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EVALUATION FORMULAS FOR A CONDITIONAL FEYNMAN INTEGRAL OVER WIENER PATHS IN ABSTRACT WIENER SPACE

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Abstract. In this paper, we introduce a simple formula for conditional Wiener integrals over $C_0(\mathbb{B})$, the space of abstract Wiener space valued continuous functions. Using this formula, we establish various formulas for a conditional Wiener integral and a conditional Feynman integral of functionals on $C_0(\mathbb{B})$ in certain classes which correspond to the classes of functionals on the classical Wiener space introduced by Cameron and Storvick. We also evaluate the conditional Wiener integral and conditional Feynman integral for functionals of the form

$$\exp\biggl\{\int_0^T \theta(s, x(s))\,\mathrm{d}\eta(s)\biggr\}$$

which are of interest in Feynman integration theories and quantum mechanics.

Keywords: Banach algebra $S''_{\mathbb{B}}$, Banach space $S''_{n,\mathbb{B}}$, conditional Wiener integral, conditional Feynman integral, simple formula for conditional Wiener integrals

MSC 2000: 28C20

1. Introduction

Let $C_0[0,T]$ denote the classical Wiener space, that is, the space of real-valued continuous functions $x(t)$ which are defined on $[0,T]$ with $x(0)=0$. The concept of conditional Wiener integrals on this space was introduced by Yeh in [18], [19]. By a conditional Wiener integral we mean the conditional expectation $E[F|X]$ of a real or complex-valued Wiener integrable function F conditioned by a Wiener measurable function X on $C_0[0,T]$, which is given as a function on the value space of X. We

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shall be concerned exclusively with X given by $X(x) = (x(t_1), \ldots, x(t_m))$, where $0 < t_1 < \ldots < t_m = T$. In [16], Park and Skoug derived a simple formula for the conditional Wiener integral with the conditioning function X. Using this formula, they expressed the conditional Wiener integral directly in terms of ordinary Wiener integrals.

Let $C_0(\mathbb{B})$ be the space of abstract Wiener space-valued continuous functions $x(t)$ which are defined on $[0,T]$ with $x(0) = 0$. In [13], the space $C_0(\mathbb{B})$ was introduced and in [17] Ryu developed several theories which appeared in classical and abstract Wiener spaces. In [20], Yoo introduced a Banach space $S''_{n,\mathbb{B}}$ and a Banach algebra $S''_{\mathbb{B}}$ on $C_0(\mathbb{B})$ which correspond to the Banach space $\mathbb{S}''_n$ and the Banach algebra $\mathbb{S}''$ on the classical Wiener space, respectively, introduced by Cameron and Storvick ([3]).

In this paper, we introduce a simple formula for a conditional Wiener integral on $C_0(\mathbb{B})$. Using the formula we establish various formulas for a conditional Wiener integral and a conditional Feynman integral of functionals in the Banach space $S''_{n,\mathbb{B}}$ and the Banach algebra $S''_{\mathbb{B}}$ on $C_0(\mathbb{B})$. Also, we evaluate the conditional Wiener integral and conditional Feynman integral of functionals of the forms

$$\begin{aligned}
&\exp\Bigl\{\int_0^T \theta(s, x(s))\,\mathrm{d}s\Bigr\},\\
&\exp\Bigl\{\int_0^T \theta(s, x(s))\,\mathrm{d}s\Bigr\}\psi(x(T)),\\
&\exp\Bigl\{\int_0^T \theta(s, x(s))\,\mathrm{d}\eta(s)\Bigr\},\\
&\exp\Bigl\{\int_0^T \theta(s, x(s))\,\mathrm{d}\eta(s)\Bigr\}\psi(x(T))
\end{aligned}$$

which are of interest in Feynman integration theories and quantum mechanics.

2. Preliminaries

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let B be a real normed linear space with norm $\|\cdot\|$ and let $\mathscr{B}(B)$ be the Borel σ-field on B. Let $X\colon (\Omega, \mathscr{A}, P) \to (B, \mathscr{B}(B))$ be a random variable. Let $F\colon \Omega \to \mathbb{C}$ be an integrable function and let P_X be the probability distribution of X on $(B, \mathscr{B}(B))$. Let $\mathscr{D}$ be the σ-field $\{X^{-1}(B_1)\colon B_1 \in \mathscr{B}(B)\}$. Let $P_{\mathscr{D}}$ be the probability measure induced by P, that is, $P_{\mathscr{D}}(E) = P(E)$ for $E \in \mathscr{D}$. For every $E \in \mathscr{D}$, let

$$Q_X(E) = \int_E F(\omega)\,\mathrm{d}P(\omega).$$

Clearly, Q_X is a complex measure on $\mathscr{D}$ such that $Q_X(E) = 0$ for every $E \in \mathscr{D}$ for which $P_{\mathscr{D}}(E) = 0$. Hence $Q_X \ll P_{\mathscr{D}}$, so that in view of the Radon-Nikodym theorem there exists a $\mathscr{D}$-measurable function $E[F|X]$ defined on Ω such that the relation

$$\int_E E[F|X](\omega)\,\mathrm{d}P_{\mathscr{D}}(\omega) = Q_X(E) = \int_E F(\omega)\,\mathrm{d}P(\omega)$$

holds for every $E \in \mathscr{D}$. Here the function $E[F|X]$ is determined uniquely $P_{\mathscr{D}}$-a.s. and it is called the conditional expectation of F given X.

Also there exists a P_X-integrable function ψ defined on B which is unique up to P_X-a.s. such that $E[F|X](\omega) = (\psi \circ X)(\omega)$ for $P_{\mathscr{D}}$-a.s. ω. ψ is also called the conditional expectation of F given X and without loss of generality, it is denoted by $E[F|X](\eta)$ for $\eta \in B$.

Throughout this paper, we will consider the function ψ as the conditional expectation of F given X.

The following lemma is useful for the proof of a simple formula for conditional Wiener integrals on $C_0(\mathbb{B})$.

Lemma 2.1 (Scalora [15]). *Let B be a real normed linear space and let $(\Omega, \mathscr{A}, P)$ be a probability space. If X_1 and X_2 defined on $(\Omega, \mathscr{A}, P)$ are independent random variables in B, then $l_1 \circ X_1$ and $l_2 \circ X_2$ are independent random variables for every $l_1, l_2 \in B^*$, where B^* is the dual space of B. Furthermore, if B is separable, then the converse is true.*

Let $(\mathscr{H}, \mathbb{B}, m)$ be an abstract Wiener space ([14]). Let $\{e_j\colon\ j \geqslant 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathscr{H}$ such that e_j's are in $\mathbb{B}^*$, the dual of the real separable Banach space $\mathbb{B}$. For each $h \in \mathscr{H}$ and $y \in \mathbb{B}$, let

$$(h, y)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (y, e_j), & \text{if the limit exists;} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ denotes the dual pairing between $\mathbb{B}$ and $\mathbb{B}^*$.

Note that for each $h (\neq 0)$ in $\mathscr{H}$, $(h, \cdot)^\sim$ is a Gaussian random variable on $\mathbb{B}$ with mean zero and variance $|h|^2$; also $(h, y)^\sim$ is essentially independent of the choice of the complete orthonormal set used in its definition and, further, $(h, \lambda y)^\sim = (\lambda h, y)^\sim = \lambda (h, y)^\sim$ for all $\lambda \in \mathbb{R}$. It is well-known that if $\{h_1, h_2, \ldots, h_n\}$ is an orthogonal set in $\mathscr{H}$, then the random variables $(h_j, \cdot)^\sim$ are independent. Moreover, if both h and y are in $\mathscr{H}$, then $(h, y)^\sim = \langle h, y\rangle$ ([12]).

Let $C_0(\mathbb{B})$ denote the set of all continuous functions on $[0, T]$ into $\mathbb{B}$ which vanish at 0. Then $C_0(\mathbb{B})$ is a real separable Banach space with the norm $\|x\|_{C_0(\mathbb{B})} \equiv$

$\sup_{0\leqslant t\leqslant T}\|x(t)\|_{\mathbb{B}}$. And from [13], the minimal σ-field making the mapping $x\to x(t)$ measurable is $\mathscr{B}(C_0(\mathbb{B}))$, the Borel σ-field on $C_0(\mathbb{B})$. Further, the Brownian motion in $\mathbb{B}$ induces a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}),\mathscr{B}(C_0(\mathbb{B})))$ which is mean-zero Gaussian ([13]).

A complex-valued measurable function defined on $C_0(\mathbb{B})$ is said to be *Wiener measurable* and a Wiener measurable function is said to be *Wiener integrable* if it is integrable.

Definition 2.2. Let $F\colon C_0(\mathbb{B})\to\mathbb{C}$ be Wiener integrable and let

$$X\colon (C_0(\mathbb{B}),\mathscr{B}(C_0(\mathbb{B})),m_{\mathbb{B}})\to(B,\mathscr{B}(B))$$

be a random variable, where B is a real normed linear space with the Borel σ-field $\mathscr{B}(B)$. The conditional expectation $E[F|X]$ of F given X defined on B is called the conditional Wiener integral of F given X.

Now we introduce the Wiener integration theorem without proof. We easily obtain this theorem by using change of variable theorem.

Theorem 2.3 (Wiener Integration Theorem)**.** *Let $\vec{t}=(t_1,t_2,\ldots,t_n)$ be given with $0=t_0\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_n\leqslant T$. Let $T_{\vec{t}}\colon \mathbb{B}^n\to\mathbb{B}^n$ be given by $T_{\vec{t}}(x_1,x_2,\ldots,x_n)=\Big(\sqrt{t_1-t_0}x_1,\sqrt{t_1-t_0}x_1+\sqrt{t_2-t_1}x_2,\ldots,\sum_{j=1}^{n}\sqrt{t_j-t_{j-1}}x_j\Big)$ and let $f\colon \mathbb{B}^n\to\mathbb{C}$ be a Borel measurable function.*

Then

$$\int_{C_0(\mathbb{B})} f(x(t_1),x(t_2),\ldots,x(t_n))\,\mathrm{d}m_{\mathbb{B}}(x)$$
$$\overset{*}{=}\int_{\mathbb{B}^n}(f\circ T_{\vec{t}})(x_1,x_2,\ldots,x_n)\,\mathrm{d}\Big(\prod_{i=1}^{n} m\Big)(x_1,x_2,\ldots,x_n),$$

where by $\overset{*}{=}$ we mean that if either side exists, then both sides exist and they are equal.

The following results are useful for the proof of Theorem 3.1. For more details, see [17].

(1) Let $0=t_0<t_1<\ldots<t_n\leqslant T$ be a partition of $[0,T]$. Then $x(t_j)-x(t_{j-1})$ $(j=1,2,\ldots,n)$ are independent as functions of x on $C_0(\mathbb{B})$.

(2) Let $\{W_t\colon 0\leqslant t\leqslant T\}$ be the Wiener process on $C_0(\mathbb{B})\times[0,T]$, where $W_t(x)=x(t)$ for $x\in C_0(\mathbb{B})$. Then, $\{W_t\colon 0\leqslant t\leqslant T\}$ is a stochastic process with stationary increments, that is, for $t_1,t_1+s,t_2,t_2+s\in[0,T]$, $W_{t_2}-W_{t_1}$ and $W_{t_2+s}-W_{t_1+s}$ have the same distribution.

(3) Let $\{W_t\colon\ 0 < t \leqslant T\}$ be the Wiener process and let $l \in \mathbb{B}^* - \{0\}$. Then $l \circ W_t$ is normally distributed with mean 0 and variance $t\|l\|^2$.

3. Simple formula for a conditional Wiener integral

In this section, we introduce a simple formula for conditional Wiener integrals on the space $C_0(\mathbb{B})$.

Let $\tau\colon\ 0 = t_0 < t_1 < \ldots < t_m = T$ be a partition of $[0,T]$ and let x be in $C_0(\mathbb{B})$. Define the polygonal function $[x]$ of x on $[0,T]$ by

$$[x](t) = \sum_{i=1}^{m} \chi_{(t_{i-1},t_i]}(t)\Big[x(t_{i-1}) + \frac{t-t_{i-1}}{t_i - t_{i-1}}(x(t_i) - x(t_{i-1}))\Big], \tag{3.1}$$

where $t \in [0,T]$. For each $\vec{\xi} = (\xi_1,\ldots,\xi_m) \in \mathbb{B}^m$, let $[\vec{\xi}]$ be the polygonal function of $\vec{\xi}$ on $[0,T]$ given as in (3.1) with $\xi_0 = 0$. Note that both $[x]\colon\ [0,T] \to \mathbb{B}$ and $[\vec{\xi}]\colon\ [0,T] \to \mathbb{B}$ are continuous.

Throughout this section, define $W_t\colon\ C_0(\mathbb{B}) \to \mathbb{B}$ by $W_t(x) = x(t)$ for $x \in C_0(\mathbb{B})$, where $0 \leqslant t \leqslant T$, and define $X_t\colon\ C_0(\mathbb{B}) \to \mathbb{B}$ by $X_t(x) = [x](t)$ for $x \in C_0(\mathbb{B})$, where $0 \leqslant t \leqslant T$.

The following theorem corresponds to the Theorem 1 in [16]. But our proof is different from that of the latter theorem.

Theorem 3.1. *Let $\{W_t\colon\ 0 \leqslant t \leqslant T\}$ be the Wiener process on $C_0(\mathbb{B}) \times [0,T]$ and define $X_\tau\colon\ C_0(\mathbb{B}) \to \mathbb{B}^m$ by $X_\tau(x) = (x(t_1),\ldots,x(t_m))$.*

Then the processes $\{W_t - X_t\colon\ 0 \leqslant t \leqslant T\}$ and $X_\tau(x) = (x(t_1),\ldots,x(t_m))$ are stochastically independent.

Proof. First, we will show that for $l \in \mathbb{B}^* - \{0\}$, $\{l(W_t)\colon\ 0 \leqslant t \leqslant T\}$ is a standard Brownian motion process and hence it is a Gaussian process. For $x \in C_0(\mathbb{B})$, $l(W_0(x)) = l(x(0)) = 0$. Let $0 \leqslant s < t \leqslant T$ and let $B \in \mathscr{B}(\mathbb{R})$. Then

$$\begin{aligned} m_{\mathbb{B}}[(l(W_t) - l(W_s))^{-1}(B)] &= m_{\mathbb{B}}[(W_t - W_s)^{-1}(l^{-1}(B))] \\ &= m_{\mathbb{B}}[(W_{t-s})^{-1}(l^{-1}(B))] \\ &= m_{\mathbb{B}}[(l(W_{t-s}))^{-1}(B)], \end{aligned}$$

where the second equality follows from the property (2) in Section 2. Thus $l(W_t) - l(W_s)$ is normally distributed with mean 0 and variance $\|l\|^2(t-s)$ by the property (3) in Section 2. By Lemma 2.1 and the property (1) in Section 2, $\{l(W_t)\colon\ 0 \leqslant t \leqslant T\}$ is a process with independent increments.

Thus for $l \in \mathbb{B}^* - \{0\}$, by the property (3) in Section 2,

$$l(W_t - X_t) = l(W_t) - l(W_{t_{i-1}}) - \frac{t - t_{i-1}}{t_i - t_{i-1}} (l(W_{t_i}) - l(W_{t_{i-1}}))$$

is normally distributed, where $t_{i-1} < t < t_i$, $i = 1, \ldots, m$.

On the other hand, for $s_1 \in (0, t_{i-1}] \cup [t_i, T]$ and $l_1 \in \mathbb{B}^* - \{0\}$, $l_1(W_{s_1})$ is normally distributed.

Let $0 \leqslant s \leqslant t \leqslant T$ and $l, l_1 \in \mathbb{B}^*$. Then

$$\begin{aligned}
\operatorname{Cov}(l(W_s), l_1(W_t)) &= \operatorname{Cov}(l(W_s), l_1(W_t) - l_1(W_s) + l_1(W_s)) \\
&= E[l(W_s) l_1(W_t - W_s)] + E[l(W_s) l_1(W_s)] \\
&= \int_{C_0(\mathbb{B})} l(x(s)) l_1(x(s)) \, \mathrm{d}m_{\mathbb{B}}(x) \\
&= \int_{\mathbb{B}} l(\sqrt{s} x_1) l_1(\sqrt{s} x_1) \, \mathrm{d}m(x_1) \\
&= s \int_{\mathbb{B}} l(x_1) l_1(x_1) \, \mathrm{d}m(x_1) = s \operatorname{Cov}(l, l_1),
\end{aligned}$$

where the second equality follows from the first part of this proof and the fourth equality follows from Theorem 2.3.

Thus, for $t \in [t_{i-1}, t_i]$, $s_1 \in [0, t_{i-1}] \cup [t_i, T]$ and $l, l_1 \in \mathbb{B}^*$,

$$\begin{aligned}
&\operatorname{Cov}(l(W_t - X_t), l_1(W_{s_1})) \\
&\quad = E\Big[\Big(l(W_t) - l(W_{t_{i-1}}) - \frac{t - t_{i-1}}{t_i - t_{i-1}} (l(W_{t_i}) - l(W_{t_{i-1}}))\Big) l_1(W_{s_1})\Big] \\
&\quad = \begin{cases}
\operatorname{Cov}(l, l_1) \Big[t - t_{i-1} - \dfrac{t - t_{i-1}}{t_i - t_{i-1}} (t_i - t_{i-1})\Big] & \text{if } s_1 \in [t_i, T] \\[2ex]
\operatorname{Cov}(l, l_1) \Big[s_1 - s_1 - \dfrac{t - t_{i-1}}{t_i - t_{i-1}} (s_1 - s_1)\Big] & \text{if } s_1 \in [0, t_{i-1}]
\end{cases} \\
&\quad = 0.
\end{aligned}$$

By the first part of this proof, $\{l(W_t - X_t), l_1(W_{s_1})\}$ are independent for any $l, l_1 \in \mathbb{B}^*$. By Lemma 2.1 $\{W_t - X_t, W_{s_1}\}$ are independent for $t \in [t_{i-1}, t_i]$ and $s_1 \in [0, t_{i-1}] \cup [t_i, T]$. In particular, for $i = 1, 2, \ldots, m$, $\{W_t - X_t \colon\ t_{i-1} \leqslant t \leqslant t_i\}$ is independent of $X_\tau(x) = (x(t_1), \ldots, x(t_m))$. Therefore the result follows. $\square$

Corollary 3.2. *If $\{W_t \colon\ 0 \leqslant t \leqslant T\}$ is the Wiener process on $C_0(\mathbb{B}) \times [0, T]$, then $\{W_t - X_t \colon\ t_{i-1} \leqslant t \leqslant t_i\}$, where $i = 1, \ldots, m$, are stochastically independent.*

Corollary 3.3. *Let X_τ be given as in Theorem* 3.1. *Define $Y\colon C_0(\mathbb{B}) \to C_0(\mathbb{B})$ by $Y(x) = x - [x]$.*

Then X_τ and Y are independent, that is, the σ-fields induced by X_τ and Y, respectively, are independent.

Proof. By Theorem 3.1, $X_\tau^{-1}(B_1) = \{x \in C_0(\mathbb{B})\colon (x(t_1), x(t_2), \dots, x(t_m)) \in B_1\}$ and $Y^{-1}(B_2) = \{x \in C_0(\mathbb{B})\colon (Y(x)(s_1), \dots, Y(x)(s_n)) \in B_2\}$ are independent, where $0 < s_1 < \dots < s_n \leqslant T$ is any partition of $[0, T]$ and B_1, B_2 are any Borel subsets of $\mathbb{B}^m$ and $\mathbb{B}^n$, respectively. Then the result follows from Problem 4 [2, p. 216]. $\square$

The following theorem corresponds to the Theorem 2 in [16].

Theorem 3.4. *Let F be integrable on $C_0(\mathbb{B})$. Let X_τ be given as in Theorem* 3.1. *Then for every Borel measurable subset B of $\mathbb{B}^m$,*

$$\mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x)\,\mathrm{d}m_{\mathbb{B}}(x) = \int_B E[F(x - [x] + [\vec{\xi}])]\,\mathrm{d}P_{X_\tau}(\vec{\xi}),$$

where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{B}^m, \mathscr{B}(\mathbb{B}^m))$.

Proof. Let $A \in \mathscr{B}(C_0(\mathbb{B}))$ and $F = \chi_A$. Then

$$\begin{aligned}\int_{X_\tau^{-1}(B)} \chi_A(x)\,\mathrm{d}m_{\mathbb{B}}(x) &= m_{\mathbb{B}}(A \cap X_\tau^{-1}(B))\\ &= \int_B m_{\mathbb{B}}(x \in A \mid X_\tau(x) = \vec{\xi})\,\mathrm{d}P_{X_\tau}(\vec{\xi})\\ &= \int_B m_{\mathbb{B}}(x - [x] + [\vec{\xi}] \in A \mid X_\tau(x) = \vec{\xi})\,\mathrm{d}P_{X_\tau}(\vec{\xi})\end{aligned}$$

for any $B \in \mathscr{B}(\mathbb{B}^m)$. Since $x - [x]$ and $X_\tau(x)$ are independent by Corollary 3.3,

$$\begin{aligned}\mu_\tau(B) &= \int_B m_{\mathbb{B}}(x - [x] + [\vec{\xi}] \in A)\,\mathrm{d}P_{X_\tau}(\vec{\xi})\\ &= \int_B E[\chi_A(x - [x] + [\vec{\xi}])]\,\mathrm{d}P_{X_\tau}(\vec{\xi}).\end{aligned}$$

The general case can be proved easily. $\square$

By the definition of the conditional Wiener integral (Definition 2.2), we have

$$E[F|X_\tau](\vec{\xi}) = E[F(x - [x] + [\vec{\xi}])] \quad \text{for } P_{X_\tau}\text{-a.s. } \vec{\xi}. \tag{3.2}$$

The equation (3.2) is called a simple formula for the conditional Wiener integral of F given X_τ on the space $C_0(\mathbb{B})$.

For $\lambda > 0$ and $\vec{\xi} \in \mathbb{B}^m$, let $F^\lambda(x) = F(\lambda^{-1/2}x)$, $X_\tau^\lambda(x) = X_\tau(\lambda^{-1/2}x)$, and suppose $E[F^\lambda | X_\tau^\lambda](\vec{\xi})$ exists. From (3.2) we have

$$E[F^\lambda | X_\tau^\lambda](\vec{\xi}) = E[F(\lambda^{-1/2}(x(\cdot) - [x](\cdot)) + [\vec{\xi}](\cdot))]$$

for a.s. $\vec{\xi} \in \mathbb{B}^m$. If, for $\vec{\xi} \in \mathbb{B}^m$, $E[F(\lambda^{-1/2}(x(\cdot) - [x](\cdot)) + [\vec{\xi}](\cdot))]$ has the analytic extension $J_\lambda(\vec{\xi})$ on $\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > 0\}$, then we write

$$J_\lambda(\vec{\xi}) = E^{\mathrm{an}\, w_\lambda}[F | X_\tau](\vec{\xi})$$

for $\lambda \in \mathbb{C}_+$. $J_\lambda(\vec{\xi})$ is a version of conditional Wiener integral.

For non-zero real q and $\vec{\xi} \in \mathbb{B}^m$, if the limit

$$\lim_{\lambda \to -\mathrm{i}q} E^{\mathrm{an}\, w_\lambda}[F | X_\tau](\vec{\xi})$$

exists, where λ approaches $-\mathrm{i}q$ through $\mathbb{C}_+$, then we write

$$\lim_{\lambda \to -\mathrm{i}q} E^{\mathrm{an}\, w_\lambda}[F | X_\tau](\vec{\xi}) = E^{\mathrm{an}\, f_q}[F | X_\tau](\vec{\xi}).$$

$E^{\mathrm{an}\, f_q}[F | X_\tau](\vec{\xi})$ is a version of conditional Feynman integral.

4. Evaluation formulas for a conditional Feynman integral

Let $\mathscr{H}$ be an infinite dimensional separable real Hilbert space. Let $\mathscr{M}(\mathscr{H})$ be the class of all complex Borel measures on $\mathscr{H}$. Let $\Delta_n = \{(s_1, s_2, \ldots, s_n) \in [0,T]^n\colon 0 = s_0 < s_1 < s_2 < \ldots < s_n \leqslant T\}$.

Let $\mathscr{M}_n'' = \mathscr{M}_n''(\Delta_n \times \mathscr{H}^n)$ be the class of all complex Borel measures on $\Delta_n \times \mathscr{H}^n$ and let $\|\mu\| = \operatorname{var}\mu$, the total variation of μ in $\mathscr{M}_n''$.

Let $S_{n,\mathbb{B}}'' = S_{n,\mathbb{B}}''(\Delta_n \times \mathscr{H}^n)$ be the space of functions of the form

$$F(x) = \int_{\Delta_n \times \mathscr{H}^n} \exp\Big\{\mathrm{i} \sum_{k=1}^n (h_k, x(s_k))^\sim\Big\}\, \mathrm{d}\mu(\vec{s}, \vec{h}) \tag{4.1}$$

for $x \in C_0(\mathbb{B})$, where $\mu \in \mathscr{M}_n''$. Here we take $\|F\|_n'' = \inf\{\|\mu\|\}$, where the infimum is taken over all μ's so that F and μ are related by (4.1).

Let $\mathscr{M}'' = \mathscr{M}''(\sum \Delta_n \times \mathscr{H}^n)$ be the class of all sequences $\{\mu_n\}$ of measures such that each $\mu_n \in \mathscr{M}_n''$ and $\sum\limits_{n=1}^\infty \|\mu_n\| < \infty$.

Let $S''_{\mathbb{B}} = S''_{\mathbb{B}}(\sum \Delta_n \times \mathscr{H}^n)$ be the space of functions on $C_0(\mathbb{B})$ of the form

$$F(x) = \sum_{n=1}^{\infty} F_n(x), \tag{4.2}$$

where each $F_n \in S''_{n,\mathbb{B}}$ and $\sum\limits_{n=1}^{\infty} \|F_n\|''_n < \infty$. The norm of F is defined by $\|F\|'' = \inf\Big\{\sum\limits_{n=1}^{\infty} \|F_n\|''_n\Big\}$, where the infimum is taken over all representations of F given by (4.2).

Theorem 4.1. *Let $F \in S''_{n,\mathbb{B}}$ be given by (4.1). Let $0 = t_0 < t_1 < \ldots < t_m = T$ be a partition of $[0,T]$ and let $X_\tau(x) = (x(t_1), \ldots, x(t_m))$.*

Then $E^{\mathrm{an}\, w_\lambda}[F|X_\tau](\vec{\xi})$ and $E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})$ exist for a.s. $\vec{\xi} \in \mathbb{B}^m$. Moreover, for $\lambda \in \mathbb{C}_+$ and for a.s. $\vec{\xi} \in \mathbb{B}^m$,

$$E^{\mathrm{an}\, w_\lambda}[F|X_\tau](\vec{\xi}) = g(\lambda, \mu, \vec{\xi}),$$

where

$$\begin{aligned} g(\lambda,\mu,\vec{\xi}) &\\ = \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m}\times\mathscr{H}^n} &\exp\bigg\{\sum_{p=1}^{m}\bigg[\mathrm{i}\sum_{k=1}^{j_p}(h_{p,k},[\vec{\xi}](s_{p,k}))^{\sim} \\ &- \frac{1}{2\lambda}\sum_{l=1}^{j_p+1}\alpha_{p,l}\bigg|\sum_{k=1}^{l-1}\frac{t_{p-1}-s_{p,k}}{t_p-t_{p-1}}h_{p,k} + \sum_{k=l}^{j_p}\frac{t_p-s_{p,k}}{t_p-t_{p-1}}h_{p,k}\bigg|^2\bigg]\bigg\}\,\mathrm{d}\mu(\vec{s},\vec{h}). \end{aligned} \tag{4.3}$$

Also for non-zero real q and for a.s. $\vec{\xi} \in \mathbb{B}^m$,

$$E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi}) = g(-\mathrm{i}q, \mu, \vec{\xi}),$$

where $\Delta'_{n;j_1,\ldots,j_m} = \{(s_{1,1},\ldots,s_{1,j_1},s_{2,1},\ldots,s_{2,j_2},\ldots,s_{m,1},\ldots,s_{m,j_m})\colon 0 = s_{1,0} < s_{1,1} < \ldots < s_{1,j_1} \leqslant t_1 < s_{2,1} < \ldots < s_{2,j_2} \leqslant t_2 < \ldots \leqslant t_{m-1} < s_{m,1} < \ldots < s_{m,j_m} \leqslant t_m = T\}$ and $\alpha_{p,l} = s_{p,l} - s_{p,l-1}$, $t_p = s_{p+1,0} = s_{p,j_p+1}$ for $p = 1,\ldots,m-1$, $t_m = s_{m,j_m+1} = T$, $\vec{h} = (h_{1,1},\ldots,h_{1,j_1},h_{2,1},\ldots,h_{2,j_2},\ldots,h_{m,1},\ldots,h_{m,j_m}) \in \mathscr{H}^n$ for $j_1+\ldots+j_m = n$ with the convention that $\sum\limits_{k=1}^{l-1}\frac{t_{p-1}-s_{p,k}}{t_p-t_{p-1}}h_{p,k} = 0$ if $l = 1$, $\sum\limits_{k=l}^{j_p}\frac{t_p-s_{p,k}}{t_p-t_{p-1}}h_{p,k} = 0$ if $l = j_p+1$ and $\sum\limits_{k=1}^{j_p}(h_{p,k},[\vec{\xi}](s_{p,k}))^{\sim} = 0$ if $j_p = 0$.

P r o o f. By Fubini's theorem, we obtain that for $\lambda > 0$ and for a.s. $\vec{\xi} \in \mathbb{B}^m$,

$$
\begin{aligned}
&E[F(\lambda^{-1/2}(x(\cdot) - [x](\cdot)) + [\vec{\xi}](\cdot))] \\
&\quad = \int_{C_0(\mathbb{B})} F(\lambda^{-1/2}(x(\cdot) - [x](\cdot)) + [\vec{\xi}](\cdot)) \,\mathrm{d}m_{\mathbb{B}}(x) \\
&\quad = \int_{C_0(\mathbb{B})} \int_{\Delta_n \times \mathscr{H}^n} \exp\Big\{\mathrm{i} \sum_{k=1}^{n} (h_k, \lambda^{-1/2}(x(s_k) - [x](s_k)) + [\vec{\xi}](s_k))^{\sim}\Big\} \\
&\qquad\qquad \mathrm{d}\mu(\vec{s}, \vec{h}) \,\mathrm{d}m_{\mathbb{B}}(x) \\
&\quad = \int_{\Delta_n \times \mathscr{H}^n} \int_{C_0(\mathbb{B})} \exp\Big\{\mathrm{i} \sum_{k=1}^{n} (h_k, \lambda^{-1/2}(x(s_k) - [x](s_k)) + [\vec{\xi}](s_k))^{\sim}\Big\} \\
&\qquad\qquad \mathrm{d}m_{\mathbb{B}}(x) \,\mathrm{d}\mu(\vec{s}, \vec{h}) \\
&\quad = \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m} \times \mathscr{H}^n} \int_{C_0(\mathbb{B})} \exp\Big\{\mathrm{i} \sum_{k=1}^{n} (h_k, \lambda^{-1/2}(x(s_k) \\
&\qquad\qquad - [x](s_k)) + [\vec{\xi}](s_k))^{\sim}\Big\} \,\mathrm{d}m_{\mathbb{B}}(x) \,\mathrm{d}\mu(\vec{s}, \vec{h}).
\end{aligned}
$$

Let $s_{p,k} = s_{j_1+\ldots+j_{p-1}+k}$ and $h_{p,k} = h_{j_1+\ldots+j_{p-1}+k}$. Then

$$
\begin{aligned}
&E[F(\lambda^{-1/2}(x(\cdot) - [x](\cdot)) + [\vec{\xi}](\cdot))] \\
&\quad = \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m} \times \mathscr{H}^n} \int_{C_0(\mathbb{B})} \exp\Big\{\mathrm{i} \sum_{p=1}^{m} \sum_{k=1}^{j_p} \big(h_{p,k}, \lambda^{-1/2}(x(s_{p,k}) \\
&\qquad\qquad - [x](s_{p,k})) + [\vec{\xi}](s_{p,k})\big)^{\sim}\Big\} \,\mathrm{d}m_{\mathbb{B}}(x) \,\mathrm{d}\mu(\vec{s}, \vec{h}) \\
&\quad = \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m} \times \mathscr{H}^n} \int_{C_0(\mathbb{B})} \prod_{p=1}^{m} \exp\Big\{\mathrm{i} \sum_{k=1}^{j_p} \Big(h_{p,k}, \lambda^{-1/2}\Big(x(s_{p,k}) \\
&\qquad - x(t_{p-1}) - \frac{s_{p,k} - t_{p-1}}{t_p - t_{p-1}}(x(t_p) - x(t_{p-1}))\Big) + [\vec{\xi}](s_{p,k})\Big)^{\sim}\Big\} \,\mathrm{d}m_{\mathbb{B}}(x) \,\mathrm{d}\mu(\vec{s}, \vec{h}).
\end{aligned}
$$

By Corollary 3.2 and Theorem 2.3,

$$
\begin{aligned}
&E[F(\lambda^{-1/2}(x(\cdot) - [x](\cdot)) + [\vec{\xi}](\cdot))] \\
&\quad = \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m} \times \mathscr{H}^n} \prod_{p=1}^{m} \Big[\int_{C_0(\mathbb{B})} \exp\Big\{\mathrm{i} \sum_{k=1}^{j_p} \Big(h_{p,k}, \lambda^{-1/2}(x(s_{p,k}) \\
&\qquad\qquad - x(t_{p-1}) - \frac{s_{p,k} - t_{p-1}}{t_p - t_{p-1}}(x(t_p) - x(t_{p-1}))\Big) + [\vec{\xi}](s_{p,k}))^{\sim}\Big\} \,\mathrm{d}m_{\mathbb{B}}(x)\Big] \\
&\qquad\qquad \mathrm{d}\mu(\vec{s}, \vec{h})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m}\times\mathscr{H}^n} \exp\Big\{\mathrm{i}\sum_{p=1}^{m}\sum_{k=1}^{j_p}(h_{p,k},[\vec{\xi}](s_{p,k}))^{\sim}\Big\}\\
&\quad \prod_{p=1}^{m}\Big[\int_{\mathbb{B}^{j_p+1}} \exp\Big\{\mathrm{i}\lambda^{-1/2}\sum_{k=1}^{j_p}\Big(h_{p,k},\sum_{v=1}^{k}\sqrt{\alpha_{p,v}}y_{p,v}-\frac{s_{p,k}-t_{p-1}}{t_p-t_{p-1}}\sum_{v=1}^{j_p+1}\sqrt{\alpha_{p,v}}y_{p,v}\Big)^{\sim}\Big\}\\
&\qquad\qquad \mathrm{d}\Big(\prod_{1}^{j_p+1} m\Big)(y_{p,1},\ldots,y_{p,j_p+1})\Big]\,\mathrm{d}\mu(\vec{s},\vec{h})\\
&= \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m}\times\mathscr{H}^n} \exp\Big\{\mathrm{i}\sum_{p=1}^{m}\sum_{k=1}^{j_p}(h_{p,k},[\vec{\xi}](s_{p,k}))^{\sim}\Big\}\\
&\quad \prod_{p=1}^{m}\Big[\int_{\mathbb{B}^{j_p+1}} \exp\Big\{\mathrm{i}\lambda^{-1/2}\sum_{l=1}^{j_p+1}\Big(\sum_{k=1}^{l-1}\sqrt{\alpha_{p,l}}\frac{t_{p-1}-s_{p,k}}{t_p-t_{p-1}}h_{p,k}\\
&\qquad\qquad +\sum_{k=l}^{j_p}\frac{t_p-s_{p,k}}{t_p-t_{p-1}}h_{p,k}\sqrt{\alpha_{p,l}},y_{p,l}\Big)^{\sim}\Big\}\,\mathrm{d}\Big(\prod_{1}^{j_p+1} m\Big)(\vec{y})\Big]\,\mathrm{d}\mu(\vec{s},\vec{h})\\
&= \sum_{j_1+\ldots+j_m=n} \int_{\Delta'_{n;j_1,\ldots,j_m}\times\mathscr{H}^n} \exp\Big\{\sum_{p=1}^{m}\Big[\mathrm{i}\sum_{k=1}^{j_p}(h_{p,k},[\vec{\xi}](s_{p,k}))^{\sim}\\
&\qquad -\frac{1}{2\lambda}\sum_{l=1}^{j_p+1}\alpha_{p,l}\Big|\sum_{k=1}^{l-1}\frac{t_{p-1}-s_{p,k}}{t_p-t_{p-1}}h_{p,k}+\sum_{k=l}^{j_p}\frac{t_p-s_{p,k}}{t_p-t_{p-1}}h_{p,k}\Big|^2\Big]\Big\}\,\mathrm{d}\mu(\vec{s},\vec{h}).
\end{aligned}
$$

Using Morera's theorem and Dominated Convergence Theorem, the results follow.

□

Theorem 4.2. *Let $F \in S''_{\mathbb{B}}$ be such that*

$$F(x)=\sum_{n=1}^{\infty}F_n(x)=\sum_{n=1}^{\infty}\int_{\Delta_n\times\mathscr{H}^n}\exp\Big\{\mathrm{i}\sum_{k=1}^{n}(h_k,x(s_k))^{\sim}\Big\}\,\mathrm{d}\mu_n(\vec{s},\vec{h}),$$

where each $F_n \in S''_{n,\mathbb{B}}$ and $\mu_n \in \mathscr{M}''_n$ with $\sum_{n=1}^{\infty}\|F_n\|''_n<\infty$. Let X_τ be given as in Theorem 4.1.

Then for any non-zero real q and a.s. $\vec{\xi}\in\mathbb{B}^m$, $E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi})$ exists and is given by

$$E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi})=\sum_{n=1}^{\infty}E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})=\sum_{n=1}^{\infty}g(-\mathrm{i}q,\mu_n,\vec{\xi}),$$

where g and the conditions are given as in Theorem 4.1.

Proof. Without loss of generality, we can assume that $\sum_{n=1}^{\infty}\|\mu_n\|<\infty$. For a.s. $\vec{\xi}\in\mathbb{B}^m$ and $\lambda>0$, by Fubini's theorem and Dominated Convergence Theorem,

$$\begin{aligned}
&E[F(\lambda^{-1/2}(x(\cdot)-[x](\cdot))+[\vec{\xi}](\cdot))]\\
&\quad=\int_{C_0(\mathbb{B})}F(\lambda^{-1/2}(x(\cdot)-[x](\cdot))+[\vec{\xi}](\cdot))\,\mathrm{d}m_{\mathbb{B}}(x)\\
&\quad=\sum_{n=1}^{\infty}\int_{\Delta_n\times\mathscr{H}^n}\int_{C_0(\mathbb{B})}\exp\Big\{\mathrm{i}\sum_{k=1}^{n}(h_k,\lambda^{-1/2}x(s_k)-\lambda^{-1/2}[x](s_k)+[\vec{\xi}](s_k))^{\sim}\Big\}\\
&\qquad\qquad\mathrm{d}m_{\mathbb{B}}(x)\,\mathrm{d}\mu_n(\vec{s},\vec{h})\\
&\quad=\sum_{n=1}^{\infty}\sum_{j_1+\ldots+j_m=n}\int_{\Delta'_{n;j_1,\ldots,j_m}\times\mathscr{H}^n}\exp\Big\{\sum_{p=1}^{m}\Big[\mathrm{i}\sum_{k=1}^{j_p}\big(h_{p,k},[\vec{\xi}](s_{p,k})\big)^{\sim}\\
&\qquad-\frac{1}{2\lambda}\sum_{l=1}^{j_p+1}\alpha_{p,l}\Big|\sum_{k=1}^{l-1}\frac{t_{p-1}-s_{p,k}}{t_p-t_{p-1}}h_{p,k}+\sum_{k=l}^{j_p}\frac{t_p-s_{p,k}}{t_p-t_{p-1}}h_{p,k}\Big|^2\Big]\Big\}\,\mathrm{d}\mu_n(\vec{s},\vec{h}).
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\|\mu_n\|<\infty$, the last series converges uniformly with respect to λ on $\mathbb{C}_+$ and each term of the series is analytic on $\mathbb{C}_+$. By Morera's theorem it is analytic on $\mathbb{C}_+$ and by Dominated Convergence Theorem, the result follows. □

Let $\mathscr{G}$ be the set of all $\mathbb{C}$-valued functions θ on $[0,T]\times\mathbb{B}$ which have the following form

$$\theta(s,y)=\int_{\mathscr{H}}\exp\{\mathrm{i}(h,y)^{\sim}\}\,\mathrm{d}\sigma_s(h) \tag{4.4}$$

where $\{\sigma_s\colon\ s\in[0,T]\}$ is a family from $\mathscr{M}(\mathscr{H})$ satisfying the following conditions:

(1) For each Borel subset E of $\mathscr{H}$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,T]$.

(2) $\|\sigma_s\|\in L_1([0,T])$.

Theorem 4.3. *Let $\theta\in\mathscr{G}$ be given by (4.4) and let X_τ be given as in Theorem 4.1. Then $F_n(x)=\big[\int_0^T\theta(s,x(s))\,\mathrm{d}s\big]^n$ and $F(x)=\exp\big\{\int_0^T\theta(s,x(s))\,\mathrm{d}s\big\}$ are elements of $S''_{\mathbb{B}}$ for $x\in C_0(\mathbb{B})$. Thus for a.s. $\vec{\xi}\in\mathbb{B}^m$ and non-zero real q, $E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})$ and $E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi})$ exist. Moreover,*

$$E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})=g(-\mathrm{i}q,\mu_n,\vec{\xi})$$

and

$$E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi})=1+\sum_{n=1}^{\infty}\frac{1}{n!}E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})=1+\sum_{n=1}^{\infty}g(-\mathrm{i}q,\mu'_n,\vec{\xi}),$$

where $\mathrm{d}\mu_n(\vec{s},\vec{h}) = n!\prod_{p=1}^{m}\prod_{k=1}^{j_p}\mathrm{d}\sigma_{s_{p,k}}(h_{p,k})\,\mathrm{d}s_{p,k}$, $\mathrm{d}\mu'_n(\vec{s},\vec{h}) = \prod_{p=1}^{m}\prod_{k=1}^{j_p}\mathrm{d}\sigma_{s_{p,k}}(h_{p,k})\,\mathrm{d}s_{p,k}$ *and both* g *and the conditions are given as in Theorem* 4.1 *for each* $n\in\mathbb{N}$.

Proof. By Theorems 4.1, 4.2 and by Theorem 3.3 in [20], we have the results. □

Let $\mathscr{F}(\mathbb{B})$ be the class of all functions on $\mathbb{B}$ of the form

$$\psi(y) = \int_{\mathscr{H}} \exp\{\mathrm{i}(h,y)^{\sim}\}\,\mathrm{d}\nu(h) \tag{4.5}$$

for $y\in\mathbb{B}$ where $\nu\in\mathscr{M}(\mathscr{H})$.

For $\lambda>0$ and $\vec{\xi}\in\mathbb{B}^m$,

$$\psi(\lambda^{-1/2}(x(T)-[x](T))+[\vec{\xi}](T)) = \psi([\vec{\xi}](T)) = \psi(\xi_m). \tag{4.6}$$

Hence we have the following results.

Theorem 4.4. *Let* $\theta\in\mathscr{G}$ *and* $\psi\in\mathscr{F}(\mathbb{B})$ *be given by* (4.4) *and* (4.5), *respectively, and let* X_τ *be given as in Theorem* 4.1.

Then the functions

$$F_n(x) = \left[\int_0^T \theta(s,x(s))\,\mathrm{d}s\right]^n \psi(x(T))$$

and

$$F(x) = \exp\left\{\int_0^T \theta(s,x(s))\,\mathrm{d}s\right\}\psi(x(T))$$

are elements of $S''_{\mathbb{B}}$ *for* $x\in C_0(\mathbb{B})$. *Thus* $E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})$ *and* $E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi})$ *exist for a.s.* $\vec{\xi}\in\mathbb{B}^m$. *Moreover, for a non-zero real* q *and for a.s.* $\vec{\xi}\in\mathbb{B}^m$,

$$\begin{aligned} E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi}) &= \psi(\xi_m) + \sum_{n=1}^{\infty}\frac{1}{n!}E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi}) \\ &= \psi(\xi_m)\left[1+\sum_{n=1}^{\infty} g(-\mathrm{i}q,\mu'_n,\vec{\xi})\right], \end{aligned}$$

where the conditions are given as in Theorem 4.3.

Proof. By Theorem 4.3 and (4.6), we have the results. □

Let η be a $\mathbb{C}$-valued Borel measure on $[0,T]$. Then $\eta = \mu_s + \mu_a + \nu$ can be decomposed uniquely into the sum of a discrete measure ν, a continuous but singular measure μ_s (with respect to the Lebesgue measure) and an absolutely continuous measure μ_a (with respect to the Lebesgue measure) ([8, p. 142]). Let $\mu = \mu_s + \mu_a$ and let δ_{τ_p} denote the Dirac measure with total mass one concentrated at τ_p. Note that μ is a continuous measure.

Let $\mathscr{G}^*$ be the set of all $\mathbb{C}$-valued functions θ on $[0,T]\times\mathbb{B}$ which have the form (4.4) where $\{\sigma_s\colon s\in[0,T]\}$ is the family from $\mathscr{M}(\mathscr{H})$ satisfying the following conditions:

(1) For each Borel subset E of $\mathscr{H}$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,T]$,

(2) $\|\sigma_s\|\in L_1([0,T],\mathscr{B}([0,T]),|\eta|)$.

Theorem 4.5. *Let $\eta = \mu + \sum\limits_{p=1}^{r} w_p\delta_{\tau_p}$, where $0\leqslant\tau_1<\ldots<\tau_r\leqslant T$ and the w_p's are in $\mathbb{C}$ for $p=1,2,\ldots,r$. Let $\theta\in\mathscr{G}^*$ be given by (4.4) and X_τ be given as in Theorem 4.1. Let $F_n(x) = \big[\int_0^T\theta(s,x(s))\,\mathrm{d}\eta(s)\big]^n$ and $F(x) = \exp\big\{\int_0^T\theta(s,x(s))\,\mathrm{d}\eta(s)\big\}$.*

Then for all non-zero real q and a.s. $\vec{\xi}\in\mathbb{B}^m$, $E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})$ and $E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi})$ exist. Moreover,

$$
\begin{aligned}
E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi}) &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})\\
&= 1+\sum_{n=1}^{\infty}\sum_{q_1+\ldots+q_m=n}\prod_{p=1}^{m}\sum_{l_{p,0}+\ldots+l_{p,r_p}=q_p}\bigg[\prod_{k=1}^{r_p}\frac{w_{p,k}^{l_{p,k}}}{l_{p,k}!}\bigg]\\
&\quad\bigg[\sum_{j_1+\ldots+j_{r_p+1}=l_{p,0}}\int_{\Delta_{l_{p,0};j_1,\ldots,j_{r_p+1}}}\int_{\mathscr{H}^{q_p}}\exp\bigg\{\mathrm{i}\sum_{\alpha=0}^{r_p}\sum_{i^*=1}^{j_{\alpha+1}+1}(h_{p,\alpha,i^*},[\vec{\xi}](s_{p,\alpha,i^*}))^{\sim}\bigg\}\\
&\quad\exp\bigg\{\frac{1}{2q\mathrm{i}}\sum_{u=0}^{r_p}\sum_{v=1}^{j_{u+1}+1}\beta_{u+1,v}\bigg|\sum_{\alpha=0}^{u-1}\sum_{i^*=1}^{j_{\alpha+1}+1}\frac{t_{p-1}-s_{p,\alpha,i^*}}{t_p-t_{p-1}}h_{p,\alpha,i^*}\\
&\qquad+\sum_{i^*=1}^{v-1}\frac{t_{p-1}-s_{p,u,i^*}}{t_p-t_{p-1}}h_{p,u,i^*}+\sum_{i^*=v}^{j_{u+1}+1}\frac{t_p-s_{p,u,i^*}}{t_p-t_{p-1}}h_{p,u,i^*}\\
&\qquad+\sum_{\alpha=u+1}^{r_p}\sum_{i^*=1}^{j_{\alpha+1}+1}\frac{t_p-s_{p,\alpha,i^*}}{t_p-t_{p-1}}h_{p,\alpha,i^*}\bigg|^2\bigg\}\\
&\quad\mathrm{d}\bigg(\prod_{\alpha=0}^{r_p}\prod_{i^*=1}^{j_{\alpha+1}}\sigma_{s_{p,\alpha,i^*}}\bigg)\bigg(\prod_{\alpha=1}^{r_p}\prod_{i^*=1}^{l_{p,\alpha}}\sigma_{\tau_{p,\alpha}}\bigg)(\vec{h},\vec{k})\,\mathrm{d}\bigg(\prod_{\alpha=0}^{r_p}\prod_{i^*=1}^{j_{\alpha+1}}\mu\bigg)(\vec{s})\bigg],
\end{aligned}
$$

where $\beta_{u,v} = s_{p,u-1,v}-s_{p,u-1,v-1}$, $t_{p-1} = s_{p,0,0}$, $t_p = s_{p,r_p,j_{r_p+1}+1}$, $\tau_{p,\alpha} = s_{p,\alpha-1,j_\alpha+1} = s_{p,\alpha,0}$, $h_{p,\alpha-1,j_\alpha+1} = \sum\limits_{i^*=1}^{l_{p,\alpha}}k_{p,\alpha,i^*}$, $h_{p,r_p,j_{r_p+1}+1} = 0$, $w_{p,k} =$

$w_{r_1+\ldots+r_{p-1}+k}$ and $\Delta_{l_{p,0};j_1,\ldots,j_{r_p+1}} = \{(s_{p,0,1},\ldots,s_{p,0,j_1},\ldots,s_{p,r_p,1},\ldots,s_{p,r_p,j_{r_p+1}})\colon 0 \leqslant t_{p-1} \leqslant s_{p,0,1} \leqslant \ldots \leqslant s_{p,0,j_1} \leqslant \tau_{p,1} < s_{p,1,1} < \ldots < s_{p,1,j_2} < \tau_{p,2} < \ldots < \tau_{p,r_p} \leqslant s_{p,r_p,1} \leqslant \ldots \leqslant s_{p,r_p,j_{r_p+1}} \leqslant t_p \leqslant T\}$ with, upon reordering τ_p's, t_{i^*}'s and renaming τ_p's, $0 \leqslant \tau_{1,1} < \tau_{1,2} < \ldots < \tau_{1,r_1} \leqslant t_1 < \tau_{2,1} < \ldots < \tau_{2,r_2} \leqslant t_2 < \ldots \leqslant t_{m-1} < \tau_{m,1} < \ldots < \tau_{m,r_m} \leqslant t_m = T$ where $r_1+\ldots+r_m = r$.

Proof. By reordering τ_p's, t_{i^*}'s and renaming τ_p's, let $0 \leqslant \tau_{1,1} < \tau_{1,2} < \ldots < \tau_{1,r_1} \leqslant t_1 < \tau_{2,1} < \ldots < \tau_{2,r_2} \leqslant t_2 < \ldots \leqslant t_{m-1} < \tau_{m,1} < \ldots < \tau_{m,r_m} \leqslant t_m = T$, where $r_1+\ldots+r_m = r$. Let $A_1 = [0,t_1]$ and $A_p = (t_{p-1},t_p]$ for $p = 2,\ldots,m$.

From the multinomial expansion theorem, the simplex trick [11], Fubini's theorem and Corollary 3.2, it follows that for $\lambda > 0$ and a.s. $\vec{\xi} \in \mathbb{B}^m$,

$$
\begin{aligned}
&E[F_n(\lambda^{-1/2}(x(\cdot)-[x](\cdot))+[\vec{\xi}](\cdot))]\\
&= \int_{C_0(\mathbb{B})}\Big[\int_0^T \theta(s,\lambda^{-1/2}(x(s)-[x](s))+[\vec{\xi}](s))\,\mathrm{d}\eta(s)\Big]^n \,\mathrm{d}m_{\mathbb{B}}(x)\\
&= \sum_{q_1+\ldots+q_m=n}\frac{n!}{q_1!\ldots q_m!}\prod_{p=1}^m\int_{C_0(\mathbb{B})}\Big[\int_{t_{p-1}}^{t_p}\theta\big(s,\lambda^{-1/2}(x(s)-[x](s))+[\vec{\xi}](s)\big)\,\mathrm{d}\mu(s)\\
&\qquad +\sum_{k=1}^{r_p} w_{p,k}\theta\big(\tau_{p,k},\lambda^{-1/2}(x(\tau_{p,k})-[x](\tau_{p,k}))+[\vec{\xi}](\tau_{p,k})\big)\Big]^{q_p}\,\mathrm{d}m_{\mathbb{B}}(x)\\
&= \sum_{q_1+\ldots+q_m=n}\frac{n!}{q_1!\ldots q_m!}\prod_{p=1}^m\int_{C_0(\mathbb{B})}\Big[\sum_{l_{p,0}+\ldots+l_{p,r_p}=q_p}\frac{q_p!}{l_{p,0}!\ldots l_{p,r_p}!}w_{p,1}^{l_{p,1}}\ldots w_{p,r_p}^{l_{p,r_p}}\\
&\qquad \Big(\int_{t_{p-1}}^{t_p}\theta\big(s,\lambda^{-1/2}(x(s)-[x](s))+[\vec{\xi}](s)\big)\,\mathrm{d}\mu(s)\Big)^{l_{p,0}}\\
&\qquad \Big[\prod_{k=1}^{r_p}\big(\theta\big(\tau_{p,k},\lambda^{-1/2}\big(x(\tau_{p,k})-[x](\tau_{p,k})\big)+[\vec{\xi}](\tau_{p,k})\big)\big)^{l_{p,k}}\Big]\Big]\,\mathrm{d}m_{\mathbb{B}}(x)\\
&= \sum_{q_1+\ldots+q_m=n} n!\prod_{p=1}^m\Big[\sum_{l_{p,0}+\ldots+l_{p,r_p}=q_p}\Big[\prod_{k=1}^{r_p}\frac{w_{p,k}^{l_{p,k}}}{l_{p,k}!}\Big]\Big[\sum_{j_1+\ldots+j_{r_p+1}=l_{p,0}}\int_{\Delta_{l_{p,0};j_1,\ldots,j_{r_p+1}}}\\
&\qquad \int_{C_0(\mathbb{B})}\Big[\prod_{\alpha=0}^{r_p}\prod_{i^*=1}^{j_{\alpha+1}}\theta\big(s_{p,\alpha,i^*},\lambda^{-1/2}(x(s_{p,\alpha,i^*})-[x](s_{p,\alpha,i^*}))+[\vec{\xi}](s_{p,\alpha,i^*})\big)\Big]\\
&\qquad \times\Big[\prod_{\alpha=1}^{r_p}(\theta(s_{p,\alpha,0},\lambda^{-1/2}(x(s_{p,\alpha,0})-[x](s_{p,\alpha,0}))+[\vec{\xi}](s_{p,\alpha,0})))^{l_{p,\alpha}}\Big]\,\mathrm{d}m_{\mathbb{B}}(x)\\
&\qquad\qquad \mathrm{d}\Big(\prod_{\alpha=0}^{r_p}\prod_{i^*=1}^{j_{\alpha+1}}\mu\Big)(\vec{s})\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{q_1+\ldots+q_m=n} n! \prod_{p=1}^{m} \Big[\sum_{l_{p,0}+\ldots+l_{p,r_p}=q_p} \Big[\prod_{k=1}^{r_p} \frac{w_{p,k}^{l_{p,k}}}{l_{p,k}!} \Big] \Big[\sum_{j_1+\ldots+j_{r_p+1}=l_{p,0}} \int_{\Delta_{l_{p,0};j_1,\ldots,j_{r_p+1}}} \\
&\quad \int_{\mathbb{B}^{l_{p,0}+r_p+1}} \prod_{\alpha=0}^{r_p} \prod_{i^*=1}^{j_{\alpha+1}} \theta\Big(s_{p,\alpha,i^*}, \lambda^{-1/2}\Big(\sum_{u=1}^{\alpha} \sum_{v=1}^{j_u+1} \sqrt{\beta_{u,v}} y_{u-1,v} + \sum_{v=1}^{i^*} \sqrt{\beta_{\alpha+1,v}} y_{\alpha,v} \\
&\qquad - \frac{s_{p,\alpha,i^*} - t_{p-1}}{t_p - t_{p-1}} \Big(\sum_{u=1}^{r_p+1} \sum_{v=1}^{j_u+1} \sqrt{\beta_{u,v}} y_{u-1,v} \Big) \Big) + [\vec{\xi}](s_{p,\alpha,i^*}) \Big) \\
&\quad \Big[\prod_{\alpha=1}^{r_p} \Big(\theta\Big(s_{p,\alpha,0}, \lambda^{-1/2}\Big(\sum_{u=1}^{\alpha} \sum_{v=1}^{j_u+1} \sqrt{\beta_{u,v}} y_{u-1,v} \\
&\qquad - \frac{s_{p,\alpha,0} - t_{p-1}}{t_p - t_{p-1}} \Big(\sum_{u=1}^{r_p+1} \sum_{v=1}^{j_u+1} \sqrt{\beta_{u,v}} y_{u-1,v} \Big) \Big) + [\vec{\xi}](s_{p,\alpha,0}) \Big) \Big)^{l_{p,\alpha}} \Big] \\
&\quad \mathrm{d}\Big(\prod_{1}^{l_{p,0}+r_p+1} m \Big) (y_{0,1}, \ldots, y_{0,j_1+1}, y_{1,1}, \ldots, y_{1,j_2+1}, \ldots, y_{r_p,1}, \ldots, y_{r_p,j_{r_p+1}+1}) \\
&\qquad \mathrm{d}\Big(\prod_{\alpha=0}^{r_p} \prod_{i^*=1}^{j_{\alpha+1}} \mu \Big)(\vec{s}) \Big] \Big],
\end{aligned}
$$

where $w_{p,k} = w_{r_1+\ldots+r_p+k}$ and the last equality follows from Theorem 2.3. From Fubini's theorem and unsymmetric Fubini's theorem ([10]), it follows that

$$
\begin{aligned}
&E[F_n(\lambda^{-1/2}(x(\cdot) - [x](\cdot)) + [\vec{\xi}](\cdot))] \\
&= n! \sum_{q_1+\ldots+q_m=n} \prod_{p=1}^{m} \Big[\sum_{l_{p,0}+\ldots+l_{p,r_p}=q_p} \Big[\prod_{k=1}^{r_p} \frac{w_{p,k}^{l_{p,k}}}{l_{p,k}!} \Big] \\
&\quad \Big[\sum_{j_1+\ldots+j_{r_p+1}=l_{p,0}} \int_{\Delta_{l_{p,0};j_1,\ldots,j_{r_p+1}}} \int_{\mathscr{H}^{q_p}} \exp\Big\{ \mathrm{i} \sum_{\alpha=0}^{r_p} \sum_{i^*=1}^{j_{\alpha+1}+1} (h_{p,\alpha,i^*}, [\vec{\xi}](s_{p,\alpha,i^*}))^{\sim} \Big\} \\
&\quad \int_{\mathbb{B}^{l_{p,0}+r_p+1}} \exp\Big\{ \mathrm{i}\lambda^{-1/2} \sum_{\alpha=0}^{r_p} \sum_{i^*=1}^{j_{\alpha+1}+1} \Big(h_{p,\alpha,i^*}, \sum_{u=1}^{\alpha} \sum_{v=1}^{j_u+1} \sqrt{\beta_{u,v}} y_{u-1,v} \\
&\qquad + \sum_{v=1}^{i^*} \sqrt{\beta_{\alpha+1,v}} y_{\alpha,v} - \frac{s_{p,\alpha,i^*} - t_{p-1}}{t_p - t_{p-1}} \sum_{u=1}^{r_p+1} \sum_{v=1}^{j_u+1} \sqrt{\beta_{u,v}} y_{u-1,v} \Big)^{\sim} \Big\} \\
&\qquad \mathrm{d}\Big(\prod_{1}^{l_{p,0}+r_p+1} m \Big)(\vec{y})\, \mathrm{d}\Big(\prod_{\alpha=0}^{r_p} \prod_{i^*=1}^{j_{\alpha+1}} \sigma_{s_{p,\alpha,i^*}} \Big) \Big(\prod_{\alpha=1}^{r_p} \prod_{i^*=1}^{l_{p,\alpha}} \sigma_{\tau_{p,\alpha}} \Big)(\vec{h}, \vec{k}) \\
&\qquad \mathrm{d}\Big(\prod_{\alpha=0}^{r_p} \prod_{i^*=1}^{j_{\alpha+1}} \mu \Big)(\vec{s}) \Big] \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= n! \sum_{q_1+\ldots+q_m=n} \prod_{p=1}^{m} \Big[\sum_{l_{p,0}+\ldots+l_{p,r_p}=q_p} \Big[\prod_{k=1}^{r_p} \frac{w_{p,k}^{l_{p,k}}}{l_{p,k}!} \Big] \\
&\Big[\sum_{j_1+\ldots+j_{r_p+1}=l_{p,0}} \int_{\Delta_{l_{p,0};j_1,\ldots,j_{r_p+1}}} \int_{\mathscr{H}^{q_p}} \exp\Big\{ \mathrm{i} \sum_{\alpha=0}^{r_p} \sum_{i^*=1}^{j_{\alpha+1}+1} \big(h_{p,\alpha,i^*}, [\vec{\xi}](s_{p,\alpha,i^*})\big)^{\sim} \Big\} \\
&\exp\Big\{ -\frac{1}{2\lambda} \sum_{u=0}^{r_p} \sum_{v=1}^{j_{u+1}+1} \beta_{u+1,v} \Big| \sum_{\alpha=0}^{u-1} \sum_{i^*=1}^{j_{\alpha+1}+1} \frac{t_{p-1}-s_{p,\alpha,i^*}}{t_p-t_{p-1}} h_{p,\alpha,i^*} \\
&\qquad + \sum_{i^*=1}^{v-1} \frac{t_{p-1}-s_{p,u,i^*}}{t_p-t_{p-1}} h_{p,u,i^*} + \sum_{i^*=v}^{j_{u+1}+1} \frac{t_p-s_{p,u,i^*}}{t_p-t_{p-1}} h_{p,u,i^*} \\
&\qquad + \sum_{\alpha=u+1}^{r_p} \sum_{i^*=1}^{j_{\alpha+1}+1} \frac{t_p-s_{p,\alpha,i^*}}{t_p-t_{p-1}} h_{p,\alpha,i^*} \Big|^2 \Big\} \\
&\qquad \mathrm{d}\Big(\prod_{\alpha=0}^{r_p} \prod_{i^*=1}^{j_{\alpha+1}} \sigma_{s_{p,\alpha,i^*}} \Big) \Big(\prod_{\alpha=1}^{r_p} \prod_{i^*=1}^{l_{p,\alpha}} \sigma_{\tau_{p,\alpha}} \Big)(\vec{h},\vec{k}) \\
&\qquad \mathrm{d}\Big(\prod_{\alpha=0}^{r_p} \prod_{i^*=1}^{j_{\alpha+1}} \mu \Big)(\vec{s}) \Big] \Big],
\end{aligned}
$$

where $h_{p,r_p,j_{r_p+1}+1} = 0$ and the last equality follows from Section 2 in this paper (or see [12]).

Using the same process as in Theorems 4.1, 4.2 and 4.3, we can get the result. □

Remark 4.6. Let $\eta = \mu + \sum_{r=1}^{\infty} w_r \delta_{\tau_r}$, where the τ_r's are in $[0,T]$ and the w_r's are in $\mathbb{C}$ for $r \in \mathbb{N}$. In the proof of the above theorem, A_p contains infinitely many τ_r's for some $p \in \{1,\ldots,m\}$. Using the following version of the $\aleph_0$-nomial formula ([11, p. 41])

$$
\Big(\sum_{r=0}^{\infty} b_r\Big)^{q_p} = \sum_{r_p=0}^{\infty} \sum_{l_{p,0}+\ldots+l_{p,r_p}=q_p,\, l_{p,r_p}\neq 0} \frac{q_p!}{l_{p,0}!\ldots l_{p,r_p}!} b_0^{l_{p,0}} \ldots b_{r_p}^{l_{p,r_p}},
$$

we can show that $E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})$ exists for a.s. $\vec{\xi} \in \mathbb{B}^m$ and for any non-zero real q.

Corollary 4.7. *Let $\eta = \sum_{p=1}^{r} w_p \delta_{\tau_p}$, where $0 \leqslant \tau_1 < \ldots < \tau_r \leqslant T$ and the w_p's are in $\mathbb{C}$ for $p = 1,\ldots,r$. Let $\theta \in \mathscr{G}^*$ be given by (4.4). Let F_n, F be given as in Theorem 4.5 and X_τ be given as in Theorem 4.1.*

Then for all non-zero real q and a.s. $\vec{\xi}\in\mathbb{B}^m$,

$$E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi}) = 1+\sum_{n=1}^{\infty}\frac{1}{n!}E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi}) = 1+\sum_{n=1}^{\infty}\sum_{q_1+\ldots+q_m=n}$$

$$\prod_{p=1}^{m}\Big\{\sum_{l_{p,1}+\ldots+l_{p,r_p}=q_p}\Big[\prod_{k=1}^{r_p}\frac{w_{p,k}^{l_{p,k}}}{l_{p,k}!}\Big]\int_{\mathscr{H}^{q_p}}\exp\Big\{\mathrm{i}\sum_{k=1}^{r_p}\sum_{\alpha=1}^{l_{p,k}}(h_{k,\alpha},[\vec{\xi}](\tau_{p,k}))^{\sim}\Big\}$$

$$\exp\Big\{\frac{1}{2q\mathrm{i}}\sum_{u=1}^{r_p+1}(\tau_{p,u}-\tau_{p,u-1})\Big|\sum_{k=1}^{u-1}\frac{t_{p-1}-\tau_{p,k}}{t_p-t_{p-1}}\sum_{\alpha=1}^{l_{p,k}}h_{k,\alpha}+\sum_{k=u}^{r_p}\frac{t_p-\tau_{p,k}}{t_p-t_{p-1}}\sum_{\alpha=1}^{l_{p,k}}h_{k,\alpha}\Big|^2\Big\}$$

$$\mathrm{d}\Big(\prod_{k=1}^{r_p}\prod_{\alpha=1}^{l_{p,k}}\sigma_{\tau_{p,k}}\Big)(\vec{h})\Big\},$$

where $w_{p,k}=w_{r_1+\ldots+r_{p-1}+k}$, with, upon introducing τ_p's, $\tau_{p,0}=t_{p-1}$, $\tau_{p,r_p+1}=t_p$, $0\leqslant\tau_{1,1}<\ldots<\tau_{1,r_1}\leqslant t_1<\ldots\leqslant t_{m-1}<\tau_{m,1}<\ldots<\tau_{m,r_m}\leqslant t_m=T$ and $r_1+\ldots+r_m=r$.

Corollary 4.8. *Let $\eta=\mu$ and $\theta\in\mathscr{G}^*$ be given by* (4.4). *Let F_n, F be given as in Theorem* 4.5 *and X_τ be given as in Theorem* 4.1.

Then for all non-zero real q and a.s. $\vec{\xi}\in\mathbb{B}^m$,

$$E^{\mathrm{an}\,f_q}[F|X_\tau](\vec{\xi}) = 1+\sum_{n=1}^{\infty}\frac{1}{n!}E^{\mathrm{an}\,f_q}[F_n|X_\tau](\vec{\xi})$$

$$=1+\sum_{n=1}^{\infty}\sum_{q_1+\ldots+q_m=n}\prod_{p=1}^{m}\Big[\int_{\Delta_{q_p}}\int_{\mathscr{H}^{q_p}}\exp\Big\{\mathrm{i}\sum_{k=1}^{q_p}(h_k,[\vec{\xi}](s_k))^{\sim}\Big\}$$

$$\exp\Big\{\frac{1}{2q\mathrm{i}}\sum_{k=1}^{q_p+1}(s_k-s_{k-1})\Big|\sum_{\alpha=1}^{k-1}\frac{t_{p-1}-s_\alpha}{t_p-t_{p-1}}h_\alpha+\sum_{\alpha=k}^{q_p}\frac{t_p-s_\alpha}{t_p-t_{p-1}}h_\alpha\Big|^2\Big\}$$

$$\mathrm{d}\Big(\prod_{k=1}^{q_p}\sigma_{s_k}\Big)(\vec{h})\,\mathrm{d}\Big(\prod_{k=1}^{q_p}\mu\Big)(\vec{s})\Big],$$

where $\Delta_{q_p}=\{(s_1,\ldots,s_{q_p})\colon\ 0\leqslant t_{p-1}=s_0<s_1<\ldots<s_{q_p}\leqslant s_{q_p+1}=t_p\}$.

By (4.6), we have the following results.

Theorem 4.9. *Let η be given as in Theorem* 4.5. *Let $\theta\in\mathscr{G}^*$ and $\psi\in\mathscr{F}(\mathbb{B})$ be given by* (4.4) *and* (4.5), *respectively. Let $G_n(x)=F_n(x)\psi(x(T))=\big[\int_0^T\theta(s,x(s))\,\mathrm{d}\eta(s)\big]^n\psi(x(T))$ and $G(x)=F(x)\psi(x(T))=\exp\big\{\int_0^T\theta(s,x(s))\,\mathrm{d}\eta(s)\big\}$ $\psi(x(T))$ where F_n, F are given as in Theorem* 4.5. *Let X_τ be given as in Theorem* 4.1.

Then for all non-zero real q and a.s. $\vec{\xi} \in \mathbb{B}^m$, $E^{\mathrm{an}\, f_q}[G_n|X_\tau](\vec{\xi})$ and $E^{\mathrm{an}\, f_q}[G|X_\tau](\vec{\xi})$ exist. Moreover,

$$\begin{aligned}
E^{\mathrm{an}\, f_q}[G|X_\tau](\vec{\xi}) &= \psi(\xi_m) + \sum_{n=1}^{\infty} \frac{1}{n!} E^{\mathrm{an}\, f_q}[G_n|X_\tau](\vec{\xi}) \\
&= \psi(\xi_m) + \sum_{n=1}^{\infty} \frac{1}{n!} E^{\mathrm{an}\, f_q}[F_n|X_\tau](\vec{\xi})\psi(\xi_m) \\
&= E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})\psi(\xi_m),
\end{aligned}$$

where $E^{\mathrm{an}\, f_q}[F_n|X_\tau](\vec{\xi})$ and $E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})$ are given as in Theorem 4.5.

Corollary 4.10. *Let η be given as in Corollary 4.7. Let $\theta \in \mathscr{G}^*$ and $\psi \in \mathscr{F}(\mathbb{B})$ be given by (4.4) and (4.5), respectively. Let G_n, G be given as in Theorem 4.9 and X_τ be given as in Theorem 4.1.*

Then for all non-zero real q and a.s. $\vec{\xi} \in \mathbb{B}^m$,

$$\begin{aligned}
E^{\mathrm{an}\, f_q}[G|X_\tau](\vec{\xi}) &= \psi(\xi_m) + \sum_{n=1}^{\infty} \frac{1}{n!} E^{\mathrm{an}\, f_q}[G_n|X_\tau](\vec{\xi}) \\
&= \psi(\xi_m) + \sum_{n=1}^{\infty} \frac{1}{n!} E^{\mathrm{an}\, f_q}[F_n|X_\tau](\vec{\xi})\psi(\xi_m) \\
&= E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})\psi(\xi_m),
\end{aligned}$$

where $E^{\mathrm{an}\, f_q}[F_n|X_\tau](\vec{\xi})$ and $E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})$ are given as in Corollary 4.7.

Corollary 4.11. *Let η be given as in Corollary 4.8. Let $\theta \in \mathscr{G}^*$ and $\psi \in \mathscr{F}(\mathbb{B})$ be given by (4.4) and (4.5), respectively. Let G_n, G be given as in Theorem 4.9 and X_τ be given as in Theorem 4.1.*

Then for all non-zero real q and a.s. $\vec{\xi} \in \mathbb{B}^m$,

$$\begin{aligned}
E^{\mathrm{an}\, f_q}[G|X_\tau](\vec{\xi}) &= \psi(\xi_m) + \sum_{n=1}^{\infty} \frac{1}{n!} E^{\mathrm{an}\, f_q}[G_n|X_\tau](\vec{\xi}) \\
&= \psi(\xi_m) + \sum_{n=1}^{\infty} \frac{1}{n!} E^{\mathrm{an}\, f_q}[F_n|X_\tau](\vec{\xi})\psi(\xi_m) \\
&= E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})\psi(\xi_m),
\end{aligned}$$

where $E^{\mathrm{an}\, f_q}[F_n|X_\tau](\vec{\xi})$ and $E^{\mathrm{an}\, f_q}[F|X_\tau](\vec{\xi})$ are given as in Corollary 4.8.

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