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ON HANKEL TRANSFORM AND HANKEL CONVOLUTION OF BEURLING TYPE DISTRIBUTIONS HAVING UPPER BOUNDED SUPPORT

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Abstract. In this paper we study Beurling type distributions in the Hankel setting. We consider the space $\mathscr{E}(w)'$ of Beurling type distributions on $(0,\infty)$ having upper bounded support. The Hankel transform and the Hankel convolution are studied on the space $\mathscr{E}(w)'$. We also establish Paley Wiener type theorems for Hankel transformations of distributions in $\mathscr{E}(w)'$.

Keywords: Beurling distributions, Hankel transformation, convolution

MSC 2000: 46F12

1. Introduction

In [2] and [3] we started the study of the Hankel transformation and the Hankel convolution on new spaces of Beurling type distributions. Our motivation were the papers of A. Beurling [8], G. Björck [9] and R. W. Braun, R. Meise and B. A. Taylor [12]. In [8] A. Beurling developed the foundations of a general theory of distributions that includes the earlier Schwartz theory [22]. G. Björck [9] completed the Beurling's investigations by introducing the space of tempered Beurling distributions. More recently, R. W. Braun, R. Meise and B. A. Taylor [12] considered a more general classes of weights and they gave a description via derivatives of the Beurling-Björck spaces. In this paper we continue our investigations about Hankel transforms and Hankel convolutions on the space $\mathscr{E}(w)'$ of Beurling type distributions on $(0,\infty)$ having upper bounded support.

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The Hankel transformation h_μ is defined by ([16])

$$h_\mu(\varphi)(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\varphi(y) y^{2\mu+1}\,\mathrm{d}y, \quad x \in (0,\infty),$$

where J_μ, as usual, represents the Bessel function of the first kind and order μ ([25]). Here we always assume that $\mu > -\frac{1}{2}$.

By $L_{1,\mu}$ we denote the space of all those complex valued and Lebesgue measurable functions f on $(0,\infty)$ such that $\int_0^\infty |f(x)| x^{2\mu+1}\,\mathrm{d}x < \infty$. Note that, since the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$, $h_\mu(\varphi)$ is a continuous bounded function on $(0,\infty)$, provided that $\varphi \in L_{1,\mu}$.

The distributional study of the Hankel transformation was started by A. H. Zemanian who, in a series of papers ([26], [27] and [28]), investigated the Hankel transforms on distributions of slow and rapid growth.

A. H. Zemanian considered a variant of the Hankel transform defined through

$$H_\mu(\varphi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\varphi(y)\,\mathrm{d}y, \quad x \in (0,\infty).$$

It is clear that h_μ and H_μ are closely connected.

In 1982 G. Altenburg [1] adapted the Zemanian's results for the h_μ transformation. He considered the space $\mathscr{H}$ constituted by all those complex valued and smooth functions φ on $(0,\infty)$ such that, for every $m, n \in \mathbb{N}$,

$$\gamma_{m,n}(\varphi) = \sup_{x\in(0,\infty)} (1+x^2)^m \left|\left(\frac{1}{x}D\right)^n \varphi(x)\right| < \infty.$$

The space $\mathscr{H}$ is endowed with the topology generated by the family $\{\gamma_{m,n}\}_{m,n\in\mathbb{N}}$ of seminorms. Thus $\mathscr{H}$ is a Fréchet space and the Hankel transformation h_μ is an automorphism of $\mathscr{H}$ ([1, Satz 5]). The Hankel transformation h_μ is defined on the dual space $\mathscr{H}'$ of $\mathscr{H}$ by transposition.

For every $a > 0$, the space $\mathscr{B}^a$ consists of all those functions $\varphi \in \mathscr{H}$ such that $\varphi(x) = 0$, $x \geqslant a$. $\mathscr{B}^a$ is equipped with the topology induced in it by $\mathscr{H}$. Thus $\mathscr{B}^a$ is a Fréchet space and it is clear that $\mathscr{B}^a$ is continuously contained in $\mathscr{B}^b$, provided that $0 < a < b$. The union space $\mathscr{B} = \bigcup\limits_{a>0} \mathscr{B}^a$ is endowed with the inductive topology.

Other significant papers concerning to distributional h_μ transformation are [20] and [21].

I. I. Hirschman [17], D. T. Haimo [15] and F. M. Cholewinski [13] investigated the convolution operation for the Hankel transformation h_μ. If f and g are in $L_{1,\mu}$ then the Hankel convolution $f \#_\mu g$ of f and g is defined by

$$(f \#_\mu g)(x) = \int_0^\infty f(y)({}_\mu\tau_x g)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}y, \quad \text{a.e. } x \in (0,\infty),$$

where $a.e.$ refers to the Lebesgue measure on $(0,\infty)$ and the Hankel translation operator ${}_\mu\tau_x$ is defined, for every $x \in (0,\infty)$, by

$$({}_\mu\tau_x g)(y) = \int_0^\infty g(z) D_\mu(x,y,z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\,\mathrm{d}z, \quad \text{a.e. } y \in (0,\infty),$$

and ${}_\mu\tau_0 g = g$.

Here the function D_μ, that is the Delsarte kernel, is given by

$$D_\mu(x,y,z) = (2^\mu\Gamma(\mu+1))^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt) t^{2\mu+1}\,\mathrm{d}t, \qquad x,y,z \in (0,\infty).$$

The Hankel transform is related to Hankel convolution and Hankel translation through the following formulas ([4, (3.1) and (3.2)])

$$h_\mu(f \#_\mu g) = h_\mu(f) h_\mu(g)$$

and

$$h_\mu({}_\mu\tau_x g) = 2^\mu \Gamma(\mu+1)(x.)^{-\mu} J_\mu(x.) h_\mu(g),$$

that hold for every $f, g \in L_{1,\mu}$ and $x \in (0,\infty)$.

In the sequel, since there is not any confusion, we will write $\#$, τ_x, $x \in [0,\infty)$, and D instead of $\#_\mu$, ${}_\mu\tau_x$, $x \in [0,\infty)$, and D_μ, respectively.

A straighforward manipulation allows to define, from the $\#$-convolution, the convolution for the Hankel transformation H_μ.

The study of the distributional Hankel convolution was started by J. de Sousa-Pinto [23] who considered only the order $\mu = 0$. More recently, in a series of papers ([4], [5] and [19]), J. J. Betancor and I. Marrero have investigated the Hankel convolution (of any order) in the Zemanian spaces. In [7] J. J. Betancor and L. Rodríguez-Mesa defined the Hankel transformation and the Hankel convolution of distributions with exponential growth.

In [2] we have introduced Beurling type function and distribution spaces that have a nice behaviour for the Hankel transform and the Hankel convolution. We now collect the main properties of those spaces that will be very useful in the sequel.

We consider, as in [12], the set $\mathscr{M}$ of functions defined as follows. A continuous, positive and non-decreasing function w defined in $[0,\infty)$ is in $\mathscr{M}$ when $w = 0$ on $[0,1]$, and it satisfies the following conditions

(α) there exists $K \geqslant 1$ such that $w(2x) \leqslant K(1 + w(x))$, $x \in [0,\infty)$,

(β) $\int_1^\infty w(x)/x^2\,\mathrm{d}x < \infty$,

(γ) $\log(1+x) = o(w(x))$, as $x \to \infty$, and
(δ) the function $\Omega(x) = w(\mathrm{e}^x)$, $x \in \mathbb{R}$, is convex.

If $w \in \mathscr{M}$ we define $w(-x) =: w(x)$, $x \in [0,\infty)$.

A useful property of the function w is the following ([12, Lemma 1.2])

$$w(x+y) \leqslant K(1 + w(x) + w(y)) \quad x, y \in \mathbb{R}. \tag{1.1}$$

Assume in the sequel that $w \in \mathscr{M}$.

Let $a > 0$. The space $\mathscr{B}^a_\mu(w)$ is constituted by all those complex valued and smooth functions φ such that $\varphi(x) = 0$, $x \geqslant a$, φ and $h_\mu(\varphi)$ are in $L_{1,\mu}$ and,

$$\delta^\mu_n(\varphi) = \int_0^\infty \mathrm{e}^{nw(x)} |h_\mu(\varphi)(x)| x^{2\mu+1}\,\mathrm{d}x < \infty,$$

for every $n \in \mathbb{N}$. On $\mathscr{B}^a_\mu(w)$ we consider the topology associated to the system $\{\delta^\mu_n\}_{n\in\mathbb{N}}$ of norms. Thus $\mathscr{B}^a_\mu(w)$ is a Fréchet space. It is obvious that $\mathscr{B}^a_\mu(w)$ is continuously contained in $\mathscr{B}^b_\mu(w)$, provided that $0 < a < b$. The union space $\mathscr{B}_\mu(w) = \bigcup_{a>0} \mathscr{B}^a_\mu(w)$ is equipped with the inductive topology.

Actually, the spaces $\mathscr{B}^a_\mu(w)$, $a > 0$, and $\mathscr{B}_\mu(w)$ are indenpendent of μ. Indeed, in [2, Proposition 2.10] we establish that, given $a > 0$, a function $\varphi \in \mathscr{B}^a$ is in $\mathscr{B}^a_\mu(w)$ if, and only if, for every $l \in \mathbb{N}$,

$$\varepsilon_l(\varphi) = \sup_{x\in(0,\infty),\, k\in\mathbb{N}} \mathrm{e}^{-l\Omega^*(x/l)} x^k \left| \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \varphi(x)\right| < \infty,$$

where the Young conjugate function Ω^* of Ω is defined, as usual, by

$$\Omega^*(x) = \sup_{y\geqslant 0}(xy - \Omega(y)), \quad x \in [0,\infty).$$

Moreover the topology generated by $\{\varepsilon_l\}_{l\in\mathbb{N}}$ on $\mathscr{B}^a_\mu(w)$, $a > 0$, coincides with the one defined by $\{\delta^\mu_n\}_{n\in\mathbb{N}}$.

Hence, in the sequel we will write $\mathscr{B}^a(w)$, $a > 0$, and $\mathscr{B}(w)$ to refer to $\mathscr{B}^a_\mu(w)$, $a > 0$, and $\mathscr{B}_\mu(w)$, respectively.

For every $x \in (0,\infty)$, the Hankel translation τ_x defines a continuous linear mapping from $\mathscr{B}_\mu(w)$ into itself ([2, Proposition 2.15, (i)]). The Hankel convolution $T \mathbin{\#} \varphi$ of $T \in \mathscr{B}_\mu(w)'$, the dual space of $\mathscr{B}_\mu(w)$, and $\varphi \in \mathscr{B}_\mu(w)$ is defined by

$$(T \mathbin{\#} \varphi)(x) = \langle T, \tau_x\varphi\rangle, \quad x \in [0,\infty).$$

By $\mathscr{E}(w)$ we denote the space of multipliers of $\mathscr{B}(w)$. $\mathscr{E}(w)$ is characterized as the set of all those functions ψ defined on $(0,\infty)$ such that, for every $a > 0$, there exists

$\varphi \in \mathscr{B}(w)$ such that $\varphi = \psi$ on $(0,a)$ ([2, Proposition 3.1]). $\mathscr{E}(w)$ is a Fréchet space when we define the topology associated with the family $\{Z^\mu_{a,n}\}_{a>0,\, n\in\mathbb{N}}$ of seminorms, where, for every $a > 0$ and $n \in \mathbb{N}$,

$$Z^\mu_{a,n}(\psi) = \inf_{\substack{\varphi\in\mathscr{B}(w),\\ \varphi=\psi \text{ on } (0,a)}} \delta^\mu_n(\varphi), \quad \psi \in \mathscr{E}(w).$$

This topology coincides with the one induced by the pointwise convergence topology of $\mathscr{L}(\mathscr{B}(w))$, the space of linear and continuous mapping from $\mathscr{B}(w)$ into itself. Moreover the space $\mathscr{E}(w)$ can be described by derivatives ([2, Proposition 3.2]). The space $\mathscr{E}(w)'$, the dual space of $\mathscr{E}(w)$, can be regarded as a subspace of $\mathscr{B}(w)'$ because $\mathscr{B}(w)$ is continuously contained in $\mathscr{E}(w)$. $\mathscr{E}(w)'$ is characterized as the space of Hankel convolution operators of $\mathscr{B}(w)$ ([2, Proposition 3.8]). Moreover, in [2, Proposition 3.7] we proved that if $T \in \mathscr{E}(w)'$, then the support of T is upper bounded, that is, there exists $a > 0$ such that $\langle T, \varphi\rangle = 0$, provided that $\varphi \in \mathscr{B}(w)$ and $\varphi(x) = 0$, $x \in (0,a)$.

If w is the function defined by $w(x) = \log(1 + x)$, $x \in [0,\infty)$, then $\mathscr{B}(w)$ coincides with $\mathscr{B}$ and the space $\mathscr{E}(w)$ is constituted by all those smooth functions f on $(0,\infty)$ such that the limit

$$\lim_{x\to 0^+} \left(\frac{1}{x}D\right)^k f(x)$$

exists, for every $k \in \mathbb{N}$. Note that in this case w does not satisfy the property (γ).

In [3] the authors defined a tempered Beurling type distributions involving the Hankel transformation h_μ. A complex and smooth function φ on $(0,\infty)$ is in $\mathscr{H}_\mu(w)$ if, and only if, $h_\mu(\varphi)$ is smooth on $(0,\infty)$ and, for every $m, n \in \mathbb{N}$,

$$\alpha_{m,n}(\varphi) = \sup_{x\in(0,\infty)} \mathrm{e}^{mw(x)} \left|\left(\frac{1}{x}D\right)^n \varphi(x)\right| < \infty,$$

and

$$\beta^\mu_{m,n}(\varphi) = \sup_{x\in(0,\infty)} \mathrm{e}^{mw(x)} \left|\left(\frac{1}{x}D\right)^n h_\mu(\varphi)(x)\right| < \infty.$$

$\mathscr{H}_\mu(w)$ is a Fréchet space when we define on it the topology associated with the family of seminorms $\{\alpha_{m,n}, \beta^\mu_{m,n}\}_{m,n\in\mathbb{N}}$. It is clear that h_μ is an automorphism of $\mathscr{H}_\mu(w)$. The Hankel transform is defined on the dual space $\mathscr{H}_\mu(w)'$ of $\mathscr{H}_\mu(w)$ by transposition. That is, the Hankel transform $h'_\mu(T)$ of $T \in \mathscr{H}_\mu(w)'$ is the element of $\mathscr{H}_\mu(w)'$ defined by

$$\langle h'_\mu(T), \varphi\rangle = \langle T, h_\mu(\varphi)\rangle, \quad \varphi \in \mathscr{H}_\mu(w).$$

We proved in ([3, Proposition 3.1]) that if $T \in \mathscr{E}(w)'$, then the Hankel transform $h'_\mu(T)$ of T coincides with the element of $\mathscr{H}_\mu(w)'$ generated by the function

$$F(x) = 2^\mu \Gamma(\mu+1)\langle T(y), (xy)^{-\mu} J_\mu(xy)\rangle, \quad x \in (0,\infty).$$

That is, for every $\varphi \in \mathscr{H}_\mu(w)$,

$$\int_0^\infty F(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x = \langle T, h_\mu(\varphi)\rangle, \quad \varphi \in \mathscr{H}_\mu(w).$$

Also, in [3] we studied the Hankel convolution on $\mathscr{H}_\mu(w)$ and $\mathscr{H}_\mu(w)'$.

We consider in this paper a class of functions $\mathscr{M}$ slightly different from the one defined in [3]. However, the theory developed in [3] can be written now in a similar way.

In this paper we complete the studies developed in [2] and [3].

As it is easy to see, if there exists $C > 0$ such that $C^{-1}w \leqslant W \leqslant Cw$, on $[0,\infty)$, then $\mathscr{B}(w) = \mathscr{B}(W)$, $\mathscr{E}(w) = \mathscr{E}(W)$ and $\mathscr{H}(w) = \mathscr{H}(W)$. Usually, it is said that w and W are equivalent when the above property is satisfied. Other functions equivalent to those in $\mathscr{M}$ are exhibited in [12, p. 221].

Note that, according to [11, p. 56], the condition $w = 0$ on $[0,1]$ imposed on the functions in $\mathscr{M}$ is not a real restriction because the spaces $\mathscr{B}(w)$ and $\mathscr{E}(w)$ do not change if we replace w by $W(x) = \max\{0, w(x) - w(1)\}$, $x \in [0,\infty)$.

This paper is organized as follows. In Section 2 we study Hankel convolution and Hankel translation on the spaces $\mathscr{E}(w)$ and its dual $\mathscr{E}(w)'$. We analyze the Hankel transform of the functionals in $\mathscr{E}(w)'$ in Section 3. We prove Paley-Wiener type theorems for the Hankel transforms on $\mathscr{E}(w)'$.

We will always denote by C a suitable positive constant not necessarily the same on each ocurrence.

2. Hankel translation and Hankel convolution on the spaces $\mathscr{E}(w)$ and $\mathscr{E}(w)'$

In this section we study the Hankel translation on the space $\mathscr{E}(w)$. Then we define the Hankel convolution on the space $\mathscr{E}(w)'$.

If $\psi \in \mathscr{E}(w)$ and $x \in (0,\infty)$ the Hankel translate $\tau_x\psi$ of ψ by x is defined by

$$(2.2) \qquad (\tau_x\psi)(y) = \int_{|x-y|}^{x+y} D(x,y,z)\psi(z)\frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}z, \quad y \in (0,\infty).$$

Note that the integral in (2.2) is absolutely convergent for every $x, y \in (0,\infty)$. Indeed, if $x, y \in (0,\infty)$, there exists $\varphi \in \mathscr{B}(w)$ such that $\varphi = \psi$ on $(0, x+y)$.

We now prove that the Hankel translations define closed operations in $\mathscr{E}(w)$.

Proposition 2.1. *Let $x \in (0,\infty)$. Then the Hankel translation τ_x defines a continuous linear mapping from $\mathscr{E}(w)$ into itself.*

Proof. Assume that $\varphi \in \mathscr{E}(w)$. Let $a > 0$. There exists $\psi \in \mathscr{B}(w)$ such that $\varphi = \psi$ on $(0, x+a)$. Then we can write

$$(\tau_x\varphi)(y) = \int_{|x-y|}^{x+y} \psi(z) D(x,y,z) \frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}z, \quad y \in (0,a).$$

Moreover, according to [2, Proposition 2.15, (i)], the function $\tau_x\psi \in \mathscr{B}(w)$. Thus we have proved that $\tau_x\varphi \in \mathscr{E}(w)$.

Suppose now $\{\varphi_\nu\}_{\nu\in\mathbb{N}}$ is a sequence in $\mathscr{E}(w)$ such that $\varphi_\nu \to 0$, as $\nu \to \infty$, in $\mathscr{E}(w)$. Let $\varepsilon, a > 0$ and $m \in \mathbb{N}$. There exists $\nu_0 \in \mathbb{N}$ such that, for every $\nu \geqslant \nu_0$, we can find $\psi_\nu \in \mathscr{B}(w)$ for which $\psi_\nu = \varphi_\nu$, on $(0, a+x)$, and

$$\delta_m^\mu(\psi_\nu) = \int_0^\infty \mathrm{e}^{mw(y)} |h_\mu(\psi_\nu)(y)| y^{2\mu+1}\,\mathrm{d}y < \varepsilon.$$

Then, by [4, (3.1)] and [17, Theorem 2.a], it follows that

$$\begin{aligned}\delta_m^\mu(\tau_x\psi_\nu) &= 2^\mu\Gamma(\mu+1)\int_0^\infty \mathrm{e}^{mw(y)} |(xy)^{-\mu} J_\mu(xy) h_\mu(\psi_\nu)(y)| y^{2\mu+1}\,\mathrm{d}y \\ &\leqslant \delta_m^\mu(\psi_\nu) < \varepsilon, \quad \nu \geqslant \nu_0.\end{aligned}$$

Moreover, $\tau_x\psi_\nu = \tau_x\varphi_\nu$, on $(0,a)$, for each $\nu \geqslant \nu_0$. Hence

$$\mathscr{L}_{a,m}^\mu(\tau_x\varphi_\nu) \leqslant \delta_m^\mu(\tau_x\psi_\nu) < \varepsilon, \quad \nu \geqslant \nu_0.$$

Thus we see that $\tau_x\varphi_\nu \to 0$, as $\nu \to \infty$, in $\mathscr{E}(w)$, and the proof is completed. □

Proposition 2.2. *Let $\varphi \in \mathscr{E}(w)$. The (nonlinear) mapping F_φ defined by $F_\varphi(x) = \tau_x\varphi$, $x \in [0,\infty)$, is continuous from $[0,\infty)$ into $\mathscr{E}(w)$.*

Proof. Let $x_0 \in [0,\infty)$. Assume that $\{x_\nu\}_{\nu\in\mathbb{N}\setminus\{0\}}$ is a sequence in $[0,\infty)$ such that $x_\nu \to x_0$, as $\nu \to \infty$. We choose $b > 0$ such that $x_\nu \in [0,b)$, $\nu \in \mathbb{N}$.

Let $a > 0$ and $m \in \mathbb{N}$. There exists $\psi \in \mathscr{B}(w)$ such that $\psi = \varphi$, on $(0, a+b)$. Then $(\tau_x\varphi)(y) = (\tau_x\psi)(y)$, $x \in (0,b)$ and $y \in (0,a)$. According to [2, Proposition 2.15, (ii)], if $\varepsilon > 0$ there exists $\nu_0 \in \mathbb{N}$ such that

$$\delta_m^\mu(\tau_{x_\nu}\psi - \tau_{x_0}\psi) < \varepsilon, \quad \nu \geqslant \nu_0.$$

Then, since $\tau_{x_\nu}\psi - \tau_{x_0}\psi \in \mathscr{B}(w)$ and $\tau_{x_\nu}\psi - \tau_{x_0}\psi = \tau_{x_\nu}\varphi - \tau_{x_0}\varphi$, on $(0,a)$, for every $\nu \in \mathbb{N}$,

$$\mathscr{L}_{a,m}^\mu(\tau_{x_\nu}\varphi - \tau_{x_0}\varphi) \leqslant \delta_m^\nu(\tau_{x_\nu}\psi - \tau_{x_0}\psi) < \varepsilon, \quad \nu \geqslant \nu_0.$$

Hence, $\tau_{x_\nu}\varphi \to \tau_{x_0}\varphi$, as $\nu \to \infty$, in $\mathscr{E}(w)$.

Thus, we have proved the continuity of the function F_φ. □

Proposition 2.1 allows us to give the following definition for the Hankel convolution on $\mathscr{E}(w)' \times \mathscr{E}(w)$. Let $T \in \mathscr{E}(w)'$ and $\varphi \in \mathscr{E}(w)$. The Hankel convolution $T \# \varphi$ of T and φ is defined through

$$(T \# \varphi)(x) = \langle T, \tau_x \varphi \rangle, \quad x \in [0, \infty).$$

Note that, according to Proposition 2.2, $T \# \varphi$ is a continuous function on $[0, \infty)$. Next we will show that $T \# \varphi \in \mathscr{E}(w)$.

Proposition 2.3. *Let $T \in \mathscr{E}(w)'$. Then the linear mapping F_T defined by*

$$F_T(\varphi) = T \# \varphi, \quad \varphi \in \mathscr{E}(w),$$

is continuous from $\mathscr{E}(w)$ into itself.

P r o o f. Since $T \in \mathscr{E}(w)'$, according to [2, Proposition 3.2], there exists $b > 0$ such that $\langle T, \varphi \rangle = 0$, for every $\varphi \in \mathscr{E}(w)$ for which $\varphi(x) = 0$, $x \in (0, b)$. Indeed, since $T \in \mathscr{E}(w)'$ [2, Proposition 3.2] implies that there exist $b, C > 0$ and $l \in \mathbb{N}$ such that

$$|\langle T, \varphi \rangle| \leqslant C \sup_{x \in (0,b),\, k \in \mathbb{N}} \mathrm{e}^{-l\Omega^*(k/l)} x^k \left| \left(\frac{1}{x} D \right)^k \varphi(x) \right|, \quad \varphi \in \mathscr{E}(w),$$

where Ω^* denotes the Young conjugate of the function Ω defined by $\Omega(x) = w(\mathrm{e}^x)$, $x \in \mathbb{R}$.

Hence, if $\varphi \in \mathscr{E}(w)$ and $\varphi(x) = 0$, $x \in (0, b)$, then $\langle T, \varphi \rangle = 0$.

Let $a > 0$ and $\varphi \in \mathscr{E}(w)$. There exists $\psi \in \mathscr{B}(w)$ such that $\varphi = \psi$, on $(0, a + b)$. Moreover, for every $x \in (0, a)$ and $y \in (0, b)$, one has

$$\begin{aligned}(\tau_x \varphi)(y) &= \int_{|x-y|}^{x+y} D(x, y, z) \varphi(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} \, \mathrm{d}z \\ &= \int_0^{a+b} D(x, y, z) \psi(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} \, \mathrm{d}z = (\tau_x \psi)(y).\end{aligned}$$

Hence, we can write

$$F_T(\varphi)(x) = \langle T, \tau_x \varphi \rangle = \langle T, \tau_x \psi \rangle = (T \# \psi)(x), \quad x \in (0, a).$$

Now according to [2, Proposition 3.8], $T \# \psi \in \mathscr{B}(w)$.

Thus we have proved that $F_T(\varphi) \in \mathscr{E}(w)$.

Suppose now that $\{\varphi_\nu\}_{\nu \in \mathbb{N}}$ is a sequence in $\mathscr{E}(w)$ such that $\varphi_\nu \to \varphi$, as $\nu \to \infty$, in $\mathscr{E}(w)$, and $F_T(\varphi_\nu) \to \psi$, as $\nu \to \infty$, in $\mathscr{E}(w)$. By invoking Proposition 2.1, for every $x \in [0, \infty)$, $\tau_x \varphi_\nu \to \tau_x \varphi$, as $\nu \to \infty$, in $\mathscr{E}(w)$. Hence, since $T \in \mathscr{E}(w)'$, for every $x \in [0,\infty)$, $(T \# \varphi_\nu)(x) \to (T \# \varphi)(x)$, as $\nu \to \infty$. Moreover, since convergence in $\mathscr{E}(w)$ implies pointwise convergence in $[0,\infty)$, $(T \# \varphi_\nu)(x) \to \psi(x)$, as $\nu \to \infty$, for every $x \in [0,\infty)$. Hence $\psi = T \# \varphi$. Thus, by invoking the closed graph theorem, we conclude that F_T is a continuous mapping from $\mathscr{E}(w)$ into itself. $\square$

Note that if $\varphi \in \mathscr{E}(w)$ then φ defines an element T_φ in the dual space $\mathscr{B}(w)'$ of $\mathscr{B}(w)$ by

$$\langle T_\varphi, \psi\rangle = \int_0^\infty \varphi(x)\psi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x, \quad \psi \in \mathscr{B}(w).$$

Indeed, let $\psi \in \mathscr{B}^a(w)$, with $a > 0$. Then, there exists $\varphi' \in \mathscr{B}(w)$ such that $\varphi' = \varphi$, on $(0,a)$. Hence according to Parseval equality for Hankel transforms, we have

$$\begin{aligned}\langle T_\varphi, \psi\rangle &= \int_0^a \varphi(x)\psi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x\\ &= \int_0^\infty \varphi'(x)\psi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x\\ &= \int_0^\infty h_\mu(\varphi')(y)h_\mu(\psi)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}y.\end{aligned}$$

Then, it follows that

$$|\langle T_\varphi, \psi\rangle| \leqslant \sup_{y\in(0,\infty)} |h_\mu(\varphi')(y)| \int_0^\infty |h_\mu(\psi)(y)|\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}y \leqslant C\delta_0^\mu(\psi).$$

Thus the continuity of T_φ is shown.

Let $\varphi \in \mathscr{E}(w)$ and $S \in \mathscr{E}(w)'$. According to [2, Section 3] we can define the Hankel convolution $T_\varphi \# S$ of T_φ and S as the element of $\mathscr{B}(w)'$ given by

$$\langle T_\varphi \# S, \psi\rangle = \langle T_\varphi, S \# \psi\rangle = \int_0^\infty \varphi(x)(S \# \psi)(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x, \quad \psi \in \mathscr{B}(w).$$

On the other hand, by Proposition 2.3 the Hankel convolution $S \# \varphi$ defined through

$$(S \# \varphi)(x) = \langle S, \tau_x\varphi\rangle, \quad x \in (0,\infty),$$

is in $\mathscr{E}(w)$, and thus in $\mathscr{B}(w)'$.

In the following we will prove that $T_{S\#\varphi} = T_\varphi \# S$. Thus the distributional convolution defined in this Section can be seen as a special case of the generalized Hankel convolution introduced in [2, Section 3].

Proposition 2.4. *Let* $\varphi \in \mathscr{E}(w)$ *and* $S \in \mathscr{E}(w)'$. *Then* $T_{S\#\varphi} = T_\varphi \# S$.

P r o o f. Let $\psi \in \mathscr{B}^a(w)$, where $a > 0$. We are going to prove that

$$\int_0^\infty \langle S, \tau_x\varphi\rangle \psi(x) x^{2\mu+1}\,\mathrm{d}x = \int_0^\infty \varphi(x)\langle S, \tau_x\psi\rangle x^{2\mu+1}\,\mathrm{d}x.$$

As was shown in the proof of Proposition 2.3, we can write

$$\langle S, \tau_x\varphi\rangle = \langle S, \tau_x\varphi'\rangle, \quad x \in (0, a),$$

for some $\varphi' \in \mathscr{B}(w)$, such that $\varphi = \varphi'$, on $(0, b)$, for some $b > 0$ such that $S \# \psi \in \mathscr{B}^b(w)$ ([2, Proposition 3.8]).

Hence, by [2, Proposition 2.22], it follows that

$$\begin{aligned}\int_0^\infty \langle S, \tau_x\varphi\rangle \psi(x) x^{2\mu+1}\,\mathrm{d}x &= \int_0^a \langle S, \tau_x\varphi'\rangle \psi(x) x^{2\mu+1}\,\mathrm{d}x \\ &= \int_0^\infty \langle S, \tau_x\varphi'\rangle \psi(x) x^{2\mu+1}\,\mathrm{d}x \\ &= \int_0^\infty \varphi'(x)\langle S, \tau_x\psi\rangle x^{2\mu+1}\,\mathrm{d}x \\ &= \int_0^\infty \varphi(x)\langle S, \tau_x\psi\rangle x^{2\mu+1}\,\mathrm{d}x.\end{aligned}$$

Thus the proof is completed. □

We can define, according to Proposition 2.3, the Hankel convolution on $\mathscr{E}(w)' \times \mathscr{E}(w)'$. Let $T, S \in \mathscr{E}(w)'$. The Hankel convolution $T \# S$ of T and S is the functional on $\mathscr{E}(w)$ defined through

$$\langle T \# S, \varphi\rangle = \langle T, S \# \varphi\rangle, \quad \varphi \in \mathscr{E}(w).$$

Note that from Proposition 2.3 it follows that $T \# S \in \mathscr{E}(w)'$.

3. The Hankel transform on the space $\mathscr{E}(w)'$.

As was mentioned in the introduction (Section 1) the authors introduced in [3] the function space $\mathscr{H}_\mu(w)$ that can be seen as a Hankel version of the space $\mathscr{S}_w$ considered by G. Björck [9]. The dual space $\mathscr{H}_\mu(w)'$ of $\mathscr{H}_\mu(w)$ can be called a space of tempered Beurling type distributions. The Hankel transformation is an automorphism of $\mathscr{H}_\mu(w)$ and this transformation is defined on $\mathscr{H}_\mu(w)'$ by transposition.

That is, if $T \in \mathscr{H}_\mu(w)'$ the Hankel transform $h'_\mu T$ of T is the element of $\mathscr{H}_\mu(w)'$ defined by

$$\langle h'_\mu(T), \varphi \rangle = \langle T, h_\mu(\varphi) \rangle, \quad \varphi \in \mathscr{H}_\mu(w).$$

The space $\mathscr{E}(w)'$ can be identified with a subspace of $\mathscr{H}_\mu(w)'$. The Hankel transform h'_μ takes a special form on $\mathscr{E}(w)'$. If $T \in \mathscr{E}(w)'$ then the Hankel transform $h'_\mu(T)$ coincides with the element of $\mathscr{H}'_\mu$ generated by the function ([3, Proposition 3.1])

$$F(x) = 2^\mu \Gamma(\mu+1) \langle T(y), (xy)^{-\mu} J_\mu(xy) \rangle, \quad x \in [0, \infty).$$

That is, for every $\varphi \in \mathscr{H}_\mu(w)$,

$$\langle T, h_\mu(\varphi) \rangle = \int_0^\infty \langle T(y), (xy)^{-\mu} J_\mu(xy) \rangle \varphi(x) x^{2\mu+1} \, \mathrm{d}x.$$

We will write

$$h'_\mu(T)(x) = 2^\mu \Gamma(\mu+1) \langle T(y), (xy)^{-\mu} J_\mu(xy) \rangle, \quad x \in [0, \infty),$$

provided that $T \in \mathscr{E}(w)'$.

Note that, according to [2, Proposition 3.4], if $T \in \mathscr{E}(w)'$ the function $h'_\mu(T)$ can be extended to the whole complex plane by defining

$$h'_\mu(T)(z) = 2^\mu \Gamma(\mu+1) \langle T(y), (yz)^{-\mu} J_\mu(yz) \rangle, \quad z \in \mathbb{C},$$

because, for every $z \in \mathbb{C}$, the function f_z defined by $f_z(y) = (yz)^{-\mu} J_\mu(yz)$, $y \in \mathbb{C}$, is in the space $\mathscr{H}_{\mathrm{e}}(\mathbb{C})$ of even and entire functions. According to [2, Proposition 3.4], $h'_\mu(T)$ is also in $\mathscr{H}_{\mathrm{e}}(\mathbb{C})$. Indeed, we have that

$$f_z(y) = \sum_{k=0}^\infty (-1)^k \frac{(yz)^{2k}}{2^{2k+\mu} k! \Gamma(\mu+k+1)}, \quad y, z \in \mathbb{C},$$

where the series converges in $\mathscr{H}_{\mathrm{e}}(\mathbb{C})$, for every $z \in \mathbb{C}$. Then, since convergence in $\mathscr{H}_{\mathrm{e}}(\mathbb{C})$ implies convergence in $\mathscr{E}(w)$, we conclude that

$$h'_\mu(T)(z) = 2^\mu \Gamma(\mu+1) \sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k+\mu} k! \, \Gamma(\mu+k+1)} \langle T(y), y^{2k} \rangle, \quad z \in \mathbb{C}.$$

Hence $h'_\mu(T)$ is an even and entire function.

We now prove a Paley-Wiener type theorem for the Hankel transforms of the distributions in $\mathscr{E}(w)'$. We characterize those even and entire functions that can be represented as Hankel transforms of functionals in $\mathscr{E}(w)'$.

Proposition 3.1. *Let $a > 0$. Assume that F is an even and entire function. The following assertions are equivalent.*

(i) *There exists $\lambda > 0$ such that, for every $\varepsilon > 0$, we can find $C_\varepsilon > 0$ for which*

$$\int_{-\infty}^{\infty} |F(\chi + \mathrm{i}\eta)| \mathrm{e}^{-\lambda w(\chi)} \, \mathrm{d}\chi \leqslant C_\varepsilon \mathrm{e}^{(a+\varepsilon)|\eta|}, \quad \eta \in \mathbb{R}.$$

(ii) *There exists $\lambda > 0$ such that, for every $\varepsilon > 0$, we can find $C_\varepsilon > 0$ for which*

$$|F(\chi + \mathrm{i}\eta)| \leqslant C_\varepsilon \mathrm{e}^{\lambda w(\chi) + (a+\varepsilon)|\eta|}, \quad \chi, \eta \in \mathbb{R}.$$

(iii) *There exists $T \in \mathscr{E}(w)'$ such that $F = h'_\mu(T)$ and that $\langle T, \varphi \rangle = 0$, provided that $\varphi \in \mathscr{B}(w)$ and $\varphi(x) = 0$, $x \leqslant a + \varepsilon$, for some $\varepsilon > 0$.*

Proof. (i) $\Rightarrow$ (ii). It is sufficient to take into account that w satisfies the property in (1.1) and to use the Cauchy integral formula as in [9, p. 365].

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. We define the functional T on $\mathscr{H}_\mu(w)$ by

$$\langle T, \varphi \rangle = \int_0^\infty F(x) h_\mu(\varphi)(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, \mathrm{d}x, \quad \varphi \in \mathscr{H}_\mu(w).$$

Then $T \in \mathscr{H}_\mu(w)'$. Indeed, since w satisfies the property (γ), we have

$$\begin{aligned} |\langle T, \varphi \rangle| &\leqslant C \int_0^\infty |F(x)| \, |h_\mu(\varphi)(x)| x^{2\mu+1} \, \mathrm{d}x \\ &\leqslant C \int_0^\infty |h_\mu(\varphi)(x)| \mathrm{e}^{\lambda w(x)} x^{2\mu+1} \, \mathrm{d}x \\ &\leqslant C \beta^\mu_{l,0}(\varphi), \quad \varphi \in \mathscr{H}_\mu(w). \end{aligned}$$

Here λ is given by (ii) and $l \in \mathbb{N}$ is chosen large enough.

We now choose a function $\psi \in \mathscr{B}^1(w)$ such that $\int_0^\infty \psi(x) x^{2\mu+1} \, \mathrm{d}x = 2^\mu \Gamma(\mu+1)$ [2, Proposition 2.18]). According to [3, Proposition 2.9], for every $\varphi \in \mathscr{H}_\mu(w)$, $\varphi \# \psi_m \to \varphi$, as $m \to \infty$, in $\mathscr{H}_\mu(w)$, where $\psi_m(x) = m^{2\mu+1} \psi(mx)$, $x \in [0, \infty)$ and $m \in \mathbb{N}$. Hence $T \# \psi_m \to T$, as $m \to \infty$, in the weak* topology of $\mathscr{H}_\mu(w)'$.

We define, for every $m \in \mathbb{N}$, $T_m = T \# \psi_m$. From the distributional interchange formula [3, Proposition 3.5] it follows that

$$h'_\mu(T_m) = h'_\mu(T) h_\mu(\psi_m), \quad m \in \mathbb{N}. \tag{3.1}$$

Moreover, for every $\varphi \in \mathscr{H}_\mu(w)$, one has

$$\begin{aligned} \langle h'_\mu(T), \varphi \rangle &= \langle T, h_\mu(\varphi) \rangle \\ &= \int_0^\infty F(x) h_\mu(h_\mu \varphi)(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, \mathrm{d}x \\ &= \int_0^\infty F(x) \varphi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, \mathrm{d}x. \end{aligned}$$

Hence $h'_\mu(T)$ coincides with the functional generated by F on $\mathscr{H}_\mu(w)$. Thus (3.1) can be rewritten as

$$h'_\mu(T_m) = Fh_\mu(\psi_m), \quad m \in \mathbb{N}. \tag{3.2}$$

By taking now into account [2, Propositions 2.6 and 2.10], where we established a Paley-Wiener type theorem for Hankel transform of the functions in $\mathscr{B}(w)$, and our hypothesis (ii), we deduce from (3.2) that $T_m \in \mathscr{B}^{a+1/m}(w)$, $m \in \mathbb{N}$.

Let $\varphi \in \mathscr{B}(w)$ such that $\varphi(x) = 0$, $x \leqslant a + \varepsilon$, for some $\varepsilon > 0$. Since $T_m(x) = 0$, provided that $x \geqslant a + \varepsilon$ and m is large enough, we have

$$\langle T, \varphi\rangle = \lim_{m\to\infty} \langle T_m, \varphi\rangle = \lim_{m\to\infty} \int_{a+\varepsilon}^{\infty} T_m(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x = 0.$$

Thus (iii) is established.

(iii) $\Rightarrow$ (i). Suppose that $F = h'_\mu T$, for some $T \in \mathscr{E}(w)'$. By proceeding as in the proof of Proposition 3.1 we can see that, for a certain $a > 0$, the following property holds: $\langle T, \varphi\rangle = 0$, provided that $\varphi \in \mathscr{E}(w)$ and $\varphi(x) = 0$, when $x \geqslant a + \varepsilon$, for some $\varepsilon > 0$.

Let $\varepsilon > 0$. According to [2, Proposition 2.18] we choose a function $\zeta \in \mathscr{B}^{a+\varepsilon}(w)$ such that $\zeta(x) = 1$, $x \in (0, a + \frac{1}{2}\varepsilon)$. Then, we can write

$$\langle T, \psi\rangle = \langle T, \zeta\psi\rangle + \langle T, (1-\zeta)\psi\rangle = \langle T, \zeta\psi\rangle, \quad \psi \in \mathscr{E}(w).$$

Since $T \in \mathscr{B}^{a+\varepsilon}(w)'$, there exists $C > 0$ and $l \in \mathbb{N}$ for which

$$|F(z)| \leqslant C\int_0^\infty \mathrm{e}^{lw(x)}|h_\mu((tz)^{-\mu}J_\mu(tz)\zeta(t))(x)|x^{2\mu+1}\,\mathrm{d}x, \quad z \in \mathbb{C}.$$

We now argue, with minor modifications, as in the proof of [9, Theorem 1.4.1, pp. 366–367] to obtain after some manipulations

$$\begin{aligned}\int_{-\infty}^{\infty} &|h_\mu(((\chi+\mathrm{i}\eta)t)^{-\mu}J_\mu((\chi+\mathrm{i}\eta)t)\zeta(t))(x)|\mathrm{e}^{-\lambda w(\chi)}\,\mathrm{d}\chi\\ &= \int_{-\infty}^{\infty} |h_\mu((xt)^{-\mu}J_\mu(xt)\zeta(t))(\chi+\mathrm{i}\eta)|\mathrm{e}^{-\lambda w(\chi)}\,\mathrm{d}\chi\\ &\leqslant C\int_{-\infty}^{\infty} |h_\mu((xt)^{-\mu}J_\mu(xt)\zeta(t))(\chi)|\mathrm{e}^{-\lambda w(\chi)/K}\,\mathrm{d}\chi\mathrm{e}^{(a+\varepsilon)|\eta|},\\ &\qquad\qquad \eta, \chi \in \mathbb{R}, \quad \text{and} \quad \lambda > 0.\end{aligned}$$

To see the last inequality we must take into account that all the functions are even and [9, Lemma 1.3.11]. Here K represents the positive constant appearing in (1.1).

We choose $k \in \mathbb{N}$ such that $\int_0^\infty \mathrm{e}^{-kw(t)}\,\mathrm{d}t < \infty$. This is possible because w satisfies the property (γ). If $\lambda = K(Kl + k)$ we can write

$$\begin{aligned}
&\int_{-\infty}^{\infty} |F(\chi + \mathrm{i}\eta)| \mathrm{e}^{-\lambda w(\chi)}\,\mathrm{d}\chi \\
&\quad \leqslant C \int_{-\infty}^{\infty} \int_0^{\infty} |h_\mu(((\chi + \mathrm{i}\eta)t)^{-\mu} J_\mu((\chi + \mathrm{i}\eta)t)\zeta(t))(x)| \\
&\qquad\qquad \times \mathrm{e}^{-\lambda w(\chi) + lw(x)} x^{2\mu+1}\,\mathrm{d}x\,\mathrm{d}\chi \\
&\quad = C \int_0^{\infty} \mathrm{e}^{lw(x)} \int_{-\infty}^{\infty} |h_\mu(((\chi + \mathrm{i}\eta)t)^{-\mu} J_\mu((\chi + \mathrm{i}\eta)t)\zeta(t))(x)| \\
&\qquad\qquad \times \mathrm{e}^{-\lambda w(\chi)}\,\mathrm{d}\chi x^{2\mu+1}\,\mathrm{d}x \\
&\quad \leqslant C \int_0^{\infty} \mathrm{e}^{-\lambda w(\chi)/K} \int_0^{\infty} \mathrm{e}^{lw(x)} \tau_\chi(|h_\mu(\zeta)|)(x) x^{2\mu+1}\,\mathrm{d}x\,\mathrm{d}\chi\,\mathrm{e}^{(a+\varepsilon)|\eta|} \\
&\quad \leqslant C \int_0^{\infty} \mathrm{e}^{(lK - \lambda/K)w(\chi)} \int_0^{\infty} x^{2\mu+1} \int_{|x-\chi|}^{x+\chi} \mathrm{e}^{lw(z)} |h_\mu(\zeta)(z)| \\
&\qquad\qquad \times D(x,\chi,z) z^{2\mu+1}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\chi\,\mathrm{e}^{(a+\varepsilon)|\eta|} \\
&\quad \leqslant C \int_0^{\infty} \mathrm{e}^{-kw(\chi)}\,\mathrm{d}\chi \int_0^{\infty} \mathrm{e}^{lw(z)} |h_\mu(\zeta)(z)| z^{2\mu+1}\,\mathrm{d}z\,\mathrm{e}^{(a+\varepsilon)|\eta|}, \quad \eta \in \mathbb{R}.
\end{aligned}$$

We have used that w satisfies the property in (1.1) and that the Hankel translation operator is a contractive operator in $L_{1,\mu}$ ([24, p. 17]).

Thus the proof is finished. □

Next we prove a useful consequence of Proposition 3.1.

Proposition 3.2. *Let $T \in \mathscr{E}(w)'$. Then, for certain $\lambda, a > 0$ the following property holds: for every $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $C > 0$ for which*

$$\Big|\Big(\frac{1}{z}D\Big)^k h'_\mu(T)(z)\Big| \leqslant C\mathrm{e}^{\lambda w(\operatorname{Re} z) + (a+\varepsilon)|\operatorname{Im} z|}, \quad z \in \mathbb{C}.$$

P r o o f. We can write

$$h'_\mu(T)(z) = 2^\mu \Gamma(\mu+1) \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{2^{2l+\mu} l!\,\Gamma(\mu+l+1)} \langle T(y), y^{2l}\rangle, \quad z \in \mathbb{C}.$$

Hence, for every $k \in \mathbb{N}$,

$$\begin{aligned}
&\Big(\frac{1}{z}D\Big)^k h'_\mu(T)(z) \\
&\quad = (-1)^k 2^\mu \Gamma(\mu+1) \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{2^{2l+\mu+k} l!\,\Gamma(\mu+k+l+1)} \langle T(y), y^{2(l+k)}\rangle \\
&\quad = 2^\mu \Gamma(\mu+1) \langle T(y), (-1)^k (zy)^{-\mu-k} J_{\mu+k}(zy) y^{2k}\rangle \\
&\quad = 2^\mu \Gamma(\mu+1)(-1)^k \langle y^{2k} T(y), (zy)^{-\mu-k} J_{\mu+k}(zy)\rangle, \quad z \in \mathbb{C}.
\end{aligned}$$

Thus we see that

$$\Big(\frac{1}{z}D\Big)^k h'_\mu(T)(z) = \frac{\Gamma(\mu+1)(-1)^k}{\Gamma(\mu+k+1)2^k} h'_{\mu+k}(T_k)(z), \quad z\in\mathbb{C} \quad \text{and} \quad k\in\mathbb{N},$$

where $T_k = y^{2k}T \in \mathscr{E}(w)'$, for every $k\in\mathbb{N}$.

Now the proof can be finished by arguing as in the proof of Proposition 3.1. □

In [3, Section 3] we defined the Hankel convolution on $\mathscr{H}_\mu(w)' \times \mathscr{H}_\mu(w)$ as follows. If $T\in\mathscr{H}_\mu(w)'$ and $\varphi\in\mathscr{H}_\mu(w)$ the Hankel convolution $T\,\#\,\varphi$ of T and φ is given by

$$(T\,\#\,\varphi)(x) = \langle T, \tau_x\varphi\rangle, \quad x\in[0,\infty).$$

In [3, Proposition 3.2] it was established that, for every $T\in\mathscr{H}_\mu(w)'$ and $\varphi\in\mathscr{H}_\mu(w)$, $T\,\#\,\varphi$ is a continuous function on $[0,\infty)$ and there exist $C>0$ and $r\in\mathbb{N}$ for which $|(T\,\#\,\varphi)(x)| \leqslant C\mathrm{e}^{rw(x)}$, $x\in[0,\infty)$. Moreover, according to Proposition 2.3, if $T\in\mathscr{E}(w)'$ and $\varphi\in\mathscr{H}_\mu(w)$ then $T\,\#\,\varphi\in\mathscr{E}(w)$.

In the following we complete the above results.

Proposition 3.3. *Let $T\in\mathscr{E}(w)'$ and $\varphi\in\mathscr{H}_\mu(w)$. Then $T\,\#\,\varphi$ is in $\mathscr{H}$.*

P r o o f. Since $\mathscr{H}_\mu(w)$ is contained in $\mathscr{E}(w)$, according to Proposition 3.3, $T\#\varphi\in\mathscr{E}(w)$. Moreover, by [3, Proposition 3.5], $h'_\mu(T\,\#\,\varphi) = h'_\mu(T)h_\mu(\varphi)$. Hence, from Proposition 3.2 and by using the Leibniz rule, we infer that, for every $m,n\in\mathbb{N}$,

$$\sup_{x\in(0,\infty)} \mathrm{e}^{mw(x)}\Big|\Big(\frac{1}{x}D\Big)^n\big(h'_\mu(T)(x)h_\mu(\varphi)(x)\big)\Big| < \infty. \tag{3.3}$$

In particular, since w satisfies the property (γ), the function $h'_\mu(T)h_\mu(\varphi)$ is in $\mathscr{H}$. Hence

$$h_\mu(h'_\mu(T)h_\mu(\varphi)) = h'_\mu(h'_\mu(T\,\#\,\varphi)) = T\,\#\,\varphi$$

is also in $\mathscr{H}$ ([1, Satz 5]). □

Next we establish a Paley-Wiener type theorem concerning, in some sense, the singular support of the distributions in $\mathscr{E}(w)'$.

Proposition 3.4. *Let $T \in \mathscr{E}(w)'$ and $a > 0$. We list three properties.*

(i) *There exists a function $f \in \mathscr{E}(w)$ on (a, ∞), that is, $f\varphi \in \mathscr{B}(w)$ provided that $\varphi \in \mathscr{B}(w)$ and $\varphi(x) = 0$, $x \leqslant a + \varepsilon$, for some $\varepsilon > 0$, such that T coincides with the distribution generated by f on (a, ∞), that is, if $\varphi \in \mathscr{B}(w)$ and $\varphi(x) = 0$, $x \in (0, a + \varepsilon)$, for a certain $\varepsilon > 0$, then*

$$\langle T, \varphi \rangle = \int_0^\infty f(x)\varphi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} \, \mathrm{d}x.$$

(ii) *There exists $n \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$, we can find $C_m > 0$ for which*

$$|h'_\mu(T)(\chi + \mathrm{i}\eta)| \leqslant C_m \mathrm{e}^{nw(\chi)+(a+1/m)|\eta|}, \quad |\eta| \leqslant mw(\chi), \quad \chi, \eta \in \mathbb{R}.$$

(iii) *There exists a smooth function f on (a, ∞) such that T coincides with the distribution generated by f on (a, ∞).*

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii). Let $m \in \mathbb{N} \setminus \{0\}$. According to [2, Proposition 2.18] we choose a function $\zeta \in \mathscr{B}^{a+1/(2m)}(w)$ such that $\zeta(x) = 1$, $x \in (0, a + 1/(4m))$. Then, we can write

$$\langle T, \varphi \rangle = \langle T, \zeta\varphi \rangle + \langle T, (1 - \zeta)\varphi \rangle, \quad \varphi \in \mathscr{B}(w).$$

It is clear that the functional $T_1 = T\zeta$ is in $\mathscr{B}(w)'$ and $\langle T_1, \psi \rangle = 0$, provided that $\psi \in \mathscr{B}(w)$ and $\psi(x) = 0$, $x \leqslant a + 1/(2m) + \varepsilon$, for a certain $\varepsilon > 0$.

On the other hand, by defining $T_2 = T - T_1$ one has

$$\langle T_2, \varphi \rangle = \langle T, (1 - \zeta)\varphi \rangle = \langle T, (1 - \zeta)\beta\varphi \rangle, \quad \varphi \in \mathscr{B}(w),$$

where β is a suitable function in $\mathscr{B}(w)$. We have taken into account that $T \in \mathscr{E}(w)'$.

Since, for every $\varphi \in \mathscr{B}(w)$, the function $(1 - \zeta)\beta\varphi \in \mathscr{B}(w)$ and $(1 - \zeta(x))\beta(x) \times \varphi(x) = 0$, $x \in (0, a + 1/(4m))$, it follows that

$$\begin{aligned} \langle T_2, \varphi \rangle &= \int_a^\infty f(x)(1 - \zeta(x))\beta(x)\varphi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} \, \mathrm{d}x \\ &= \langle f(1 - \zeta)\beta, \varphi \rangle, \quad \varphi \in \mathscr{B}(w), \end{aligned}$$

or, in other words, $T_2 = f(1 - \zeta)\beta$. Moreover, since $f \in \mathscr{E}(w)$ on (a, ∞), $f(1 - \zeta)\beta$ is in $\mathscr{B}(w)$.

From Proposition 3.1 we infer that there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$|h'_\mu(T_1)(\chi + \mathrm{i}\eta)| \leqslant C\mathrm{e}^{nw(\chi)+(a+1/m)|\eta|}, \quad \chi, \eta \in \mathbb{R}.$$

Moreover, by [2, Proposition 2.6], for every $l \in \mathbb{N}$ there exists $C > 0$ such that

$$|h'_\mu(T_2)(\chi + \mathrm{i}\eta)| \leqslant C\mathrm{e}^{-lw(\chi)+b|\eta|}, \quad \chi, \eta \in \mathbb{R},$$

for a certain $b > 0$.

Hence, we can write, for every $l \in \mathbb{N}$,

$$|h'_\mu(T)(\chi + \mathrm{i}\eta)| \leqslant C(\mathrm{e}^{nw(\chi)+(a+1/m)|\eta|} + \mathrm{e}^{(-l+bm)w(\chi)}),$$

for each $\chi, \eta \in \mathbb{R}$ and $|\eta| \leqslant mw(\chi)$.

Then, by choosing l large enough we obtain

$$|h'_\mu(T)(\chi + \mathrm{i}\eta)| \leqslant C\mathrm{e}^{nw(\chi)+(a+1/m)|\eta|}, \quad \chi, \eta \in \mathbb{R} \quad \text{and} \quad |\eta| \leqslant mw(\chi).$$

(ii) $\Rightarrow$ (iii). According to [10, Remark 1.2, (c)], we can assume that w is a smooth function having bounded derivatives in $[0, \infty)$.

Our objective is to find a smooth function f on (a, ∞) such that

$$\langle T, \varphi \rangle = \int_a^\infty f(x)\varphi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} \,\mathrm{d}x,$$

provided that $\varphi \in \mathscr{B}(w)$ and $\varphi(x) = 0$, $x \leqslant a + \varepsilon$, for some $\varepsilon > 0$.

We choose $\varphi \in \mathscr{B}^1(w)$ such that $\int_0^\infty \varphi(x)x^{2\mu+1} \,\mathrm{d}x = 2^\mu \Gamma(\mu + 1)$ and we define, for every $k \in \mathbb{N}$, $\varphi_k(x) = k^{2\mu+2}\varphi(kx)$, $x \in [0, \infty)$. It is clear that $\varphi_k \in \mathscr{B}(w)$, for each $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. We can write by [4, (3.1)]

$$\begin{aligned}(T \,\#\, \varphi_k)(x) &= \langle T, \tau_x \varphi_k \rangle = \langle h'_\mu(T), h_\mu(\tau_x \varphi_k) \rangle \\ &= \int_0^\infty h'_\mu(T)(y) h_\mu(\varphi_k)(y)(xy)^{-\mu} J_\mu(xy) y^{2\mu+1} \,\mathrm{d}y, \quad x \in (0, \infty).\end{aligned}$$

We now prove that, for each $x \in (0, \infty)$ and $m \in \mathbb{N}$,

$$\begin{aligned}\int_0^\infty & h'_\mu(T)(y) h_\mu(\varphi_k)(y)(xy)^{-\mu} J_\mu(xy) y^{2\mu+1} \,\mathrm{d}y \\ &= \frac{1}{2} \int_{\Gamma_m(w)} h'_\mu(T)(y) h_\mu(\varphi_k)(y)(xy)^{-\mu} H_\mu^{(1)}(xy) y^{2\mu+1} \,\mathrm{d}y,\end{aligned} \tag{3.4}$$

where $\Gamma_m(w)$ represents a Hörmander type path ([18]), that is, a representation of $\Gamma_m(w)$ is the following

$$z(t) = t + mw(t)\mathrm{i}, \quad |t| \geqslant 1,$$

and

$$z(t) = t + mw(1)\mathrm{i}, \quad |t| \leqslant 1,$$

for every $m \in \mathbb{N}$. Here $H_\mu^{(1)}$ denotes the Hankel function of the first kind and order μ ([25, p. 73]).

Let $x \in (0, \infty)$ and $m \in \mathbb{N}$. To see (3.4) we proceed firstly as in [14, Lemma 6.1]. Thus we show that, for every $m \in \mathbb{N}$,

$$\begin{aligned}\int_0^\infty & h'_\mu(T)(y)h_\mu(\varphi_k)(y)(xy)^{-\mu}J_\mu(xy)y^{2\mu+1}\,\mathrm{d}y \\ &= \frac{1}{2}\int_{-\infty}^\infty h'_\mu(T)(y+\mathrm{i}mw(1))h_\mu(\varphi_k)(y+\mathrm{i}mw(1)) \\ &\qquad \times (x(y+\mathrm{i}mw(1)))^{-\mu}H_\mu^{(1)}(x(y+\mathrm{i}mw(1)))(y+\mathrm{i}mw(1))^{2\mu+1}\,\mathrm{d}y.\end{aligned}$$

We now use the well known Cauchy theorem. According to [2, Proposition 2.6] and Proposition 3.1 and by invoking [14, (5.3.c) and (5.3.d)], we can deduce that, for a certain $d > 0$ and for every $n \in \mathbb{N}$, there exists $C > 0$ such that

$$|h'_\mu(T)(\chi+\mathrm{i}\eta)h_\mu(\varphi_k)(\chi+\mathrm{i}\eta)(x(\chi+\mathrm{i}\eta))^{-\mu}H_\mu^{(1)}(x(\chi+\mathrm{i}\eta))| \leqslant C\mathrm{e}^{-x\eta+d|\eta|-nw(\chi)},$$

for $\chi, \eta \in \mathbb{R}$ and $\chi^2+\eta^2 \geqslant 1$.

Hence, if $\chi, \eta \in \mathbb{R}$, $\chi^2+\eta^2 \geqslant 1$ and $0 \leqslant \eta \leqslant mw(\chi)$, one has, for each $n \in \mathbb{N}$,

$$|h'_\mu(T)(\chi+\mathrm{i}\eta)h_\mu(\varphi_k)(\chi+\mathrm{i}\eta)(x(\chi+\mathrm{i}\eta))^{-\mu}H_\mu^{(1)}(x(\chi+\mathrm{i}\eta))| \leqslant C\mathrm{e}^{-nw(\chi)}.$$

Thus, we can deduce that

$$\int_{\mathscr{C}_{R,m}} h'_\mu(T)(z)h_\mu(\varphi_k)(z)(xz)^{-\mu}H_\mu^{(1)}(xz)z^{2\mu+1}\,\mathrm{d}z \to 0, \quad \text{as } R \to \infty,$$

where $\mathscr{C}_{R,m}$ represents the piece of the circle $\chi^2+\eta^2 = R^2$ between $\eta = mw(1)$ and the path $\eta = mw(\chi)$.

Hence we have shown that (3.4) holds.

We have proved that, for every $x \in (0, \infty)$ and $m \in \mathbb{N}$,

$$(T \,\#\, \varphi_k)(x) = \frac{1}{2}\int_{\Gamma_m(w)} h'_\mu(T)(y)h_\mu(\varphi_k)(y)(xy)^{-\mu}H_\mu^{(1)}(xy)y^{2\mu+1}\,\mathrm{d}y. \tag{3.5}$$

Suppose now that $\varepsilon > 0$ and $x \geqslant a+\varepsilon$. By invoking again Proposition 3.1 and [14, (5.3.c)], we can write, for a certain $n \in \mathbb{N}$,

$$\begin{aligned}|(x(\chi+\mathrm{i}\eta))^{-\mu}H_\mu^{(1)}(x(\chi+\mathrm{i}\eta))h'_\mu(T)(\chi+\mathrm{i}\eta)| &\leqslant C\mathrm{e}^{(m(a+\varepsilon/2-x)+n)w(\chi)} \\ &\leqslant C\mathrm{e}^{(-\varepsilon m/2+n)w(\chi)},\end{aligned}$$

for each $\eta = mw(\chi)$, $\eta^2+\chi^2 \geqslant 1$ and $m \in \mathbb{N}$.

Moreover, from [14, (5.3.b)] one has

$$|h_\mu(\varphi_k)(z)| = |h_\mu(\varphi)(z/k)| \leqslant C\mathrm{e}^{|\mathrm{Im}\, z|/k} \int_0^1 |\varphi(x)| x^{2\mu+1}\, \mathrm{d}x, \quad z \in \mathbb{C} \text{ and } k \in \mathbb{N}.$$

Then, by taking into account that w satisfies the property (γ) and the additional smoothness properties assumed at the beginning of the proof, we can choose a large enough $m \in \mathbb{N}$ so that the integral in (3.5) is absolutely convergent and there exists a continuous function g defined on $\mathbb{R}$ such that g is absolutely integrable on $\mathbb{R}$ and

$$|h'_\mu(T)(y) h_\mu(\varphi_k)(y)(xy)^{-\mu} H_\mu^{(1)}(xy) y^{2\mu+1}|_{y=mw(t)} \leqslant g(t), \quad t \in \mathbb{R},$$

and $k \in \mathbb{N}$ is large enough. Here m and g do not depend on $x \geqslant a+\varepsilon$.

Also it is not hard to see that

$$h_\mu(\varphi_k)(y) = h_\mu(\varphi)(y/k) \to h_\mu(\varphi)(0) = 1, \quad \text{as } k \to \infty,$$

for every $y \in (0,\infty)$.

Hence, the dominated convergence theorem allows us to conclude that if $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that, for every $x > a+\varepsilon$,

$$(T \# \varphi_k)(x) \to \frac{1}{2} \int_{\Gamma_{m_\varepsilon}(w)} h'_\mu(T)(y)(xy)^{-\mu} H_\mu^{(1)}(xy) y^{2\mu+1}\, \mathrm{d}y, \quad \text{as } k \to \infty.$$

Let $\psi \in \mathscr{B}^b(w)$, with $b > a$, such that $\psi(x) = 0$, $x \leqslant a+\varepsilon$, for some $\varepsilon > 0$. We can write

$$\begin{aligned} (3.6)\quad \langle T, \psi\rangle &= \lim_{k\to\infty} \langle T \# \varphi_k, \psi\rangle \\ &= \lim_{k\to\infty} \int_0^\infty (T \# \varphi_k)(x)\psi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, \mathrm{d}x \\ &= \frac{1}{2} \int_0^\infty \psi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \int_{\Gamma_{m_\varepsilon}(w)} h'_\mu(T)(y)(xy)^{-\mu} H_\mu^{(1)}(xy) y^{2\mu+1}\, \mathrm{d}y\, \mathrm{d}x, \end{aligned}$$

for a certain $m_\varepsilon \in \mathbb{N}$.

We now define a function f on (a,∞) as follows. For every $k \in \mathbb{N}$, we choose $m_{1/k}$ such that $m_{1/k} < m_{1/(k+1)} - 1$, and we define the function f_k by

$$f_k(x) = \frac{1}{2} \int_{\Gamma_{m_{1/k}}(w)} h'_\mu(T)(y)(xy)^{-\mu} H_\mu^{(1)}(xy) y^{2\mu+1}\, \mathrm{d}y, \quad x \in (a+1/k, \infty).$$

Note that if $x \in (a+1/k,\infty)$ and $l \in \mathbb{N}$, $l > k$, then $f_k(x) = f_l(x)$. The function f is defined by

$$f(x) = f_k(x), \quad x \in (a+1/k,\infty) \quad \text{and} \quad k \in \mathbb{N}.$$

According to (3.6) we can write

$$\langle T, \psi \rangle = \int_a^\infty f(x)\psi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, \mathrm{d}x,$$

provided that $\psi \in \mathscr{B}(w)$ and $\psi(x) = 0$, $x \leqslant a + \varepsilon$, for some $\varepsilon > 0$.

Moreover, f is smooth on (a, ∞). Indeed, let $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that

$$f(x) = \frac{1}{2} \int_{\Gamma_{m_{1/k}}(w)} h'_\mu(T)(y)(xy)^{-\mu} H_\mu^{(1)}(xy) y^{2\mu+1} \, \mathrm{d}y, \quad x \in (a+\varepsilon, \infty) \text{ and } k \geqslant k_0.$$

According to [14, (5.1.b)], by proceeding as above, we can show that, for every $l \in \mathbb{N}$

$$\Big(\frac{1}{x} D\Big)^l f(x) = \frac{(-1)^l}{2} \int_{\Gamma_{m_{1/k}}(w)} h'_\mu(T)(y)(xy)^{-\mu-l} H_{\mu+l}^{(1)}(xy) y^{2l+2\mu+1} \, \mathrm{d}y,$$
$$x \in (a+\varepsilon, \infty),$$

where k is large enough.

Thus the proof is finished. □

As was mentioned in Section 1, if $w(x) = \log(1+x)$, $x \in [0, \infty)$, then the space $\mathscr{B}(w)$ coincides with the space $\mathscr{B}$. In that case, the space $\mathscr{E}(w)$ of multipliers of $\mathscr{B}$ is the space $\mathscr{E}$ of complex valued and smooth functions on $(0, \infty)$ such that, for every $k \in \mathbb{N}$, the limit

$$\lim_{x \to 0} \Big(\frac{1}{x} D\Big)^k f(x)$$

exists. In particular, if f is a smooth function on (a, b), where $0 < a < b \leqslant \infty$, then $f\varphi \in \mathscr{B}$, for every $\varphi \in \mathscr{B}$ such that $\varphi(x) = 0$, $x \notin (c, d)$, where $a < c < d < b$. The converse of the last assertion is also true.

Although the function $w(x) = \log(1+x)$, $x \in [0, \infty)$, does not satisfy the property (γ), by proceeding as in the proof of Proposition 3.4 we can obtain the following result that seems to be new in the theory of Hankel transformation on Zemanian spaces.

Proposition 3.5. *Let $T \in \mathscr{E}'$ and $a > 0$. The following two assertions are equivalent.*

(i) *There exists a smooth function f on (a, ∞) such that*

$$\langle T, \varphi \rangle = \int_a^\infty f(x)\varphi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, \mathrm{d}x,$$

for every $\varphi \in \mathscr{B}$ such that $\varphi(x) = 0$, $x \leqslant a + \varepsilon$, for some $\varepsilon > 0$.

(ii) *There exists* $n \in \mathbb{N}$ *such that, for every* $m \in \mathbb{N}$, *we can find* $C_m > 0$, *for which*

$$|h'_\mu(T)(\chi+\mathrm{i}\eta)| \leqslant C_m(1+|\chi|)^n \mathrm{e}^{(a+1/m)|\eta|}, \quad \chi, \eta \in \mathbb{R} \quad \text{and} \quad |\eta| \leqslant m\log(1+|\chi|).$$

Remark 3. We do not know if the function f defined in the proof of (ii) $\Rightarrow$ (iii) in Proposition 4.4 has the following property: $f\varphi \in \mathscr{B}(w)$, for every $\varphi \in \mathscr{B}(w)$ such that $\varphi(x) = 0$, $x \leqslant a+\varepsilon$, for some $\varepsilon > 0$. We conjecture that f satisfies this property and then the statements (i) and (ii) in Proposition 3.4 are equivalent. Note that the Proposition 3.5 states that this property holds when $w(x) = \log(1+x)$, $x \in [0,\infty)$.

An immediate consequence of Proposition 3.5 is the following.

Proposition 3.6. *Let* $T \in \mathscr{E}'$ *and* $a > 0$. *Assume that* P *is a polynomial. Then the following two assertions are equivalent.*

(i) *There exists a smooth function* f *on* (a,∞) *such that*

$$\langle T, \varphi\rangle = \int_a^\infty f(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x,$$

for every $\varphi \in \mathscr{B}$ *such that* $\varphi(x) = 0$, $x \leqslant a+\varepsilon$, *for some* $\varepsilon > 0$.

(ii) *There exists a smooth function* f *on* (a,∞) *for which*

$$\langle P(\Delta_\mu)T, \varphi\rangle = \int_a^\infty f(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,\mathrm{d}x,$$

for every $\varphi \in \mathscr{B}$ *such that* $\varphi(x) = 0$, $x \leqslant a+\varepsilon$, *for some* $\varepsilon > 0$.

P r o o f. It is sufficient to take into account [1, Lemma 8, (b), (6)] and Proposition 3.5. □

Note that from Proposition 3.6 it follows, in particular, that if $T \in \mathscr{E}'$ and $P(\Delta_\mu)T$ is smooth on $(0,\infty)$, for some polynomial P, then T is also smooth on $(0,\infty)$.

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