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NORMAL VIETORIS IMPLIES COMPACTNESS: A SHORT PROOF

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Abstract. One of the most celebrated results in the theory of hyperspaces says that if the Vietoris topology on the family of all nonempty closed subsets of a given space is normal, then the space is compact (Ivanova-Keesling-Velichko). The known proofs use cardinality arguments and are long. In this paper we present a short proof using known results concerning Hausdorff uniformities.

Keywords: hyperspaces, Vietoris topology, locally finite topology, Hausdorff metric, compactness, normality, countable compactness

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Suppose (X,τ) is a T_1 space and $CL(X)$, the family of all nonempty closed subsets of X, is assigned the Vietoris topology τ_V. Suppose $(CL(X),\tau_V)$ is normal. One of the most spectacular results in Hyperspaces due to Ivanova, Keesling and Velichko ([4], [6] and [8]) implies that (X,τ) is compact. In this paper we provide an alternative short proof using some recent results in Hyperspaces.

We use the notation

$V^+ = \{E \in CL(X)\colon\ E \subset V\}$,

$V^- = \{E \in CL(X)\colon\ E \cap V \neq \emptyset\}$,

for $\mathcal{A} \subset \tau$, $\mathcal{A}^- = \bigcap\{V^-\colon\ V \in \mathcal{A}\}$.

The Vietoris topology τ_V is generated by sets of the form $\{V^+\colon\ V \in \tau\}$ and $\mathcal{A}^-$ where $\mathcal{A} \subset \tau$ is finite ([1]).

Let $\mathcal{U}$ be a compatible uniformity on X ([3]). For each $U \in \mathcal{U}$, let $\hat{U} = \{(A,B)\colon$ $A,B \in CL(X),\ A \subseteq U[B]$ and $B \subseteq U[A]\}$. Then, $\{\hat{U}\colon\ U \in \mathcal{U}\}$ is a base for a uniformity $\mathbf{U}_H$ on $CL(X)$ called the Hausdorff uniformity associated with $\mathcal{U}$ ([7], [2]).

We note the following:

(a) Since X is embedded in $(CL(X), \tau_V)$ as a closed subset, (X, τ) itself is normal.

(b) Each real valued continuous function f on a space gives rise to a continuous pseudometric $d_f(x, y) = |f(x) - f(y)|$.

(c) The finest totally bounded uniformity $\mathcal{U}_0$ on the normal space X is generated by pseudometrics arising from all the members of $C^\star(X)$ (the set of all continuous functions f from X to the real interval $[0, 1]$). Moreover, the Hausdorff uniformity $\mathbf{U}_{0H}$ on $CL(X)$ associated with $\mathcal{U}_0$ is compatible with the Vietoris topology τ_V ([2]).

(d) If $\mathcal{F}$ is a nonconvergent ultrafilter, then each $F \in \mathcal{F}$ has more than one point (otherwise it would be a principal ultrafilter; a contradiction).

Proof. Suppose $(CL(X), \tau_V)$ is normal but not compact. Then it has a nonconvergent ultrafilter $\mathcal{F}$ which is Cauchy with respect to $\mathbf{U}_{0H}$. Choose distinct elements $\{x_F, y_F\}$ from each element $F \in \mathcal{F}$. Then $\{(x_F, y_F)\colon F \in \mathcal{F}\}$ is a Cauchy net with respect to $\mathbf{U}_{0H}$. Obviously $A = \{x_F\colon F \in \mathcal{F}\}$ and $B = \{y_F\colon F \in \mathcal{F}\}$ are disjoint closed sets in X and so there is a continuous function $f\colon X \to [0, 1]$ with $f(A) = 0$ and $f(B) = 1$. This shows that the net $\{(x_F, y_F)\colon F \in \mathcal{F}\}$ is not *small* (see [3]) with respect to the entourage in $\mathbf{U}_{0H}$ corresponding to the pseudometric d_f on X; a contradiction.

References

[1] *G. Beer*: Topologies on Closed and Closed Convex Sets. Kluwer Academic Publishers, 1993.

[2] *A. Di Concilio, S. A. Naimpally and P. L. Sharma*: Proximal hypertopologies. Proceedings of the VI Brasilian Topological Meeting, Campinas, Brazil (1988). Unpublished.

[3] *R. Engelking*: General Topology. Helderman Verlag, Berlin, 1989, Revised and completed version.

[4] *V. M. Ivanova*: On the theory of the space of subsets. Dokl. Akad. Nauk. SSSR *101* (1955), 601–603.

[5] *J. Keesling*: Normality and properties related to compactness in hyperspaces. Proc. Amer. Math. Soc. *24* (1970), 760–766.

[6] *J. Keesling*: On the equivalence of normality and compactness in hyperspaces. Pacific J. Math. *33* (1970), 657–667.

[7] *S. A. Naimpally and P. L. Sharma*: Fine uniformity and the locally finite hyperspace topology on 2^X. Proc. Amer. Math. Soc. *103* (1988), 641–646.

[8] *N. V. Velichko*: On spaces of closed subsets. Sibirskii Matem. Z. *16* (1975), 627–629; English traslation: Siberian Math. J. *16* (1975), 484–486.

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