Jaroslav Ježek

A note on idempotent modifications of groups

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 1, 229–231

Persistent URL: http://dml.cz/dmlcz/127879

Czechoslovak Mathematical Journal, 54 (129) (2004), 229–231

A NOTE ON IDEMPOTENT MODIFICATIONS OF GROUPS

J. Ježek, Praha

(Received July 19, 2001)

Abstract. The idempotent modification of a group is always a subdirectly irreducible algebra.

Keywords: simple algebra, idempotent, group

MSC 2000: 08B26

The *idempotent modification* of an algebra A is the algebra A' obtained from A by (preserving the underlying set and) modifying the basic operations in the following way: if f is an n-ary basic operation of A, then the operation f' defined by

$$f'(a_1,\dots,a_n)=\begin{cases} a_1 & \text{if } a_1=\dots=a_n,\\ f(a_1,\dots,a_n) & \text{otherwise}\end{cases}$$

is a basic operation of A'.

Let us consider the following property of a class C of algebras: the idempotent modification of an arbitrary algebra from C is subdirectly irreducible. The aim of this paper is to prove that the variety of groups enjoys this property.

Theorem 1. *The idempotent modification of a group is a subdirectly irreducible algebra.*

The proof will be divided into several lemmas. Let $(G,\cdot)$ be a group and $(G,\cdot)$ be its idempotent modification, i.e.,

$$a\circ b=\begin{cases} a & \text{if } a=b,\\ ab & \text{otherwise.}\end{cases}$$

While working on this paper the author was partially supported by the Grant Agency of the Czech Republic, grant 201/99/0263 and by the institutional grant MSM113200007.

Let $\sim$ be a congruence of $(G, \circ)$.

Lemma 2. *$a \sim 1$ if and only if $a^{-1} \sim 1$.*

Proof. It is sufficient to prove that $a \sim 1$ implies $a^{-1} \sim 1$. If $a^{-1} = a$, there is nothing to prove. Let $a^{-1} \neq a$. Then $a^{-1} \circ a \sim a^{-1} \circ 1$ gives $1 \sim a^{-1}$. $\square$

Lemma 3. *$a \sim 1$ implies $a^2 \sim 1$.*

Proof. This is clear if $a^2 = 1$. Let $a^2 \neq 1$. We have $a \circ a^2 \sim 1 \circ a^2$, i.e., $a^3 \sim a^2$. If $a^3 = 1$, we are done. So, let $a^3 \neq 1$. We have $a^3 \circ a^{-1} \sim a^2 \circ a^{-1} = a^2 a^{-1} = a \sim 1$. If $a^3 \neq a^{-1}$, this means that $a^2 \sim 1$. If $a^3 = a^{-1}$, then $a^3 \sim 1$ by Lemma 2, and this together with $a^3 \sim a^2$ gives $a^2 \sim 1$. $\square$

Lemma 4. *$\{a\colon\ a \sim 1\}$ is a subgroup of G.*

Proof. By Lemma 2, it is sufficient to prove that $a \sim 1$ and $b \sim 1$ imply $ab \sim 1$. This is clear if $a \neq b$. If $a = b$, it follows from Lemma 3. $\square$

Lemma 5. *If $a \sim b$ where $a \neq b$ and $a^2 \neq 1$, then $a \sim b \sim 1$.*

Proof. We have $a \circ a \sim a \circ b$, i.e., $a \sim ab$. Hence $a^{-1} \circ a \sim a^{-1} \circ ab$, i.e., $1 \sim a^{-1} \circ ab$. If $a^{-1} \neq ab$, we get $1 \sim b$ and we are done. If $a^{-1} = ab$ then $a \sim ab = a^{-1}$, so that $a \circ a \sim a \circ a^{-1}$ and thus $a \sim 1$. $\square$

Lemma 6. *If $a \sim b$ where $a \neq b$ and $a^2 \neq 1$, then $x \sim 1$ for all $x \in G$ such that $x^2 \neq 1$.*

Proof. We have $a \sim b \sim 1$ by Lemma 5. Let $x^2 \neq 1$. We have $a \circ x \sim b \circ x$. If either $x = a$ or $x = b$, then $x \sim 1$ and we are done. Otherwise, $ax \sim bx$. Hence $a^{-1} \circ ax \sim a^{-1} \circ bx$. If $a^{-1} = ax$, then $x = a^{-2}$ and $x \sim 1$ by Lemma 4. Otherwise, $x \sim a^{-1} \circ bx$. If $x \neq a^{-1} \circ bx$, then we are done by Lemma 5. Let $x = a^{-1} \circ bx$. If $a^{-1} \neq bx$, then $x = a^{-1}bx$, so that $a = b$, a contradiction. Hence $a^{-1} = bx$. But then $x = a^{-1} \sim 1$. $\square$

Lemma 7. *If $a \sim b$ where $a \neq b$, then $x \sim 1$ for all $x \in G$ such that $x^2 \neq 1$.*

Proof. By Lemma 6, it is sufficient to consider the case when $a^2 = b^2 = 1$. Let $x^2 \neq 1$. We have $a \circ x \sim b \circ x$, i.e., $ax \sim bx$. Hence $a \circ ax \sim a \circ bx$, i.e., $x \sim a \circ bx$. If $x \neq a \circ bx$, we can use Lemma 6. So, let $x = a \circ bx$.

If $a \neq bx$, we get $x = abx$, so that $ab = 1$ and $a = b$, a contradiction. Hence $a = bx$, i.e., $x = ba$. Since $a \circ a \sim b \circ a$, we have $a \sim ba = x$ and we can use Lemma 6. $\square$

Lemma 8. *If* $\sim$ *is nontrivial, then* $x^2 = 1$ *for all* $x \in G$.

Proof. Suppose that $\sim$ is nontrivial and there exists an element $x \in G$ with $x^2 \neq 1$. By Lemma 7, the block of $\sim$ containing 1 contains all such elements x. Let y be an element outside this block, so that $y^2 = 1$ and $y \neq 1$. We have $y \circ 1 \sim y \circ x$, i.e., $y \sim yx$. Hence $y \circ y \sim y \circ yx$, i.e., $y \sim yyx = x$, a contradiction. □

Lemma 9. *Let* G *be a group satisfying* $x^2 = 1$ *for all* x. *Then* $(G - \{1\})^2 \cup \mathrm{id}$ *is the only nontrivial congruence of* $(G, \circ)$.

Proof. Clearly, this relation is a congruence of $(G, \circ)$. Let $\sim$ be a nontrivial congruence of $(G, \circ)$. If $x \sim 1$ for an element $x \neq 1$, then for any element $y \notin \{x, 1\}$ we have $xy \sim y$, $xy \circ y \sim y$, $xyy \sim y$, $x \sim y$, $y \sim 1$. If $x \sim y$ for two distinct elements x, y different from 1, then for any $z \notin \{x, y, 1\}$ we have $xz \sim yz$, $xxz \sim x \circ yz$, $z \sim x \circ yz$; if $x = yz$, we get $z \sim x$; otherwise, we get $z \sim xyz$, $z \sim xyzz = xy \sim x$. □

We have finished the proof of Theorem 1. In fact, we have proved more:

Theorem 10. *The idempotent modification of a group* G *is always simple, unless the group satisfies* $x^2 = 1$ *for all* x*; in this last case, the congruence lattice of the idempotent modification is the three-element chain.*

It would be interesting to find other varieties with the property of Theorem 1. In particular, we can ask: Does there exist a variety V of quasigroups, not contained in the variety of groups, such that the idempotent modification of any quasigroup from V is subdirectly irreducible?

References

[1] *R. McKenzie, G. McNulty and W. Taylor*: Algebras, Lattices, Varieties, Volume I. Wadsworth & Brooks/Cole, Monterey, 1987.

Author's address: J. Ježek, MFF UK, Sokolovská 83, 186 75 Praha 8, e-mail: `jezek@karlin.mff.cuni.cz`.