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ON UNIFORMLY LOCALLY COMPACT QUASI-UNIFORM HYPERSPACES

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Abstract. We characterize those Tychonoff quasi-uniform spaces $(X,\mathscr{U})$ for which the Hausdorff-Bourbaki quasi-uniformity is uniformly locally compact on the family $\mathscr{K}_0(X)$ of nonempty compact subsets of X. We deduce, among other results, that the Hausdorff-Bourbaki quasi-uniformity of the locally finite quasi-uniformity of a Tychonoff space X is uniformly locally compact on $\mathscr{K}_0(X)$ if and only if X is paracompact and locally compact. We also introduce the notion of a co-uniformly locally compact quasi-uniform space and show that a Hausdorff topological space is σ-compact if and only if its (lower) semicontinuous quasi-uniformity is co-uniformly locally compact. A characterization of those Hausdorff quasi-uniform spaces $(X,\mathscr{U})$ for which the Hausdorff-Bourbaki quasi-uniformity is co-uniformly locally compact on $\mathscr{K}_0(X)$ is obtained.

Keywords: Hausdorff-Bourbaki quasi-uniformity, hyperspace, locally compact, cofinally complete, uniformly locally compact, co-uniformly locally compact

MSC 2000: 54E15, 54B20, 54D45

1. Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ will denote the set of real numbers, the set of integer numbers and the set of positive integer numbers, respectively.

Our basic reference for quasi-uniform and quasi-metric spaces is [9], for general topology it is [6], and for hyperspaces it is [1].

Let us recall that a quasi-uniformity on a set X is a filter $\mathscr{U}$ on $X\times X$ such that
(i) for each $U\in\mathscr{U}$, $\{(x,x)\colon x\in X\}\subseteq U$;
(ii) for each $U\in\mathscr{U}$ there is $V\in\mathscr{U}$ such that $V^2\subseteq U$.

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A quasi-uniform space is a pair $(X, \mathscr{U})$ such that X is a (nonempty) set and $\mathscr{U}$ is a quasi-uniformity on X.

Each quasi-uniformity $\mathscr{U}$ on X generates a topology $\mathscr{T}(\mathscr{U}) = \{G \subseteq X\colon$ for each $x \in G$ there is $U \in \mathscr{U}$ such that $U(x) \subseteq G\}$, where, as usual, $U(x) = \{y \in X\colon (x,y) \in U\}$.

If $\mathscr{U}$ is a quasi-uniformity on X then the filter $\mathscr{U}^{-1} = \{U^{-1}\colon U \in \mathscr{U}\}$ is also a quasi-uniformity on X, called the conjugate of $\mathscr{U}$ (as usual, $U^{-1} = \{(x,y)\colon (y,x) \in U\}$). Furthermore, the filter $\mathscr{U}^s := \mathscr{U} \vee \mathscr{U}^{-1}$ is the coarsest uniformity on X finer than $\mathscr{U}$.

Given a topological space X we denote by $\mathscr{P}_0(X)$ (resp. $\mathscr{CL}_0(X)$, $\mathscr{K}_0(X)$), the family of all nonempty subsets (resp. nonempty closed subsets, nonempty compact subsets) of X. If $(X, \mathscr{U})$ is a quasi-uniform space, $\mathscr{CL}_0(X)$ (resp. $\mathscr{K}_0(X)$) denotes the family of all nonempty $\mathscr{T}(\mathscr{U})$-closed subsets (resp. nonempty $\mathscr{T}(\mathscr{U})$-compact subsets) of X.

The Hausdorff-Bourbaki quasi-uniformity of a quasi-uniform space $(X, \mathscr{U})$ is defined as the quasi-uniformity $\mathscr{U}_*$ on $\mathscr{P}_0(X)$ which has as a base the family of sets of the form

$$U_H = \{(A,B) \in \mathscr{P}_0(X) \times \mathscr{P}_0(X)\colon\ B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\},$$

whenever $U \in \mathscr{U}$ ([2], [14]).

The restriction of $\mathscr{U}_*$ to $\mathscr{CL}_0(X)$ and to $\mathscr{K}_0(X)$ is also denoted by $\mathscr{U}_*$ if no confusion arises.

A filter $\mathscr{F}$ on a (quasi-)uniform space $(X, \mathscr{U})$ is weakly Cauchy ([5], [8], [9]) provided that for each $U \in \mathscr{U}$, $\bigcap_{F \in \mathscr{F}} U^{-1}(F) \neq \emptyset$. A (quasi-)uniformity $\mathscr{U}$ on a set X is called cofinally complete ([7], [13]) if every weakly Cauchy filter on $(X, \mathscr{U})$ has a cluster point. In this case, we say that $(X, \mathscr{U})$ is a cofinally complete quasi-uniform space.

It is well known that a Tychonoff space is paracompact if and only if its fine uniformity is cofinally complete ([5], [7], [8]).

In our context a topological space X will be called locally compact if each point of X has a neighborhood whose closure is compact.

A (quasi-)uniformity $\mathscr{U}$ on a set X is said to be uniformly locally compact ([8], [9]) if there is $U \in \mathscr{U}$ such that for each $x \in X$, $\overline{U(x)}$ is compact. In this case, we say that $(X, \mathscr{U})$ is a uniformly locally compact (quasi-)uniform space.

It is well known ([7], [8], [9]) that a Tychonoff (quasi-)uniform space is uniformly locally compact if and only if it is locally compact and cofinally complete, and, hence, the fine uniformity of a Tychonoff space X is uniformly locally compact if and only if X is paracompact and locally compact.

Several authors have discussed the preservation of local compactness and uniform local compactness by the Vietoris topology and the Hausdorff-Bourbaki (quasi-)uniformity, respectively. Thus, Michael proved in [15] (see also [4]) that a Tychonoff space X is locally compact if and only if the Vietoris topology of X is locally compact on $\mathscr{K}_0(X)$. In [3] Burdick characterized uniform local compactness of $(\mathscr{CL}_0(X), \mathscr{U}_*)$ in the case that $\mathscr{U}$ is a uniformity on X and showed that the Hausdorff-Bourbaki uniformity of the Euclidean uniformity on $\mathbb{R}$ is not locally compact on $\mathscr{CL}_0(X)$. However, its restriction to $\mathscr{K}_0(\mathbb{R})$ is uniformly locally compact because it was proved in [13] that a uniform space $(X, \mathscr{U})$ is uniformly locally compact if and only if $(\mathscr{K}_0(X), \mathscr{U}_*)$ is uniformly locally compact. In [13], p. 140, a uniformly locally compact quasi-uniform (actually, quasi-metric) space $(X, \mathscr{U})$ is constructed such that $(\mathscr{K}_0(X), \mathscr{U}_*)$ is not cofinally complete and, thus, not uniformly locally compact.

These results and examples suggest the problem of characterizing those (Tychonoff) quasi-uniform spaces $(X, \mathscr{U})$ for which the Hausdorff-Bourbaki quasi-uniformity is uniformly locally compact on $\mathscr{K}_0(X)$. Here we solve this problem and deduce conditions under which the Hausdorff-Bourbaki quasi-uniformity of the locally finite quasi-uniformity and of the (lower) semicontinuous quasi-uniformity, respectively, is uniformly locally compact on $\mathscr{K}_0(X)$. In particular, we prove that the Hausdorff-Bourbaki quasi-uniformity of the locally finite quasi-uniformity of a Tychonoff space X is uniformly locally compact on $\mathscr{K}_0(X)$ if and only if X is paracompact and locally compact. The quasi-metric case is also discussed. Finally, we introduce the notion of a co-uniformly locally compact quasi-uniform space and show that a Hausdorff space is σ-compact if and only if its semicontinuous quasi-uniformity is co-uniformly locally compact. A characterization of Hausdorff quasi-uniform spaces $(X, \mathscr{U})$ for which the Hausdorff-Bourbaki quasi-uniformity is co-uniformly locally compact on $\mathscr{K}_0(X)$ is obtained.

2. Uniform local compactness of $(\mathscr{K}_0(X), \mathscr{U}_*)$

The following lemmas will be useful in establishing our main result.

Lemma 1 ([8], [9]). *A quasi-uniform space is uniformly locally compact if and only if it is locally compact and cofinally complete.*

Let us recall that a quasi-uniformity $\mathscr{U}$ on a set X is precompact ([9]) provided that for each $U \in \mathscr{U}$ there is a finite subset A of X such that $U(A) = X$. $\mathscr{U}$ is said to be totally bounded ([9]) if $\mathscr{U}^s$ is a totally bounded uniformity on X. Of course, $\mathscr{U}$ is totally bounded if and only if $\mathscr{U}^{-1}$ is so. It is well known that every totally bounded quasi-uniformity is precompact but the converse does not hold in general.

Lemma 2. *Let $(X,\mathscr{U})$ be a compact quasi-uniform space such that for each $K \in \mathscr{K}_0(X)$, $\mathscr{U}^{-1}|K$ is precompact. Then $\mathscr{U}^{-1}$ is hereditarily precompact.*

Proof. Let A be a nonempty subset of X. Since $(X,\mathscr{U})$ is compact, the closure $\overline{A}$ of A in $(X,\mathscr{T}(\mathscr{U}))$ belongs to $\mathscr{K}_0(X)$. So $\mathscr{U}^{-1}|\overline{A}$ is precompact. It immediately follows that $\mathscr{U}^{-1}|A$ is precompact. □

Following [19], a filter $\mathscr{F}$ on a quasi-uniform space $(X,\mathscr{U})$ is left K-Cauchy provided that for each $U \in \mathscr{U}$ there is $F_U \in \mathscr{F}$ such that $U(x) \in \mathscr{F}$ for all $x \in F_U$. $(X,\mathscr{U})$ is said to be left K-complete if every left K-Cauchy filter is convergent ([19]).

Since every left K-Cauchy filter converges to all of its cluster points ([19]) and each left K-Cauchy filter is weakly Cauchy, every cofinally complete quasi-uniform space is left K-complete.

Lemma 3 ([11]). *Let $(X,\mathscr{U})$ be a T_1 quasi-uniform space and let A be a precompact subspace of $(X,\mathscr{U})$. If $(\mathscr{K}_0(X),\mathscr{U}_*)$ is left K-complete, hence $\mathscr{U}^{-1}|A$ is precompact.*

Lemma 4. *Let $(X,\mathscr{U})$ be a T_1 quasi-uniform space. Then $(\mathscr{K}_0(X),\mathscr{U}_*)$ is compact if and only if $(X,\mathscr{U})$ is compact and $\mathscr{U}^{-1}$ is hereditarily precompact.*

Proof. It is proved in [12] that if $(X,\mathscr{U})$ is a T_1 quasi-uniform space, then $(\mathscr{P}_0(X),\mathscr{U}_*)$ is compact if and only if $(X,\mathscr{U})$ is compact and $\mathscr{U}^{-1}$ is hereditarily precompact, and it is proved in [11] that $(\mathscr{K}_0(X),\mathscr{U}_*)$ is compact if and only if $(\mathscr{P}_0(X),\mathscr{U}_*)$ is compact. Combining these results we obtain the conclusion. □

Theorem 1. *Let $(X,\mathscr{U})$ be a Tychonoff quasi-uniform space. Then $(\mathscr{K}_0(X),\mathscr{U}_*)$ is uniformly locally compact if and only if $(X,\mathscr{U})$ is uniformly locally compact and for each $K \in \mathscr{K}_0(X)$, $\mathscr{U}^{-1}|K$ is precompact.*

Proof. Suppose that $(\mathscr{K}_0(X),\mathscr{U}_*)$ is uniformly locally compact. By [13], Proposition 3.1 and Remark 3.2, $(X,\mathscr{U})$ is uniformly locally compact. Now let $K \in \mathscr{K}_0(X)$. Then K is a precompact subspace of $(X,\mathscr{U})$. Since, by Lemma 1, $(\mathscr{K}_0(X),\mathscr{U}_*)$ is cofinally complete and every cofinally complete quasi-uniform space is left K-complete, it follows from Lemma 3 that $\mathscr{U}^{-1}|K$ is precompact.

Conversely, by assumption there is $U \in \mathscr{U}$ such that for each $x \in X$, $\overline{U(x)}$ is compact. Let $V \in \mathscr{U}$ be such that $V^3 \subseteq U$. We shall prove that for each $K \in \mathscr{K}_0(X)$, $\overline{V_H(K)}$ is compact in $(\mathscr{K}_0(X),\mathscr{U}_*)$.

Let $K \in \mathscr{K}_0(X)$. We first show that $\overline{V(K)}$ is a compact subset of $(X,\mathscr{U})$. Indeed, let $(z_\lambda)_{\lambda\in\Lambda}$ be a net in $\overline{V(K)}$. Then there is a finite subset $\{x_1,\ldots,x_n\}$ of K such that $K \subseteq \bigcup_{i=1}^{n} V(x_i)$. On the other hand, for each $\lambda \in \Lambda$ there is $a_\lambda \in K$ with

$z_\lambda \in \overline{V^2(a_\lambda)}$. (In fact, fix $\lambda \in \Lambda$. Since $z_\lambda \in \overline{V(K)}$, for each $W \in \mathscr{U}$ there exists $b_W \in K$ such that $W(z_\lambda) \cap V(b_W) \neq \emptyset$; by compactness of K, the net $(b_W)_{W \in \mathscr{U}}$ has a cluster point $a_\lambda \in K$; so $z_\lambda \in \overline{V^2(a_\lambda)}$.)

Since $a_\lambda \in V(x_i)$ for some $i \in \{1, \ldots, n\}$, we deduce that $V^2(a_\lambda) \subseteq V^3(x_i)$ and thus $\{z_\lambda \colon \lambda \in \Lambda\} \subseteq \bigcup_{i=1}^{n} \overline{V^3(x_i)}$. Hence, the net $(z_\lambda)_{\lambda \in \Lambda}$ is contained in the compact set $\bigcup_{i=1}^{n} \overline{U(x_i)}$, so it has a cluster point which obviously belongs to $\overline{V(K)}$. We conclude that $\overline{V(K)}$ is a compact subset of $(X, \mathscr{U})$.

By our hypothesis, $\mathscr{U}^{-1}\big|K'$ is precompact for each $K' \in \mathscr{K}_0(X)$ and, in particular, for each $K' \in \mathscr{K}_0(\overline{V(K)})$. Therefore, by Lemma 2, $\mathscr{U}^{-1}\big|\overline{V(K)}$ is hereditarily precompact and by Lemma 4, $(\mathscr{K}_0(\overline{V(K)}), \mathscr{U}_*)$ is compact.

Finally, let $(K_\lambda)_{\lambda \in \Lambda}$ be a net in $\overline{V_H(K)}$. Then $K_\lambda \subseteq \overline{V(K)}$ and thus $K_\lambda \in \mathscr{K}_0(\overline{V(K)})$ for all $\lambda \in \Lambda$. Hence, there is $C \in \mathscr{K}_0(\overline{V(K)})$ which is a cluster point of $(K_\lambda)_{\lambda \in \Lambda}$ with respect to $\mathscr{T}(\mathscr{U}_*)$. Obviously $C \in \mathscr{K}_0(X)$ and $C \in \overline{V_H(K)}$. We have shown that $\overline{V_H(K)}$ is compact in $(\mathscr{K}_0(X), \mathscr{U}_*)$ for all $K \in \mathscr{K}_0(X)$. Hence $(\mathscr{K}_0(X), \mathscr{U}_*)$ is uniformly locally compact. □

Corollary 1 ([13])**.** *Let $(X, \mathscr{U})$ be a Tychonoff uniform space. Then $(\mathscr{K}_0(X), \mathscr{U}_*)$ is uniformly locally compact if and only if $(X, \mathscr{U})$ is uniformly locally compact.*

Next we apply Theorem 1 to study uniform local compactness on $\mathscr{K}_0(X)$ of the Hausdorff-Bourbaki quasi-uniformity corresponding to some canonical quasi-uniformities of a Tychonoff space X.

Following [9] we denote by $\mathscr{P}$, $\mathscr{LF}$, $\mathscr{PF}$ and $\mathscr{SC}$ the Pervin quasi-uniformity, the locally finite quasi-uniformity, the point finite quasi-uniformity and the (lower) semicontinuous quasi-uniformity, respectively, of a (topological) space X.

Let us recall that a cover $\mathscr{G}$ of a topological space X is said to be compact finite if each compact subset of X meets only finitely many members of $\mathscr{G}$.

Let $\mathscr{A}$ be the collection of all compact finite open covers $\mathscr{G}$ of a topological space X. For each $\mathscr{G} \in \mathscr{A}$ let $U_{\mathscr{G}} = \bigcup_{x \in X} (\{x\} \times \bigcap\{G \in \mathscr{G} \colon x \in G\})$. Then $\{U_{\mathscr{G}} \colon \mathscr{G} \in \mathscr{A}\}$ is a subbase for a transitive quasi-uniformity $\mathscr{CF}$ for X, called the *compact finite* quasi-uniformity of X.

It is clear that $\mathscr{P} \subseteq \mathscr{LF} \subseteq \mathscr{CF} \subseteq \mathscr{PF}$.

On the other hand, observe that if X is a locally compact space then $\mathscr{LF} = \mathscr{CF}$.

Lemma 5. *Let X be a space. Then $\mathscr{CF}\big|K$ is totally bounded for all $K \in \mathscr{K}_0(X)$.*

P r o o f. Let $\mathscr{A}$ be the collection of all finite open covers $\mathscr{G}$ of X. For each $\mathscr{G} \in \mathscr{A}$ let $U_{\mathscr{G}} = \bigcup_{x \in X} (\{x\} \times \bigcap\{G \in \mathscr{G} \colon x \in G\})$. It is well known ([9]) that $\{U_{\mathscr{G}} \colon \mathscr{G} \in \mathscr{A}\}$

is a subbase for the Pervin quasi-uniformity $\mathscr{P}$. So we immediately deduce that for each $K \in \mathscr{K}_0(X)$, $\mathscr{C}\mathscr{F}\big|K = \mathscr{P}\big|K$. Then the result follows from the well-known facts that $\mathscr{P}$ is totally bounded and that total boundedness is a hereditary property. □

Proposition 1. *Let X be a Tychonoff space. Then $(\mathscr{K}_0(X), \mathscr{C}\mathscr{F}_*)$ is uniformly locally compact if and only if $(X, \mathscr{C}\mathscr{F})$ is uniformly locally compact.*

P r o o f. It is an immediate consequence of Theorem 1 and Lemma 5. □

Proposition 2. *Let X be a space. Then $\mathscr{P}\mathscr{F} = \mathscr{C}\mathscr{F}$ if and only if $\mathscr{P}\mathscr{F}^{-1}\big|K$ is precompact for all $K \in \mathscr{K}_0(X)$.*

P r o o f. Suppose that $\mathscr{P}\mathscr{F}^{-1}\big|K$ is precompact for all $K \in \mathscr{K}_0(X)$. Let $K \in \mathscr{K}_0(X)$, $\mathscr{G}$ a point finite open cover of X and $U \in \mathscr{P}\mathscr{F}$ be such that $U(x) = \bigcap\{G \in \mathscr{G} \colon x \in G\}$ for all $x \in X$. Then there exists a finite subset F of K such that $K \subseteq U^{-1}(F)$. Let $G_0 \in \mathscr{G}$ be such that $G_0 \cap K \neq \emptyset$, then there exists $x \in F$ such that $G_0 \cap U^{-1}(x) \neq \emptyset$. Let $y \in G_0 \cap U^{-1}(x)$. Then $x \in U(y) = \bigcap\{G \in \mathscr{G} \colon y \in G\}$, so $x \in G_0$. It follows that $F \cap G_0 \neq \emptyset$. Since $\mathscr{G}$ is point finite and F is finite, $\mathscr{G}$ is compact finite. We conclude that $\mathscr{P}\mathscr{F} = \mathscr{C}\mathscr{F}$.

The converse follows from Lemma 5. □

Corollary 2. *Let X be a Tychonoff space such that $\mathscr{P}\mathscr{F} \neq \mathscr{C}\mathscr{F}$. Then $(\mathscr{K}_0(X), \mathscr{P}\mathscr{F}_*)$ is not uniformly locally compact.*

Denote by $\mathscr{F}\mathscr{N}$ the fine uniformity of a Tychonoff space X. It then follows from Corollary 1 and results cited in Section 1 that $(\mathscr{K}_0(X), \mathscr{F}\mathscr{N}_*)$ is uniformly locally compact if and only if X is paracompact and locally compact.

Here, we obtain the following result.

Proposition 3. *Let X be a Tychonoff space. Then $(\mathscr{K}_0(X), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact if and only if X is paracompact and locally compact.*

P r o o f. Suppose that $(\mathscr{K}_0(X), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact. By Theorem 1, $(X, \mathscr{L}\mathscr{F})$ is uniformly locally compact. Therefore X is locally compact and $\mathscr{L}\mathscr{F}$ is cofinally complete by Lemma 1. Hence X is paracompact by the third corollary of Theorem 2.2 in [8] that states that $\mathscr{L}\mathscr{F}$ is cofinally complete if and only if X is paracompact.

Conversely, if X is a paracompact locally compact space, then $\mathscr{L}\mathscr{F}$ is cofinally complete, so by Lemma 1, $\mathscr{L}\mathscr{F}$ is a uniformly locally compact quasi-uniformity. Furthermore, since $\mathscr{L}\mathscr{F} = \mathscr{C}\mathscr{F}$, it follows from Lemma 5 that for each $K \in \mathscr{K}_0(X)$, $\mathscr{L}\mathscr{F}^{-1}\big|K$ is precompact. Therefore $(\mathscr{K}_0(X), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact by Theorem 1. □

Remark 1. a) Note that the proof of Proposition 3 shows that if X is a (Tychonoff) paracompact locally compact space, then $(\mathscr{K}_0(X), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact and $\mathscr{L}\mathscr{F} = \mathscr{C}\mathscr{F}$.

b) We also have (compare Proposition 1) that for a Tychonoff space X, $(\mathscr{K}_0(X), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact if and only if $(X, \mathscr{L}\mathscr{F})$ is uniformly locally compact.

c) Since a space is metacompact if and only if $\mathscr{P}\mathscr{F}$ is cofinally complete ([8]), the existence of metacompact locally compact spaces that are not paracompact shows that uniform local compactness of $(X, \mathscr{P}\mathscr{F})$ does not imply uniform local compactness of $(\mathscr{K}_0(X), \mathscr{P}\mathscr{F}_*)$ in general.

Lemma 6. *Let X be a locally compact Tychonoff space, and suppose that X is not discrete. Then there exists a compact subspace K of X such that $\mathscr{S}\mathscr{C}^{-1}\big|K$ is not precompact.*

Proof. Let x be a non-isolated point of X and let K be a compact neighborhood of x. Then there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of open neighborhoods of x such that $G_n \subseteq K$ and $\overline{G_{n+1}} \subseteq G_n$ with $G_n \neq G_{n+1}$ for each $n \in \mathbb{N}$. Let $G = \bigcap\limits_{n\in\mathbb{N}} G_n = \bigcap\limits_{n\in\mathbb{N}} \overline{G_n}$, which is clearly a nonempty compact set, and let $A_{-n} = G_n \setminus G$ and $A_n = X$ for each $n \in \mathbb{N}$. It is clear that A_n is open and $A_n \subseteq A_{n+1}$ whenever $n \in \mathbb{Z}$, $\bigcup\limits_{n\in\mathbb{Z}} A_n = X$ and $\bigcap\limits_{n\in\mathbb{Z}} A_n = \emptyset$, and hence $\mathscr{A} = \{A_n \colon n \in \mathbb{Z}\}$ is an open spectrum. Let $U \in \mathscr{S}\mathscr{C}$ be such that $U(x) = \bigcap\{A \in \mathscr{A} \colon x \in A\}$ for all $x \in X$ (see [9], Theorem 2.12). It is easy to check that, for any finite subset F of X, $U^{-1}(F) = X \setminus A_n$ for some $n \in \mathbb{Z}$, and hence $\mathscr{S}\mathscr{C}^{-1}\big|K$ is not precompact, since $A_n \cap K \neq \emptyset$ for each $n \in \mathbb{Z}$. □

Lemma 7 ([8], [9]). *A Tychonoff space is Lindelöf if and only if $\mathscr{S}\mathscr{C}$ is cofinally complete.*

Proposition 4. *Let X be a Tychonoff space. Then $(\mathscr{K}_0(X), \mathscr{S}\mathscr{C}_*)$ is uniformly locally compact if and only if X is discrete and countable.*

Proof. Suppose that X is discrete and countable. By Lemmas 1 and 7, $(X, \mathscr{S}\mathscr{C})$ is uniformly locally compact. Clearly $\mathscr{S}\mathscr{C}^{-1}\big|K$ is precompact for all $K \in \mathscr{K}_0(X)$. Then $(\mathscr{K}_0(X), \mathscr{S}\mathscr{C}_*)$ is uniformly locally compact by Theorem 1.

Conversely, it follows from Theorem 1 that $(X, \mathscr{S}\mathscr{C})$ is uniformly locally compact and $\mathscr{S}\mathscr{C}^{-1}\big|K$ is precompact for all $K \in \mathscr{K}_0(X)$. Therefore X is discrete and Lindelöf by Lemmas 6 and 7, so it is discrete and countable. □

Example 1. From Propositions 3 and 4 it follows that $(\mathscr{K}_0(\mathbb{R}), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact while $(\mathscr{K}_0(\mathbb{R}), \mathscr{S}\mathscr{C}_*)$ is not.

Example 2. The result that $(\mathscr{K}_0(\mathbb{R}), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact can be extended to any Tychonoff topological group X of pointwise countable type whose left uniformity is cofinally complete. In fact, it was proved in [20] that such a topological group is locally compact and since every Tychonoff topological group of pointwise countable type is paracompact, it follows from Proposition 3 that $(\mathscr{K}_0(X), \mathscr{L}\mathscr{F}_*)$ is uniformly locally compact.

By $\mathscr{T}_V$ we shall denote the Vietoris topology of a topological space X.

Remark 2. It is proved in [18] that for a quasi-uniform space $(X, \mathscr{U})$, one has $\mathscr{T}_V = \mathscr{T}(\mathscr{U}_*)$ on $\mathscr{K}_0(X)$ if and only if $\mathscr{U}^{-1}|K$ is precompact for all $K \in \mathscr{K}_0(X)$.

Hence, by Theorem 1 and Proposition 3, the Hausdorff-Bourbaki quasi-uniformity on $\mathscr{K}_0(X)$ of the locally finite quasi-uniformity of any (Tychonoff) paracompact locally compact space X is compatible with the Vietoris topology of X.

Let (X, d) be a bounded quasi-pseudo-metric space. Then the Hausdorff-Bourbaki quasi-pseudo-metric d_* on $\mathscr{P}_0(X)$ is defined by

$$d_*(A, B) = \max\Big\{\sup_{b\in B} d(A, b), \sup_{a\in A} d(a, B)\Big\}$$

whenever $A, B \in \mathscr{P}_0(X)$ ([2], [14]).

The quasi-pseudo-metric d_* generates on $\mathscr{P}_0(X)$ the Hausdorff-Bourbaki quasi-uniformity of the quasi-uniformity $\mathscr{U}_d$, where $\mathscr{U}_d$ is the quasi-uniformity on X generated by d (see [9], page 3), i.e. $\mathscr{U}_{d_*} = (\mathscr{U}_d)_*$.

Let us recall ([16]) that a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-pseudo-metric space (X, d) is right K-Cauchy provided that for each $\varepsilon > 0$ there is n_ε such that $d(x_m, x_n) < \varepsilon$ whenever $m \geqslant n \geqslant n_\varepsilon$.

A quasi-metric space (X, d) is said to be uniformly locally compact if $(X, \mathscr{U}_d)$ is a uniformly locally compact quasi-uniform space.

Thus Theorem 1 can be restated for quasi-metric spaces, as follows.

Theorem 2. *Let (X, d) be a Tychonoff quasi-metric space. Then $(\mathscr{K}_0(X), d_*)$ is uniformly locally compact if and only if (X, d) is uniformly locally compact and each convergent sequence in X has a right K-Cauchy subsequence.*

P r o o f. Let (X, d) be a Tychonoff quasi-metric space such that $(\mathscr{K}_0(X), d_*)$ is uniformly locally compact. By Theorem 1, (X, d) is uniformly locally compact and for each $K \in \mathscr{K}_0(X)$, $(\mathscr{U}_d)^{-1}|K$ is precompact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X which is $\mathscr{T}(\mathscr{U}_d)$-convergent to a point $x \in X$. Let $Y = \{x\} \cup \{x_n \colon n \in \mathbb{N}\}$. Then $(Y, \mathscr{U}_d|Y)$

is a compact quasi-uniform space and for each $K \in \mathscr{K}_0(Y)$, $(\mathscr{U}_d)^{-1}\big|K$ is precompact because such a K also belongs to $\mathscr{K}_0(X)$. Thus $(\mathscr{U}_d)^{-1}\big|Y$ is hereditarily precompact by Lemma 2. Hence, the sequence $(x_n)_{n\in\mathbb{N}}$ has a right K-Cauchy subsequence by [10], Theorem 3.

Conversely, suppose that there is $K \in \mathscr{K}_0(X)$ such that $(\mathscr{U}_d)^{-1}\big|K$ is not precompact. Then there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in K and a $U \in \mathscr{U}_d$ such that $x_{n+1} \notin \bigcup_{i=1}^{n} U^{-1}(x_i)$ for all $n \in \mathbb{N}$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ has a cluster point (in K), it follows from our hypothesis that $(x_n)_{n\in\mathbb{N}}$ has a right K-Cauchy subsequence, which contradicts that $x_{n+1} \notin \bigcup_{i=1}^{n} U^{-1}(x_i)$ for all $n \in \mathbb{N}$. We conclude that for each $K \in \mathscr{K}_0(X)$, $(\mathscr{U}_d)^{-1}\big|K$ is precompact. Hence $(\mathscr{K}_0(X), d_*)$ is uniformly locally compact by Theorem 1. $\square$

Remark 3. Let $(X, \mathscr{U})$ be a Tychonoff uniform space. Since $(\mathscr{K}_0(X), \mathscr{U}_*)$ is (uniformly) locally compact if and only if $(X, \mathscr{U})$ is (uniformly) locally compact ([4], [15] and Corollary 1 above), it seems natural to conjecture, in the light of Lemma 1, that cofinal completeness is also preserved by the Hausdorff-Bourbaki uniformity on $\mathscr{K}_0(X)$, and thus, Corollary 1 follows in an easy and elegant way, as a factorization. Unfortunately, this is not the case, not even for metric spaces as is shown in [13], Remark 3.6. Furthermore, for a metric space (X, d), cofinal completeness of $(\mathscr{K}_0(X), d_*)$ is equivalent to its uniform local compactness and thus to uniform local compactness of (X, d) ([13], Proposition 3.4 and Corollary 3.5). In this direction, it seems interesting to note that if X is a metrizable topological group which admits a compatible cofinally complete metric d, then X is locally compact ([17]) and, hence, $(\mathscr{K}_0(X), d_*)$ is uniformly locally compact.

3. Co-uniform local compactness of $(\mathscr{K}_0(X), \mathscr{U}_*)$

In [21] the third author introduced and studied the notion of a D-co-Lebesgue quasi-uniform space. In particular, it is proved that the fine transitive quasi-uniformity of a space X is D-co-Lebesgue if and only if X is metacompact, and that a quasi-uniform space is D-co-Lebesgue if and only if it is cofinally co-complete, where a quasi-uniform space $(X, \mathscr{U})$ is called cofinally co-complete if every weakly Cauchy filter on $(X, \mathscr{U}^{-1})$ has a cluster point in $(X, \mathscr{T}(\mathscr{U}))$.

In this context it seems natural to propose an appropriate notion of "co-uniform local compactness". Thus, we say that a quasi-uniformity $\mathscr{U}$ on a set X is *co-uniformly locally compact* if there exists $U \in \mathscr{U}$ such that for each $x \in X$, $\overline{U^{-1}(x)}$ is compact. In this case, the quasi-uniform space $(X, \mathscr{U})$ is said to be *co-uniformly locally compact.* (Here, closure means closure in $(X, \mathscr{T}(\mathscr{U}))$.)

In Theorem 3 below we shall show that the characterization of uniform local compactness of $(\mathscr{K}_0(X), \mathscr{U}_*)$ obtained in Theorem 1 admits an analogous result for co-uniformly locally compact Hausdorff quasi-uniform spaces.

Similarly to the proofs of Proposition 5.32 and Theorem 5.33 of [9] we can show the following result.

Proposition 5. *Let $(X, \mathscr{U})$ be a quasi-uniform space. Then*

1. *$(X, \mathscr{U})$ is cofinally co-complete if and only if every directed open cover of $(X, \mathscr{U})$ is a quasi-uniform cover in $(X, \mathscr{U}^{-1})$.*
2. *If $(X, \mathscr{U})$ is locally compact and cofinally co-complete, then $(X, \mathscr{U})$ is co-uniformly locally compact.*
3. *If $(X, \mathscr{U})$ is co-uniformly locally compact, then $(X, \mathscr{U})$ is cofinally co-complete.*

Our next proposition shows that a co-uniformly locally compact quasi-uniform space need not be locally compact. We shall use the following observation.

Remark 4. It is well known ([9], Corollary 2.15) that if X is a space, then for each $U \in \mathscr{SC}$ there is a countable subset D of X such that $U(D) = X$. However, it can be easily proved that for each $U \in \mathscr{SC}$ there is a countable subset D of X such that $(U \cap U^{-1})(D) = X$. This fact will be used in the proof of Proposition 6 below.

Proposition 6. *A Hausdorff space is σ-compact if and only if $\mathscr{SC}$ is co-uniformly locally compact.*

P r o o f. Suppose that X is a Hausdorff σ-compact space. Then there is an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $X = \bigcup_{n=1}^{\infty} K_n$. Define $A_{-n} = X \setminus K_n$ and $A_n = X$ for each $n \in \mathbb{N}$. Thus $\{A_n \colon n \in \mathbb{Z}\}$ is an open spectrum. So $U = \bigcup_{n\in\mathbb{Z}} (A_n \setminus A_{n-1}) \times A_n$ is an element of $\mathscr{SC}$ (see [9], Theorem 2.12). Now let $x \in X$. There is $n \in \mathbb{N}$ such that $x \in A_{-n} \setminus A_{-(n+1)}$. Therefore $U^{-1}(x) = X \setminus A_{-(n+1)} = K_{n+1}$. We conclude that $\mathscr{SC}$ is co-uniformly locally compact.

Conversely, let X be a (not necessarily Hausdorff) space such that $\mathscr{SC}$ is co-uniformly locally compact. Let $U \in \mathscr{SC}$ such that $\overline{U^{-1}(x)}$ is compact for all $x \in X$. By Remark 4, there is a countable subset D of X such that $(U \cap U^{-1})(D) = X$. Hence $X = \bigcup_{x\in D} U^{-1}(x) = \bigcup_{x\in D} \overline{U^{-1}(x)}$. Therefore X is σ-compact. $\square$

However, we obtain the following result.

Proposition 7. *Let X be a space such that $\mathscr{L}\mathscr{F}$ is co-uniformly locally compact. Then X is locally compact.*

Proof. Let $\mathscr{G}$ be a locally finite open cover of X such that $\overline{U^{-1}(x)}$ is compact for all $x \in X$, where $U = \bigcup\limits_{x \in X} \{x\} \times \bigcap\{G \in \mathscr{G} \colon x \in G\}$. Let $x \in X$ and let A be an open neighborhood of x such that $A \subseteq U(x)$ and A meets only a finite number of elements of $\mathscr{G}$, namely $G_1, \ldots, G_n$.

For each finite subset J of $\{1, \ldots, n\}$ such that $A \cap \Big(\bigcap\limits_{j \in J} G_j\Big) \neq \emptyset$, let $x_J \in A \cap \Big(\bigcap\limits_{j \in J} G_j\Big)$. We show that $A \subseteq \bigcup\limits_J U^{-1}(x_J)$. Indeed, let $y \in A$ and $J_y = \{j \in \{1, \ldots, n\} \colon y \in G_j\}$. Then $x_{J_y} \in \bigcap\limits_{j \in J_y} G_j$, so $y \in U^{-1}(x_J)$. Hence $A \subseteq \bigcup\limits_J U^{-1}(x_J)$, and thus $\overline{A}$ is a compact neighborhood of x. □

Let us recall ([9]) that a quasi-uniformity $\mathscr{U}$ on a set X is said to be point symmetric if $\mathscr{T}(\mathscr{U}) \subseteq \mathscr{T}(\mathscr{U}^{-1})$.

Lemma 8. *Let $(X, \mathscr{U})$ be a cofinally co-complete T_1 quasi-uniform space. Then $\mathscr{U}$ is point symmetric.*

Proof. Let $\mathscr{F}$ be a filter on X which is $\mathscr{T}(\mathscr{U}^{-1})$-convergent to a point $x \in X$. Then $\mathscr{F}$ is a weakly Cauchy filter on $(X, \mathscr{U}^{-1})$ and thus it has a $\mathscr{T}(\mathscr{U})$-cluster point $y \in X$. Since $(X, \mathscr{U})$ is T_1, it follows that $x = y$, so $\mathscr{F}$ is $\mathscr{T}(\mathscr{U})$-convergent to x. We conclude that $\mathscr{U}$ is point symmetric. □

Lemma 9. *Let $(X, \mathscr{U})$ be a quasi-uniform space such that $(\mathscr{K}_0(X), \mathscr{U}_*)$ is point symmetric. Then $\mathscr{U}^{-1}\big|K$ is precompact for each $K \in \mathscr{K}_0(X)$.*

Proof. Let $K \in \mathscr{K}_0(X)$ and $U \in \mathscr{U}$. Then, there exists $V \in \mathscr{U}$ such that $V_H^{-1}(K) \subseteq U_H(K)$. Since K is compact there exists a finite subset K' of K such that $K \subseteq V(K')$, so $K \in V_H(K')$ and hence $K' \in V_H^{-1}(K) \subseteq U_H(K)$. Thus $K \subseteq U^{-1}(K')$. We conclude that $\mathscr{U}^{-1}\big|K$ is precompact. □

Theorem 3. *Let $(X, \mathscr{U})$ be a Hausdorff quasi-uniform space. Then $(\mathscr{K}_0(X), \mathscr{U}_*)$ is co-uniformly locally compact if and only if $(X, \mathscr{U})$ is co-uniformly locally compact and $\mathscr{U}^{-1}\big|K$ is precompact for each $K \in \mathscr{K}_0(X)$.*

Proof. Suppose that $(\mathscr{K}_0(X), \mathscr{U}_*)$ is co-uniformly locally compact. By Proposition 5, $(\mathscr{K}_0(X), \mathscr{U}_*)$ is cofinally co-complete. Furthermore, it is a Hausdorff quasi-uniform space because Hausdorffness is preserved by the Vietoris topology $\mathscr{T}_V$ of $(X, \mathscr{T}(\mathscr{U}))$ on $\mathscr{K}_0(X)$, and $\mathscr{T}_V \subseteq \mathscr{T}(\mathscr{U}_*)$ on $\mathscr{K}_0(X)$ (see [18]). Therefore $\mathscr{U}_*$ is point symmetric on $\mathscr{K}_0(X)$ by Lemma 8. It follows from Lemma 9 that $\mathscr{U}^{-1}\big|K$ is precompact for each $K \in \mathscr{K}_0(X)$.

Let us prove that $(X, \mathscr{U})$ is co-uniformly locally compact. Let $U \in \mathscr{U}$ be such that $\overline{U_H^{-1}(K)}$ is compact for each $K \in \mathscr{K}_0(X)$. Then $\overline{U^{-1}(x)}$ is compact for each $x \in X$. Indeed, given $x \in X$ let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in $\overline{U^{-1}(x)}$. It easily follows that $\{x_\lambda\} \in \overline{U_H^{-1}(\{x\})}$ for each $\lambda \in \Lambda$ and since $\overline{U_H^{-1}(\{x\})}$ is compact, there exists $C \in \overline{U_H^{-1}(\{x\})}$ which is a cluster point of $(\{x_\lambda\})_{\lambda\in\Lambda}$. Therefore y is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$ for each $y \in C$. We have shown that $\overline{U^{-1}(x)}$ is compact, and hence $(X, \mathscr{U})$ is co-uniformly locally compact.

Conversely, suppose that $(X, \mathscr{U})$ is co-uniformly locally compact and $\mathscr{U}^{-1}\big|K$ is precompact for each $K \in \mathscr{K}_0(X)$. Let $U \in \mathscr{U}$ be such that $\overline{U^{-1}(x)}$ is compact for each $x \in X$ and let $V \in \mathscr{U}$ with $V^3 \subseteq U$.

Let $K \in \mathscr{K}_0(X)$. We shall prove that $\overline{V_H^{-1}(K)}$ is compact. Let $(K_\lambda)_{\lambda\in\Lambda}$ be a net in $\overline{V_H^{-1}(K)}$. Then $K_\lambda \subseteq \overline{V^{-1}(K)}$ for all $\lambda \in \Lambda$, and thus $K_\lambda \in \mathscr{K}_0(\overline{V^{-1}(K)})$.

Next we show that $(\mathscr{K}_0(\overline{V^{-1}(K)}), \mathscr{U}_*)$ is compact. Since $\mathscr{U}^{-1}\big|K$ is precompact, there exists a finite subset $\{x_1, \dots, x_n\}$ of K such that $K \subseteq \bigcup\limits_{i=1}^{n} V^{-1}(x_i)$. Hence $\overline{V^{-1}(K)} \subseteq \bigcup\limits_{i=1}^{n} \overline{U^{-1}(x_i)}$. Therefore $\overline{V^{-1}(K)}$ is compact. Since, by our hypothesis, $\mathscr{U}^{-1}\big|K'$ is precompact whenever $K' \in \mathscr{K}_0(\overline{V^{-1}(K)})$, it follows from Lemmas 2 and 4 that $(\mathscr{K}_0(\overline{V^{-1}(K)}), \mathscr{U}_*)$ is compact.

Consequently, the net $(K_\lambda)_{\lambda\in\Lambda}$ has a cluster point C in $\mathscr{K}_0(\overline{V^{-1}(K)})$, and hence $\overline{V_H^{-1}(K)}$ is compact. We conclude that $(\mathscr{K}_0(X), \mathscr{U}_*)$ is co-uniformly locally compact. □

The following result should be compared with Propositions 1 and 3 and Remark 1b).

Corollary 3. *Let X be a Hausdorff space. Then*

a) *$(\mathscr{K}_0(X), \mathscr{CF}_*)$ is co-uniformly locally compact if and only if $(X, \mathscr{CF})$ is co-uniformly locally compact.*

b) *$(\mathscr{K}_0(X), \mathscr{LF}_*)$ is co-uniformly locally compact if and only if $(X, \mathscr{LF})$ is co-uniformly locally compact.*

Proof. a) Apply Theorem 3 and Lemma 5.

b) If $(\mathscr{K}_0(X), \mathscr{LF}_*)$ is co-uniformly locally compact, then $(X, \mathscr{LF})$ is co-uniformly locally compact by Theorem 3. Conversely, if $(X, \mathscr{LF})$ is co-uniformly locally compact, then X is a locally compact space by Proposition 7, so $\mathscr{CF} = \mathscr{LF}$. Hence $(\mathscr{K}_0(X), \mathscr{LF}_*)$ is co-uniformly locally compact by part a). □

Related to the proof of the "only if" part in Theorem 3 above, we give an example of a T_1 quasi-uniform (actually, quasi-metric) space $(X, \mathscr{U})$ such that $(\mathscr{K}_0(X), \mathscr{U}_*)$ is not T_1.

Example 3. Let $X = \{1/n\colon n \in \mathbb{N}\}$ and let d be the quasi-metric on X given by $d(1/n, 1/m) = 1/m$ if $n \neq m$, and $d(1/n, 1/n) = 0$, whenever $n, m \in \mathbb{N}$. Clearly the topology generated by d is the co-finite topology on X.

Let $Y = \{1/2n\colon n \in \mathbb{N}\}$. Then $X, Y \in \mathscr{K}_0(X)$ and an easy computation shows that $d_*(X, Y) = 0$. Hence the Hausdorff-Bourbaki quasi-pseudo-metric d_* is not a quasi-metric on $\mathscr{K}_0(X)$.

Lemma 10. *Let X be a space and $K \in \mathscr{K}_0(X)$. Then $\mathscr{SC}^{-1}\big|K$ is precompact if and only if every lower semicontinuous real-valued function on X is bounded on K.*

P r o o f. Suppose that $\mathscr{SC}^{-1}\big|K$ is precompact and let f be a lower semicontinuous real-valued function on X. Thus $U \in \mathscr{SC}$ where $U = \{(x, y)\colon f(x) - f(y) < 1\}$. Let K' be a finite subset of K such that $K \subseteq U^{-1}(K')$. Put $M = \max\{f(z)\colon z \in K'\}$. Then $f(x) \leqslant 1 + M$ for all $x \in K$. Since f is also lower semicontinuous on K, it is lower bounded on K ([9], Proposition 3.17). Hence f is bounded on K.

Conversely, let f be a lower semicontinuous real-valued function f on X and let $\varepsilon > 0$. Put $U = \{(x, y)\colon f(x) - f(y) < \varepsilon\}$. By assumption, $f(K)$ is a bounded subset of $\mathbb{R}$, so there is a finite subset K' of K such that $f(K) \subseteq \bigcup_{z \in K'}]f(z) - \varepsilon, f(z) + \varepsilon[$. Hence $K \subseteq U^{-1}(K')$. Consequently, $\mathscr{SC}^{-1}\big|K$ is precompact. □

Proposition 8. *Let X be a Hausdorff space. Then $(\mathscr{K}_0(X), \mathscr{SC}_*)$ is co-uniformly locally compact if and only if X is a σ-compact space such that every lower semicontinuous real-valued function on X is bounded on each compact subset of X.*

P r o o f. Apply Proposition 6, Theorem 3 and Lemma 10. □

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