslant 1$ we have

$$\varphi(t,\lambda x) \leqslant \lambda^\eta \varphi(t,x).$$

We have $\varphi(t,x) = \varphi\big(t, \|x\|M^{-1}Mx\|x\|^{-1}\big) \leqslant \big(\|x\|/M\big)^\eta \varphi\big(t, Mx/\|x\|\big)$. Let $\xi(t) = \max\{\varphi(t,y)\colon \|y\| = M\}$. Clearly $\xi \in L^2(T)$ and we have that for all $t \in T \setminus N_0$ and all $\|x\| \geqslant M$

$$\varphi(t,x) \leqslant \Big(\frac{\|x\|}{M}\Big)^\eta \xi(t). \tag{14}$$

For $t \in T \setminus N_0$, $\varphi(t,\cdot)$ is bounded on $\overline{B}_M(0) = \{x \in \mathbb{R}^N : \|x\| \leqslant M\}$. Therefore because $\xi \leqslant 0$ from (14) we infer that (13) holds.

Let $V\colon W^{1,2}_{\text{per}}(T,\mathbb{R}^N) \to \mathbb{R}$ be the locally Lipschitz energy functional defined by

$$V(x) = \frac{1}{2}\|x'\|_2^2 + \int_0^b \varphi(t,x(t))\,\mathrm{d}t.$$

**Claim 1.** *$V$ satisfies the nonsmooth* PS-*condition.*

To this end let $\{x_n\}_{n\geqslant 1} \subseteq W^{1,2}_{\text{per}}(T,\mathbb{R}^N)$ be a sequence such that

$$|V(x_n)| \leqslant M_1 \quad \text{for all } n \geqslant 1 \quad \text{and} \quad m(x_n) \to 0 \ \text{as } n \to \infty.$$

Let $x_n^* \in \partial V(x_n)$ such that $m(x_n) = \|x_n^*\|_*$, $n \geqslant 1$. Let $A\colon W^{1,2}_{\text{per}}(T,\mathbb{R}^N) \to W^{1,2}_{\text{per}}(T,\mathbb{R}^N)^*$ be the monotone continuous (hence maximal monotone) linear operator defined by $\langle A(x), y\rangle = \int_0^b (x'(t), y'(t))_{\mathbb{R}^N}\,\mathrm{d}t$, where $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair $(W^{1,2}_{\text{per}}(T,\mathbb{R}^N), W^{1,2}_{\text{per}}(T,\mathbb{R}^N)^*)$. We have $x_n^* = A(x_n) + u_n$, with

$u_n \in L^2(T, \mathbb{R}^N)$, $u_n(t) \in \partial\varphi(t, x_n(t))$ a.e. on $T$, $n \geqslant 1$. From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,2}_{\rm per}(T, \mathbb{R}^N)$ we have that

$$\frac{\eta}{2}\|x_n'\|_2^2 + \int_0^b \eta\varphi(t, x_n(t))\,\mathrm{d}t \leqslant M_1\eta \tag{15}$$

and

$$-\|x_n'\|_2^2 - \int_0^b (u_n(t), x_n(t))_{\mathbb{R}^N}\,\mathrm{d}t \leqslant \varepsilon_n\|x_n\|, \quad \text{where } \varepsilon_n \downarrow 0. \tag{16}$$

Adding (15) and (16), we obtain

$$\Big(\frac{\eta}{2} - 1\Big)\|x_n'\|_2^2 + \int_0^b (\eta\varphi(t, x_n(t)) - (u_n(t), x_n(t)))_{\mathbb{R}^N}\,\mathrm{d}t \leqslant M_1\eta + \varepsilon_n\|x_n\|, \tag{17}$$

$$\Big(\frac{\eta}{2} - 1\Big)\|x_n'\|_2^2 + \int_0^b (\eta\varphi(t, x_n(t)) - \varphi^0(t, x_n(t); x_n(t)))\,\mathrm{d}t \leqslant M_1\eta + \varepsilon_n\|x_n\|.$$

Note that

$$\begin{aligned}\int_0^b (\eta\varphi(t, x_n(t)) - \varphi^0(t, x_n(t); x_n(t)))\,\mathrm{d}t &\\ = \int_{\{\|x_n(t)\|<M\}} &(\eta\varphi(t, x_n(t)) - \varphi^0(t, x_n(t); x_n(t)))\,\mathrm{d}t\\ + \int_{\{\|x_n(t)\|\geqslant M\}} &(\eta\varphi(t, x_n(t)) - \varphi^0(t, x_n(t); x_n(t)))\,\mathrm{d}t.\end{aligned} \tag{18}$$

By virtue of hypothesis $\mathrm{H}(\varphi)_5$(iii), we see that there exists $\beta_1 > 0$ such that

$$-\beta_1 \leqslant \int_{\{\|x_n(t)\|<M\}} (\eta\varphi(t, x_n(t)) - \varphi^0(t, x_n(t); x_n(t)))\,\mathrm{d}t \quad \text{for all } n \geqslant 1. \tag{19}$$

Also from hypothesis $\mathrm{H}(\varphi)_5$(iv) we have

$$0 \leqslant \int_{\{\|x_n(t)\|\geqslant M\}} (\eta\varphi(t, x_n(t)) - \varphi^0(t, x_n(t); x_n(t)))\,\mathrm{d}t. \tag{20}$$

Using (19) and (20) in (18), we obtain

$$-\beta_1 \leqslant \int_0^b (\eta\varphi(t, x_n(t)) - \varphi^0(t, x_n(t); x_n(t)))\,\mathrm{d}t. \tag{21}$$

Using (21) in (17) we obtain

$$\Big(\frac{\eta}{2} - 1\Big)\|x_n'\|_2^2 \leqslant \beta_2(1 + \|x_n\|) \quad \text{for some } \beta_2 > 0 \text{ and all } n \geqslant 1. \tag{22}$$

From (22), we shall establish that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded. Suppose that this is not the case. We may assume that $\|x_n\| \to \infty$. Let $y_n = x_n/\|x_n\|$, $n \geqslant 1$. By passing to a subsequence if necessary, we may assume that $y_n \xrightarrow{w} y$ in $W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$ and $y_n \to y$ in $C(T,\mathbb{R}^N)$ as $n \to \infty$. Divide (22) by $\|x_n\|^2$. We have

$$\begin{aligned}
&\Big(\frac{\eta}{2}-1\Big)\|y_n'\|_2^2 \leqslant \beta_2\Big(\frac{1}{\|x_n\|^2}+\frac{1}{\|x_n\|}\Big),\\
\Rightarrow\ &\Big(\frac{\eta}{2}-1\Big)\|y'\|_2^2 \leqslant 0 \quad \text{(recall that since } y_n \xrightarrow{w} y \text{ in } L^2(T,\mathbb{R}^N),\\
&\qquad\qquad \|y'\|_2^2 \leqslant \liminf_{n\to\infty}\|y_n'\|_2^2),\\
&\qquad \Rightarrow y'=0, \quad \text{i.e. } y=c\in\mathbb{R}^N \text{ (recall } \eta>2) \text{ and } y_n \to c \text{ in } W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N).
\end{aligned}$$

We have $x_n = \overline{x}_n + \hat{x}_n$ and $y_n = \overline{y_n} + \hat{y_n}$ with $\overline{y_n} = \overline{x}_n/\|x_n\|$ and $\hat{y_n} = \hat{x}_n/\|x_n\|$, $n \geqslant 1$. Hence $\hat{y_n} = \hat{x}_n/\|x_n\| \to 0$ in $W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$ and $\overline{y_n} = \overline{x}_n/\|x_n\| \to c$ in $\mathbb{R}^N$ as $n\to\infty$. Suppose that $c=0$. Then $y_n \to 0$ in $W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$, a contradiction since $\|y_n\| = 1$ for all $n \geqslant 1$. So $c \neq 0$. This means that for all $t \in T$ we have $\|x_n(t)\| \to \infty$ as $n \to \infty$. In fact we claim that the convergence is uniform on $T$, i.e. $\min_{t\in T}\|x_n(t)\| \to \infty$ as $n\to\infty$. To this end, since $y_n \to c$ in $C(T,\mathbb{R}^N)$, given $0<\varepsilon<\|c\|$ we can find $n_0 \geqslant 1$ such that for all $n \geqslant n_0$ and $t\in T$, we have

$$\|y_n(t)-c\| < \varepsilon \Rightarrow 0 < \|c\|-\varepsilon < \|y_n(t)\|.$$

Because $\|x_n\| \to \infty$, given $\beta_3 > 0$, we can find $n_1 \geqslant 1$ such that for all $n \geqslant n_1$ we have

$$\|x_n\| \geqslant \beta_3 > 0.$$

So for $n \geqslant n_2 = \max\{n_0, n_1\}$ and for $t\in T$, we have

$$\begin{aligned}
&\frac{\|x_n(t)\|}{\beta_3} \geqslant \frac{\|x_n(t)\|}{\|x_n\|} \geqslant \|y_n(t)\| > \|c\|-\varepsilon = \theta > 0\\
&\Rightarrow \|x_n(t)\| > \theta\beta_3 \quad \text{for all } n \geqslant 1 \text{ and all } t\in T.
\end{aligned}$$

Since $\beta_3 > 0$ was arbitrary and $\theta > 0$, we conclude that $\min_{t\in T}\|x_n(t)\| \to \infty$ as $n\to\infty$. So without any loss of generality we may assume that $\|x_n(t)\| > 0$ for all $n \geqslant 1$ and all $t \in T$. Then from the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$

we have

$$
\begin{aligned}
& -\frac{2M_1}{\|x_n\|^2} \leqslant \|y_n'\|_2^2 + \int_0^b \frac{2\varphi(t,x_n(t))}{\|x_n\|^2}\,\mathrm{d}t, \\
\Rightarrow\ & -\frac{2M_1}{\|x_n\|^2} \leqslant \|y_n'\|_2^2 + \int_0^b \frac{2\varphi(t,x_n(t))}{\|x_n(t)\|^2}\|y_n(t)\|^2\,\mathrm{d}t \\
\Rightarrow\ & -\frac{2M_1}{\|x_n\|^2} \leqslant \beta_4 + \int_0^b \frac{2\alpha_1(t) - 2\alpha_2(t)\|x_n(t)\|^\eta}{\|x_n(t)\|^2}\|y_n(t)\|^2\,\mathrm{d}t,
\end{aligned}
$$

for some $\beta_4 > 0$ and all $n \geqslant 1$ (see (13)).

Passing to the limit as $n \to \infty$ and since $\eta > 2$ and $\min_{t\in T}\|x_n(t)\| \to \infty$, we obtain a contradiction ($0 \leqslant -\infty$). Therefore $\{x_n\}_{n\geqslant 1} \subseteq W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$. Arguing as before, we can show that $x_n \to x$ in $W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)$, which proves Claim 1.

Now in order to eventually apply Proposition 11, we consider the following cylinder set:

$$
\begin{aligned}
C_R = \Big\{ x \in W^{1,2}_{\mathrm{per}}(T,\mathbb{R}^N)\colon\ & x(t) = c + \lambda \frac{\sqrt{b}}{\pi\sqrt{2}} e_1 \cos\frac{2\pi}{b}t, \\
& c \in \mathbb{R}^N,\ \|c\| \leqslant R,\ 0 \leqslant \lambda \leqslant R \Big\}
\end{aligned}
$$

with $e_1 = (1,0,\dots,0) \in \mathbb{R}^N$. The number $R > 0$ will be determined in the sequel. For $x \in C_R$ we have

$$
V(x) = \frac{1}{2}\lambda^2 + \int_0^b \varphi(t,x(t))\,\mathrm{d}t.
$$

**Claim 2.** *For $R > 0$ large, we have $V|_{\partial C_R} \leqslant 0$.*

For $x$ on the lower base (i.e. $\lambda = 0$) of the cylinder $C_R$, we have $x = \xi \in \mathbb{R}^N$ and

$$
V(x) = \int_0^b \varphi(t,\xi)\,\mathrm{d}t < 0 \quad (\text{since } \varphi < 0 \text{ by hypothesis } \mathrm{H}(\varphi)_5).
$$

Note that if $x \in C_R$, we have

$$
\int_0^b \|x(t)\|^2\,\mathrm{d}t = b\|c\|^2 + \lambda^2.
$$

Also because of (13) for every $x \in C_R$ we have

$$
\int_0^b \varphi(t,x(t))\,\mathrm{d}t \leqslant \beta_5 - \beta_6\|x\|_\eta^\eta \leqslant \beta_7 - \beta_8\|x\|_2^\eta \quad \text{for some } \beta_5,\beta_6,\beta_7,\beta_8 > 0.
$$

Thus for $x \in C_R$ we can write that

$$V(x) = \frac{\lambda^2}{2} + \int_0^b \varphi(t, x(t))\,\mathrm{d}t \leqslant \frac{\lambda^2}{2} - \beta_8(b\|c\|^2 + \lambda^2)^{\eta/2} - \beta_7.$$

So for $x \in C_R$ on the lateral boundary $\|c\| = R$ and the upper base $\lambda = R$ of the cylinder we have

$$V(x) \leqslant \frac{R^2}{2} - \beta_8 b R^\eta + \beta_7.$$

Since $\eta > 2$, choose $R > 0$ large enough we shall have

$$V|_{\partial C_R} \leqslant 0,$$

which proves Claim 2.

Next let $E = \{y \in Y\colon\ \|y'\|_2 = 2\sqrt{3}/\sqrt{b}\}$ (recall that on $Y$ $\|y'\|_2$ is an equivalent norm).

**Claim 3.** $0 \leqslant \inf\limits_E V$.

Recall that from Sobolev's inequality, for any $y \in E$ we have

$$\|y\|_\infty^2 \leqslant \frac{b}{12}\|y'\|_2^2 = \frac{b}{12}\frac{12}{b} = 1 \Rightarrow \|y\|_\infty \leqslant 1.$$

Using hypothesis $\mathrm{H}(\varphi)_5$(v) we have $\varphi(t, y(t)) \geqslant -6/b^2$ a.e. on $T$ and so for all $y \in E$ we have

$$V(y) = \frac{1}{2}\|y'\|_2^2 + \int_0^b \varphi(t, y(t))\,\mathrm{d}t \geqslant \frac{6}{b} - \frac{6}{b^2}b = 0,$$

which proves Claim 3.

Because of Claims 1, 2, 3 we can apply Proposition 11 and obtain $x \in W^{1,2}_{\mathrm{per}}(T, \mathbb{R}^N)$ such that $0 \in \partial V(x)$ and $V(x) \geqslant 0$. Evidently $x$ is nonconstant because for $\xi \in \mathbb{R}^N$, $V(\xi) < 0$ (hypothesis $\mathrm{H}(\varphi)_5$(v)). As before we can show that $x \in C^1(T, \mathbb{R}^N)$ and that it solves (9). $\square$

**Remark.** This theorem is a partial extension of [12, Theorem 3].

Let $h \in L^1(T)$, $h \geqslant 0$ with $\|h\|_1 \geqslant b$ and $c(t) = h(t) - 1 - 6/b^2 + \ln 2/b^2$. Then it is easy to check that the function

$$\varphi(t, x) = \begin{cases} -\dfrac{6}{b^2} + \dfrac{1}{b^2}\|x\| \ln(\|x\| + 1) & \text{if } x \leqslant 1, \\ -h(t)\|x\|^\eta + \|x\| + c(t) & \text{if } x > 1, \end{cases} \qquad 2 < \eta < \infty,$$

satisfies hypotheses $\mathrm{H}(\varphi)_5$.

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