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THE GENERALIZED HOLDITCH THEOREM FOR THE HOMOTHETIC MOTIONS ON THE PLANAR KINEMATICS

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Abstract. W. Blaschke and H. R. Müller [4, p. 142] have given the following theorem as a generalization of the classic Holditch Theorem: Let E/E' be a 1-parameter closed planar Euclidean motion with the rotation number ν and the period T. Under the motion E/E', let two points $A = (0,0)$, $B = (a+b,0) \in E$ trace the curves $k_A, k_B \subset E'$ and let F_A, F_B be their orbit areas, respectively. If F_X is the orbit area of the orbit curve k of the point $X = (a,0)$ which is collinear with points A and B then

$$F_X = \frac{[aF_B + bF_A]}{a+b} - \pi\nu ab.$$

In this paper, under the 1-parameter closed planar homothetic motion with the homothetic scale $h = h(t)$, the generalization given above by W. Blaschke and H. R. Müller is expressed and

$$F_X = \frac{[aF_B + bF_A]}{a+b} - h^2(t_0)\pi\nu ab,$$

is obtained, where $\exists t_0 \in [0,T]$.

Keywords: Holditch Theorem, homothetic motion, Steiner formula

MSC 2000: 53A17

1. Introduction

Let E and E' be moving and fixed Euclidean planes and $\{O;\mathbf{e_1},\mathbf{e_2}\}$ and $\{O';\mathbf{e'_1},\mathbf{e'_2}\}$ be their coordinate systems, respectively. By taking $\mathbf{OO'} = \mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2}$ for $u_1, u_2 \in \mathbb{R}$, the motion defined by the transformation

$$\mathbf{x}' = h\mathbf{x} - \mathbf{u} \tag{1.1}$$

is called 1-*parameter planar homothetic motion* and denoted by E/E', where h is a homothetic scale of the motion E/E' and $\mathbf{x}$ and $\mathbf{x}'$ are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X \in E$, respectively. The homothetic scale h and the vectors $\mathbf{x}$, $\mathbf{x}'$ and $\mathbf{u}$ are continuously differentiable functions of a real parameter t. Furthermore, at the initial time $t = 0$ the coordinate systems $\{O; \mathbf{e_1}, \mathbf{e_2}\}$ and $\{O'; \mathbf{e'_1}, \mathbf{e'_2}\}$ are coincident. Taking $\varphi = \varphi(t)$ as the rotation angle between $\mathbf{e_1}$ and $\mathbf{e'_1}$, the equation

$$\begin{aligned} \mathbf{e_1} &= \cos\varphi \mathbf{e'_1} + \sin\varphi \mathbf{e'_2} \\ \mathbf{e_2} &= -\sin\varphi \mathbf{e'_1} + \cos\varphi \mathbf{e'_2} \end{aligned} \tag{1.2}$$

can be written. If

$$\begin{aligned} u_j(t+T) &= u_j(t), \quad j = 1, 2 \\ \varphi(t+T) &= \varphi(t) + 2\pi\nu, \quad \forall t \in [0, T] \end{aligned} \tag{1.3}$$

then the motion E/E' is called 1-*parameter closed planar homothetic motion* with the period $T > 0$ and the rotation number $\nu \in Z$. To avoid the cases of the pure translation and the pure rotation we assume that

$$\dot{\varphi}(t) = \frac{\mathrm{d}\varphi}{\mathrm{d}t} \neq 0.$$

Under the 1-parameter closed planar homothetic motions, if $P = (p_1, p_2)$ is the pole point of the motion at the time t then the sliding velocity of a fixed point $X = (x_1, x_2) \in E$ with respect to E' is

$$\mathrm{d}\mathbf{x}' = \{(x_1 - p_1)\,\mathrm{d}h - (x_2 - p_2)h\,\mathrm{d}\varphi\}\mathbf{e_1} + \{(x_1 - p_1)h\,\mathrm{d}\varphi + (x_2 - p_2)\,\mathrm{d}h\}\mathbf{e_2}. \tag{1.4}$$

Furthermore, the orbit area F_X of the point X, given by Gauss area formula [3], is

$$F_X = \frac{1}{2}\oint (x'_1\,\mathrm{d}x'_2 - x'_2\,\mathrm{d}x'_1), \tag{1.5}$$

where the integration is taken along the closed orbit curve of X. Then, we obtain

$$\begin{aligned} 2F_X = {}& (x_1^2 + x_2^2)\int_0^T h^2(t)\,\mathrm{d}\varphi(t) - 2x_1\int_0^T p_1(t)h^2(t)\,\mathrm{d}\varphi(t) \\ & - 2x_2\int_0^T p_2(t)h^2(t)\,\mathrm{d}\varphi(t) + \int_0^T \{u_1(t)p_1(t)h(t)\,\mathrm{d}\varphi(t) \\ & + u_2(t)p_2(t)h(t)\,\mathrm{d}\varphi(t) + u_1(t)p_2(t)\,\mathrm{d}h(t) - u_2(t)p_1(t)\,\mathrm{d}h(t)\} \\ & + x_1\int_0^T \{u_2(t)\,\mathrm{d}h(t) - 2p_2(t)h(t)\,\mathrm{d}h(t) + h(t)\,\mathrm{d}u_2(t)\} \\ & + x_2\int_0^T \{-u_1(t)\,\mathrm{d}h(t) + 2p_1(t)h(t)\,\mathrm{d}h(t) - h(t)\,\mathrm{d}u_1(t)\}, \quad [1]. \end{aligned} \tag{1.6}$$

Moreover, using the mean value theorem of integral-calculus for the closed interval $0 \leqslant t \leqslant T$, there exists at least a point $t_0 \in [0, T]$ such that

$$\int_0^T h^2(t)\,\mathrm{d}\varphi(t) = \int_0^T h^2(t)\dot{\varphi}(t)\,\mathrm{d}t = 2h^2(t_0)\pi\nu. \tag{1.7}$$

By taking $\nu \neq 0$, the Steiner point $S = (s_1, s_2)$ for the closed planar homothetic motion can be written as

$$s_j = \frac{\int_0^T h^2(t)p_j(t)\,\mathrm{d}\varphi(t)}{\int_0^T h^2(t)\,\mathrm{d}\varphi(t)}, \quad j = 1, 2. \tag{1.8}$$

Thus, from Eqs. (1.6), (1.7) and (1.8) we get

$$F_X = F_0 + h^2(t_0)\pi\nu(x_1^2 + x_2^2 - 2x_1s_1 - 2x_2s_2) + \mu_1x_1 + \mu_2x_2, \quad [1], \tag{1.9}$$

where F_0 is the orbit area of the origin of the moving coordinate system and

$$\begin{aligned} \mu_1 &= \frac{1}{2}\int_0^T \{-2h(t)p_2(t)\,\mathrm{d}h(t) + h(t)\,\mathrm{d}u_2(t) + u_2(t)\,\mathrm{d}h(t)\}, \\ \mu_2 &= \frac{1}{2}\int_0^T \{2h(t)p_1(t)\,\mathrm{d}h(t) - h(t)\,\mathrm{d}u_1(t) - u_1(t)\,\mathrm{d}h(t)\}. \end{aligned} \tag{1.10}$$

Eq. (1.9) is called *the Steiner area formula* for the 1-parameter closed planar homothetic motion.

2. The generalized Holditch theorem for the closed planar homothetic motions

Theorem 1. *Let E/E' be 1-parameter planar homothetic motion with the rotation number ν. Let F_A, F_B denote the orbit areas of the orbit curves $k_A, k_B \subset E'$ of the points $A = (0, 0)$, $B = (a + b, 0) \in E$, respectively. If F_X is the orbit area of the orbit curve k of the point $X = (a, 0)$, which is collinear with points A and B, then*

$$F_X = \frac{[aF_B + bF_A]}{a + b} - h^2(t_0)\pi\nu ab. \tag{2.1}$$

Proof. From Eq. (1.9), for the orbit areas F_A, F_B and F_X, we obtain

$$F_A = F_0, \tag{2.2}$$

$$F_B = F_0 + h^2(t_0)\pi\nu[(a + b)^2 - 2s_1(a + b)] + \mu_1(a + b), \tag{2.3}$$

and

$$F_X = F_0 + h^2(t_0)\pi\nu(a^2 - 2s_1 a) + \mu_1 a. \tag{2.4}$$

From Eqs. (2.2) and (2.3), we have

$$\frac{aF_B + bF_A}{a+b} - h^2(t_0)\pi\nu ab = F_0 + h^2(t_0)\pi\nu(a^2 - 2s_1 a) + \mu_1 a. \tag{2.5}$$

Then, from Eqs. (2.4) and (2.5), we get Eq. (2.1). □

Special case 1. *In the case of the homothetic scale $h \equiv 1$, we get*

$$F_X = \frac{[aF_B + bF_A]}{a+b} - \pi\nu ab, \tag{2.6}$$

which was given by [4].

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