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# OSCILLATION AND NONOSCILLATION OF NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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*Abstract.* In this paper, oscillattion and nonoscillation criteria are established for neutral differential equations with positive and negative coefficients. Our criteria improve and extend many results known in the literature.

*Keywords*: oscillation, neutral differential equations, positive and negative coefficients

*MSC 2000*: 34K15, 34K40, 34C10

## 1. Introduction

Consider the neutral delay differential equation with positive and negative coefficients

$$[x(t)-R(t)x(t-r)]'+\sum_{i=1}^{m}P_i(t)x(t-\tau_i)-\sum_{j=1}^{n}Q_j(t)x(t-\sigma_j)=0,\quad t\geqslant t_0, \tag{1.1}$$

where $P_i, Q_j, R \in C([t_0,\infty),\mathbb{R}^+)$, $r\in(0,\infty)$ and $\tau_i,\sigma_j\in\mathbb{R}^+$ for $i=1,2,\dots,m$ and $j=1,2,\dots,n$.

When $m=n=1$, Eq. (1.1) reduces to

$$[x(t)-R(t)x(t-r)]'+P(t)x(t-\tau)-Q(t)x(t-\sigma)=0,\quad t\geqslant t_0, \tag{1.2}$$

where $P,Q,R\in C([t_0,\infty),\mathbb{R}^+)$, $r\in(0,\infty)$ and $\tau,\sigma\in\mathbb{R}^+$. In recent years, the oscillation of Eq. (1.2) has been investigated by many authors. See, for example, [2],

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[4]–[6], [9], [10], [12], [16] and the references cited therein. However, to the best of our knowledge, there is little in the way of results for the oscillation and nonoscillation of neutral differential equations with positive and negative coefficients with more than one delay.

Our aim in this paper is to establish oscillation and also nonoscillation criteria for Eq. (1.1). Our results improve and extend many results known in the literature.

The following assumptions will be used throughout the paper without further notice.

- ($A_1$) There exist a positive integer number $p \leqslant m$ and a partition of the set $\{1, 2, \ldots, n\}$ into $p$ disjoint subsets $J_1, J_2, \ldots, J_p$ such that $j \in J_i$ implies that $\sigma_j \leqslant \tau_i$;
- ($A_2$) $H_i(t) := P_i(t) - \sum\limits_{k \in J_i} Q_k(t - \tau_i + \sigma_k) \geqslant 0$ ($\not\equiv 0$) for $i = 1, 2, \ldots, p$, $H_i(t) := P_i(t)$ for $i = p+1, \ldots, m$;
- ($A_3$) $\varrho = \max\{r, \tau_i, \sigma_j \colon 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ and $\delta = \min\{r, \tau_i, \sigma_j \colon 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$.

A function $x(t) \in C([t_1 - \varrho, \infty), R)$ is said to be a solution of Equation (1.1) for some $t_1 \geqslant t_0$ if $x(t) - R(t)x(t-r)$ is continuously differentiable on $[t_1, \infty)$ and satisfies (1.1) for $t > t_1$.

As is customary, a solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

For convenience, we will assume that all inequalities concerning the values of functions are satisfied eventually for all large $t$.

## 2. Lemmas

We need the following lemmas for the proofs of our main results.

**Lemma 2.1.** *Assume that*

$$R(t) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\,\mathrm{d}s \leqslant 1. \tag{2.1}$$

*Let $x(t)$ be an eventually positive solution of the differential inequality*

$$[x(t) - R(t)x(t-r)]' + \sum_{i=1}^{m} P_i(t)x(t-\tau_i) - \sum_{j=1}^{n} Q_j(t)x(t-\sigma_j) \leqslant 0 \tag{2.2}$$

*and set*

$$z(t) = x(t) - R(t)x(t-r) - \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s. \tag{2.3}$$

*Then*

$$z'(t) \leqslant 0, \quad z(t) > 0, \quad \text{and} \quad z'(t) + \sum_{i=1}^{m} H_i(t)z(t-\tau_i) \leqslant 0. \tag{2.4}$$

P r o o f. Assume that $t_1 \geqslant t_0 + \varrho$ is such that $x(t)$ is positive for $t \geqslant t_1$. Then by (2.2) and (2.3), we get

$$\begin{aligned} z'(t) = & -\sum_{i=1}^{p} P_i(t)x(t-\tau_i) + \sum_{i=1}^{p}\sum_{k\in J_i} Q_k(t-\tau_i+\sigma_k)x(t-\tau_i) \\ & - \sum_{i=p+1}^{m} P_i(t)x(t-\tau_i). \end{aligned} \tag{2.5}$$

In view of $x(t) \geqslant z(t)$, (2.5) yields

$$z'(t) + \sum_{i=1}^{m} H_i(t)z(t-\tau_i) \leqslant 0.$$

Now we prove $z(t) > 0$. For otherwise, there would exist a $t_2 \geqslant t_1$ such that $z(t_2) \leqslant 0$. Then eventually $z(t) < 0$ because $z'(t) \leqslant 0$ and so there exist $t_3 \geqslant t_2$ and $\mu > 0$ such that $z(t) \leqslant -\mu$ for $t \geqslant t_3$. Hence

$$\begin{aligned} x(t) \leqslant & -\mu + R(t)x(t-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s \\ \leqslant & -\mu + \left(R(t) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\,\mathrm{d}s\right)\max_{t-\varrho\leqslant s\leqslant t} x(s) \\ \leqslant & -\mu + \max_{t-\varrho\leqslant s\leqslant t} x(s). \end{aligned}$$

Lemma 1.5.4 in [6] implies that $x(t)$ cannot be a nonnegative function on $[t_3, \infty)$, thus contradicting $x(t) > 0$. The proof is complete. □

**Lemma 2.2.** *Assume that*

$$R(t) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\,\mathrm{d}s \geqslant 1. \tag{2.6}$$

*Let $x(t)$ be an eventually positive solution of* (2.2) *and let $z(t)$ be defined by* (2.3). *Then the oscillation of all solutions of the second order ordinary differential equation*

$$y''(t) + \varrho^{-1}\sum_{i=1}^{m} H_i(t)y(t) = 0, \quad t \geqslant t_0 \tag{2.7}$$

*implies that $z'(t) \leqslant 0$ and $z(t) < 0$ eventually.*

P r o o f. From (2.5) we have

$$z'(t) \leqslant -\sum_{i=1}^{m} H_i(t)x(t-\tau_i) \leqslant 0. \tag{2.8}$$

Therefore, if $z(t) < 0$ does not hold eventually, then $z(t) > 0$ eventually. Let $t_1 > t_0+\varrho$ be such that $x(t-\varrho) > 0$, $z(t) > 0$ for $t \geqslant t_1$. Set $M = 2^{-1}\min\{x(t)\colon\ t_1 = \varrho \leqslant t \leqslant t_1\}$. Then $x(t) > M$ for $t_1 - \varrho \leqslant t \leqslant t_1$. We claim that

$$x(t) > M, \quad t \geqslant t_1. \tag{2.9}$$

If (2.9) does not hold, then there exists a $t^* > t_1$ such that $x(t) > M$ for $t_1-\varrho \leqslant t < t^*$ and $x(t^*) = M$. By (2.3) and (2.6) we get

$$\begin{aligned} M = x(t^*) &= z(t^*) + R(t)x(t-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s \\ &> \Bigl(R(t) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\,\mathrm{d}s\Bigr)M \geqslant M. \end{aligned}$$

This is a contradiction and so (2.9) holds. Let $\lim\limits_{t\to\infty} z(t) = a$. There exist two possible cases:

*Case I.* $a = 0$. There exists a $T_1 > t_1$ such that $z(t) < M/2$ for $t \geqslant T_1$. Then for any $\bar t > T_1$, we have

$$\frac{1}{\varrho}\int_{\bar t}^{t+\varrho} z(s)\,\mathrm{d}s \leqslant M < x(t), \quad t \in [\bar t, \bar t + \varrho].$$

*Case II.* $a > 0$. Then $z(t) \geqslant a$ for $t \geqslant t_1$. From (2.3) and (2.9) we get

$$x(t) \geqslant a + R(t)x(t-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s \geqslant a + M, \quad t \geqslant t_1.$$

By induction, it is easy to see that $x(t) \geqslant ka + M$ for $t \geqslant t_1 + (k-1)\varrho$ and so $\lim\limits_{t\to\infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\frac{1}{\varrho}\int_{T}^{t+\varrho} z(s)\,\mathrm{d}s \leqslant 2z(T) < x(t), \quad t \in [T, T+\varrho].$$

Combining the cases I and II we see that

$$x(t) > \frac{1}{\varrho}\int_{T}^{t+\varrho} z(s)\,\mathrm{d}s, \quad t \in [T, T+\varrho].$$

Now we prove that

$$x(t) > \frac{1}{\varrho}\int_T^{t+\varrho} z(s)\,\mathrm{d}s, \quad t \geqslant T+\varrho. \tag{2.10}$$

Otherwise, there would exist a $t^* > T + \varrho$ such that

$$\begin{aligned} x(t^*) &= \frac{1}{\varrho}\int_T^{t^*+\varrho} z(s)\,\mathrm{d}s,\\ x(t) &> \frac{1}{\varrho}\int_T^{t+\varrho} z(s)\,\mathrm{d}s \quad \text{for} \quad t \in (T+\varrho, t^*). \end{aligned}$$

Then, from (2.3) and (2.6), we have

$$\begin{aligned} \frac{1}{\varrho}\int_T^{t^*+\varrho} z(s)\,\mathrm{d}s &= z(t^*) + R(t^*)x(t^*-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t^*-\tau_i+\sigma_k}^{t^*} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s\\ &> \frac{1}{\varrho}\int_{t^*}^{t^*+\varrho} z(s)\,\mathrm{d}s + \Bigl(R(t^*) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t^*-\tau_i+\sigma_k}^{t^*} Q_k(s)\,\mathrm{d}s\Bigr)\frac{1}{\varrho}\int_T^{t^*} z(s)\,\mathrm{d}s\\ &\geqslant \frac{1}{\varrho}\int_T^{t^*+\varrho} z(s)\,\mathrm{d}s. \end{aligned}$$

This is a contradiction and so (2.10) holds. Thus, for $t > T + \varrho$, we obtain

$$x(t-\tau_i) > \frac{1}{\varrho}\int_T^{t} z(s)\,\mathrm{d}s. \tag{2.11}$$

Substituting (2.11) into (2.8) leads to

$$z'(t) + \sum_{i=1}^{m} H_i(t)\Bigl(\frac{1}{\varrho}\int_T^t z(s)\,\mathrm{d}s\Bigr) \leqslant 0, \quad t > T+\varrho.$$

Set

$$y(t) = \int_T^t z(s)\,\mathrm{d}s, \quad t > T+\varrho.$$

Then $y'(t) = z(t)$, $y''(t) = z'(t)$ and

$$y''(t) + \frac{1}{\varrho}\sum_{i=1}^{m} H_i(t)y(t) \leqslant 0, \quad t > T+\varrho.$$

By Lemma 2.4 in [11], Eq. (2.7) has an eventually positive solution. This is a contradiction and the proof is complete. □

**Lemma 2.3.** *Assume that*

$$R(t)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)\,\mathrm{d}s\equiv 1. \tag{2.12}$$

*Then the fact that the inequality* (2.2) *has an eventually positive solution* $x(t)$ *implies that Eq.* (1.1) *has a solution* $\bar{x}(t)$ *which satisfies* $0<\bar{x}(t)\leqslant x(t)$ *eventually.*

Proof. Let $z(t)$ be defined by (2.3). By Lemma 2.1 there exists a $t_1>t_0$ such that $x(t-\varrho)>0$, $z(t)>0$ and $z'(t)\leqslant 0$ for $t\geqslant t_1$. Set $M=2^{-1}\min\{x(t)\colon t_1=\varrho\leqslant t\leqslant t_1\}$. Then $x(t)>M$ for $t\geqslant t_1-\varrho$. From (2.3) and (2.4) we have

$$\begin{aligned} x(t)\geqslant R(t)x(t-r)&+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)x(s-\sigma_k)\,\mathrm{d}s\\ &+\int_t^{\infty}\sum_{i=1}^{m}H_i(s)x(s-\tau_i)\,\mathrm{d}s,\quad t\geqslant t_1. \end{aligned} \tag{2.13}$$

Define a sequence of functions $\{x_v(t)\}$ by $x_0(t)=x(t)$ and for $v=1,2,\ldots$ by

$$\begin{aligned} x_v(t)&=R(t)x_{v-1}(t-r)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)x_{v-1}(s-\sigma_k)\,\mathrm{d}s\\ &\quad+\int_t^{\infty}\sum_{i=1}^{m}H_i(s)x_{v-1}(s-\tau_i)\,\mathrm{d}s,\quad t\geqslant t_1+\varrho,\\ x_v(t)&=M+\frac{x_v(t_1+\varrho)-M}{x(t_1+\varrho)-M}(x(t)-M),\quad t_1\leqslant t<t_1+\varrho. \end{aligned} \tag{2.14}$$

Then, from (2.13) and (2.14), we have for $t\geqslant t_1+\varrho$

$$\begin{aligned} x_0(t)=x(t)&\geqslant x_1(t)\\ &=R(t)x(t-r)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)x(s-\sigma_k)\,\mathrm{d}s\\ &\quad+\int_t^{\infty}\sum_{i=1}^{m}H_i(s)x(s-\tau_i)\,\mathrm{d}s\\ &\geqslant\Big(R(t)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)\,\mathrm{d}s\Big)M=M. \end{aligned}$$

For $t_1\leqslant t<t_1+\varrho$ we have

$$x_0(t)=x(t)\geqslant M+\frac{x_1(t_1+\varrho)-M}{x(t_1+\varrho)-M}(x(t)-M)=x_1(t)\geqslant M.$$

Thus, $x_0(t) \geqslant x_1(t) \geqslant M$ for $t \geqslant t_1$. By induction, one can easily prove that

$$x_v(t) \geqslant x_{v+1}(t) \geqslant M, \quad t \geqslant t_1, \ \ v = 1, 2, \ldots$$

Therefore, $\{x_v(t)\}$ has a pointwise limit function $\bar{x}(t)$ with $0 < M \leqslant \lim_{v\to\infty} x_v(t) = \bar{x}(t) \leqslant x(t)$ for $t \geqslant t_1$. By the Monotone Convegence Theorem we have

$$\begin{aligned}\bar{x}(t) = R(t)\bar{x}(t-r) &+ \sum_{i=1}^{p} \sum_{k\in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\bar{x}(s-\sigma_k)\,\mathrm{d}s \\ &+ \int_t^\infty \sum_{i=1}^{m} H_i(s)\bar{x}(s-\tau_i)\,\mathrm{d}s, \quad t \geqslant t_1 + \varrho.\end{aligned}$$

This implies that

$$[\bar{x}(t) - R(t)\bar{x}(t-r)]' + \sum_{i=1}^{m} P_i(t)\bar{x}(t-\tau_i) - \sum_{j=1}^{n} Q_j(t)\bar{x}(t-\sigma_j) = 0, \quad t \geqslant t_1 + \varrho.$$

The proof is complete. □

**Lemma 2.4.** *Assume that* (2.12) *holds with* $\delta > 0$. *Then Eq.* (1.1) *has an eventually positive solution if the second order ordinary differential equation*

$$y''(t) + \delta^{-1} \sum_{i=1}^{m} H_i(t)y(t) = 0, \quad t \geqslant t_0 \tag{2.15}$$

*has an eventually positive solution.*

Proof. Let $y(t)$ be an eventually positive solution of (2.15). Then there exists a $t_1 > t_0$ such that $y(t) > 0$, $y''(t) \leqslant 0$ and $y'(t) > 0$ for $t \geqslant t_1$. Define a function $x(t)$ by

$$\begin{aligned}&x(t) = \delta^{-1}y(t_1), \quad t_1 \leqslant t \leqslant t_1 + \varrho - \delta, \\ &x(t) = \delta^{-1}[y(t_1) + (t - t_1 - \varrho + \delta)y'(t_1 + \varrho)], \quad t_1 + \varrho - \delta \leqslant t \leqslant t_1 + \varrho,\end{aligned}$$

and

$$\begin{aligned}x(t) = y'(t) + R(t)x(t-r) + \sum_{i=1}^{p} \sum_{k\in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s, \\ t_1 + \varrho + l\delta < t \leqslant t_1 + \varrho + (l+1)\delta, \quad l = 0, 1, \ldots.\end{aligned}$$

Then $x(t)$ is continuous and positive for $t \geqslant t_1$, and

$$(2.16) \quad y'(t) = x(t) - R(t)x(t-r) - \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s, \quad t \geqslant t_1.$$

Since $y'(t) > 0$ and $y''(t) \leqslant 0$, we have for $t_1 + \varrho - \delta \leqslant t \leqslant t_1 + \varrho$

$$y(t) - y(t_1) = y'(\xi)(t-t_1) \geqslant y'(t_1+\varrho)(t-t_1) \geqslant (t - t_1 - \varrho + \delta)y'(t_1+\varrho),$$

and so

$$x(t) \leqslant \frac{1}{\delta} y(t), \quad t_1 \leqslant t \leqslant t_1 + \varrho.$$

For $t_1 + \varrho \leqslant t \leqslant t_1 + \varrho + \delta$, we have

$$\begin{aligned} x(t) &= y'(t) + R(t)x(t-r) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)x(s-\sigma_k)\,\mathrm{d}s \\ &\leqslant \frac{1}{\delta}(y(t) - y(t-\delta)) + \left( R(t) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\,\mathrm{d}s \right) \frac{1}{\delta} y(t-\delta) \\ &= \frac{1}{\delta} y(t). \end{aligned}$$

By induction, one can prove in general that for $l = 0, 1, \ldots$

$$x(t) \leqslant \frac{1}{\delta} y(t), \quad t_1 + \varrho + l\delta < t \leqslant t_1 + \varrho + (l+1)\delta.$$

Therefore

$$x(t) \leqslant \frac{1}{\delta} y(t), \quad t \geqslant t_1$$

and so

$$(2.17) \qquad x(t-\tau_i) \leqslant \frac{1}{\delta} y(t-\tau_i) < \frac{1}{\delta} y(t), \quad t \geqslant t_1 + \varrho, \ \ i = 1, 2, \ldots, m.$$

Substituting (2.16) and (2.17) into (2.15) we obtain

$$[x(t) - R(t)x(t-r)]' + \sum_{i=1}^{m} P_i(t)x(t-\tau_i) - \sum_{j=1}^{n} Q_j(t)x(t-\sigma_j) \leqslant 0.$$

By Lemma 2.3, Eq. (1.1) has an eventually positive solution. The proof is complete. □

**Lemma 2.5** ([1], [7]). *Consider the ordinary differential equation*

$$y''(t) + p(t)y(t) = 0, \quad t \geqslant t_0, \tag{2.18}$$

*where* $p(t) \in C([t_0, \infty), \mathbb{R}^+)$. *Then*

(i) *All solutions of* (2.18) *oscillate if*

$$\liminf t \int_t^\infty p(s)\,\mathrm{d}s > \frac{1}{4}.$$

(ii) *Eq.* (2.18) *has an eventually positive solution if*

$$t \int_t^\infty p(s)\,\mathrm{d}s \leqslant \frac{1}{4} \quad \textit{for large } t.$$

## 3. Results and proofs

**Theorem 3.1.** *Assume that* (2.1) *holds,* $\tau_p = \max\{\tau_1, \tau_2, \ldots, \tau_m\}$ *and*

$$\limsup_{t\to\infty} \int_t^{t+\tau_p} H_p(s)\,\mathrm{d}s > 0.$$

*If*

$$\begin{aligned} \int_{t_0}^\infty \sum_{i=1}^m H_i(t) \ln\biggl[\mathrm{e}\biggl(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\biggr) \\ + 1 - \operatorname{sgn}\biggl(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\biggr)\biggr]\,\mathrm{d}t = \infty, \end{aligned} \tag{3.1}$$

*then all solutions of* (1.1) *oscillate.*

Proof. On the contrary, assume that (1.1) has an eventually positive solution $x(t)$ and let $z(t)$ be defined by (2.3). It follows from Lemma 2.1 that (2.4) holds. From Corollary 3.2.2 in [6], we have that the delay differential equation

$$y'(t) + \sum_{i=1}^m H_i(t)y(t-\tau_i) = 0 \tag{3.2}$$

has an eventually positive solution $y(t)$. Let $\lambda(t) = -y'(t)/y(t)$. Then $\lambda(t) \geqslant 0$ and it satisfies

$$\lambda(t) = \sum_{i=1}^{m} H_i(t) \exp\Big(\int_{t-\tau_i}^{t} \lambda(s)\,\mathrm{d}s\Big) \tag{3.3}$$

or

$$\lambda(t) \sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s = \sum_{i=1}^{m} H_i(t) \Big(\sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big) \cdot \exp\Big(\int_{t-\tau_i}^{t} \lambda(s)\,\mathrm{d}s\Big).$$

One can easily show that

$$\varphi(u)u\mathrm{e}^{x} \geqslant \varphi(u)x + \varphi(u)\ln(\mathrm{e}u + 1 - \operatorname{sgn} u) \quad \text{for } u \geqslant 0 \text{ and } x \in R, \tag{3.4}$$

where $\varphi(0) = 0$ and $\varphi(u) \geqslant 0$ for $u > 0$.

Employing inequality (3.4) on the right-hand side of (3.3) we get

$$\begin{aligned}\lambda(t) \sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s &\geqslant \sum_{i=1}^{m} H_i(t) \int_{t-\tau_i}^{t} \lambda(s)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{m} H_i(t) \ln\Big[\mathrm{e}\Big(\sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big) + 1 - \operatorname{sgn}\Big(\sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)ds\Big)\Big]\end{aligned}$$

or

$$\begin{aligned}\lambda(t) \sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s - \sum_{i=1}^{m} H_i(t) \int_{t-\tau_i}^{t} \lambda(s)\,\mathrm{d}s &\geqslant \sum_{i=1}^{m} H_i(t) \ln\Big[\mathrm{e}\Big(\sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big) \\ &\quad + 1 - \operatorname{sgn}\Big(\sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big)\Big].\end{aligned} \tag{3.5}$$

Then for $N > T$

$$\begin{aligned}\int_{T}^{N} \lambda(t) \sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s\,\mathrm{d}t - \int_{T}^{N} \sum_{i=1}^{m} H_i(t) \int_{t-\tau_i}^{t} \lambda(s)\,\mathrm{d}s\,\mathrm{d}t &\geqslant \int_{T}^{N} \sum_{i=1}^{m} H_i(t) \ln\Big[\mathrm{e}\Big(\sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big) \\ &\quad + 1 - \operatorname{sgn}\Big(\sum_{i=1}^{m} \int_{t}^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big)\Big]\,\mathrm{d}t.\end{aligned} \tag{3.6}$$

By interchanging the order of integration, we find that

$$\int_T^N H_i(t) \int_{t-\tau_i}^t \lambda(s)\,\mathrm{d}s\,\mathrm{d}t \geqslant \int_T^{N-\tau_i} \int_s^{s+\tau_i} H_i(t)\lambda(s)\,\mathrm{d}t\,\mathrm{d}s = \int_T^{N-\tau_i} \lambda(t) \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\,\mathrm{d}t. \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\begin{aligned}\sum_{i=1}^m \int_{N-\tau_i}^N \lambda(t) \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\,\mathrm{d}t \\ \geqslant \int_T^N \sum_{i=1}^m H_i(t) \ln\Big[\mathrm{e}\Big(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big) \\ + 1 - \operatorname{sgn}\Big(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big)\Big]\,\mathrm{d}t.\end{aligned} \tag{3.8}$$

On the other hand, since (3.2) has an eventually positive solution, by Lemma 2 in [8] we have

$$\int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s < 1, \quad i = 1, 2, \ldots, m \tag{3.9}$$

eventually. Then by (3.8) and (3.9) we obtain

$$\begin{aligned}\sum_{i=1}^m \int_{N-\tau_i}^N \lambda(t)dt \geqslant \int_T^N \sum_{i=1}^m H_i(t) \ln\Big[\mathrm{e}\Big(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big) \\ + 1 - \operatorname{sgn}\Big(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big)\Big]\,\mathrm{d}t\end{aligned}$$

or

$$\begin{aligned}\sum_{i=1}^m \ln \frac{y(N-\tau_i)}{y(N)} \geqslant \int_T^N \sum_{i=1}^m H_i(t) \ln\Big[\mathrm{e}\Big(\sum_{i=1}^n \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big) \\ + 1 - \operatorname{sgn}\Big(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s\Big)\Big]\,\mathrm{d}t.\end{aligned}$$

By the assumption

$$\lim_{t\to\infty} \prod_{i=1}^m \frac{y(t-\tau_i)}{y(t)} = \infty.$$

This implies

$$\lim_{t\to\infty} \frac{y(t-\tau_p)}{y(t)} = \infty. \tag{3.10}$$

However, by Lemma 1 in [8] we have

$$\liminf_{t\to\infty} \frac{y(t-\tau_p)}{y(t)} < \infty.$$

This contradicts (3.10) and completes the proof. □

**Remark 3.1.** We note that when $R(t) \equiv 0$, Theorem 3.1 improves Theorem 3.2 in [3] because the condition $\sum_{i=1}^{m} \int_t^{t+\tau_i} H_i(s)\,\mathrm{d}s > 0$ is no longer required.

**Theorem 3.2.** *Assume that* (2.12) *holds and that*

$$\liminf_{t\to\infty} t \int_t^{\infty} \sum_{i=1}^{m} H_i(s)\,\mathrm{d}s > \frac{\varrho}{4}. \tag{3.11}$$

*Then all solutions of* (1.1) *oscillate.*

Proof. Suppose that Eq. (1.1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (2.3). Then by Lemma 2.1 we have $z(t) > 0$ eventually. On the other hand, by Lemma 2.5, (3.11) implies that all solutions of Eq. (2.7) oscillate. By Lemma 2.2, it follows that $z(t) < 0$. This contradiction completes the proof. □

**Theorem 3.3.** *Assume that* (2.6) *and* (3.11) *hold and that*

$$R(t-\tau_i)H_i(t) \leqslant hH_i(t-r), \quad i = 1, 2, \ldots, m. \tag{3.12}$$

*Also suppose that $H_i(t)/Q_j(t-\tau_i+\sigma_j)$ is nonincreasing and satisfies*

$$H_i(t)Q_j(t-\tau_i) \leqslant h_j H_i(t-\sigma_j), \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \tag{2.13}$$

*where $h$, $h_j$ $(j = 1, 2, \ldots, n)$ are nonnegative constants satisfying*

$$h + \sum_{i=1}^{p} \sum_{k\in J_i} h_k(\tau_i - \sigma_k) = 1. \tag{3.14}$$

*Then every solution of* (1.1) *oscillates.*

P r o o f. Assume the contrary. Eq. (1.1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (2.3). Then by Lemma 2.2 we have $z(t) < 0$ eventually. From (2.8), (3.12) and (3.13) we have

$$\begin{aligned}
z'(t) &\leqslant -\sum_{i=1}^{m} H_i(t)x(t-\tau_i)\\
&= -\sum_{i=1}^{m} H_i(t)[z(t-\tau_i) + R(t-\tau_i)x(t-r-\tau_i)\\
&\quad + \sum_{l=1}^{p}\sum_{k\in J_l}\int_{t-\tau_l+\sigma_k}^{t} Q_k(s-\tau_i)x(s-\tau_i-\sigma_k)\,\mathrm{d}s]\\
&\geqslant -\sum_{i=1}^{m} H_i(t)z(t-\tau_i) - h\sum_{i=1}^{m} H_i(t-r)x(t-r-\tau_i)\\
&\quad -\sum_{l=1}^{p}\sum_{k\in J_l}\sum_{i=1}^{m} h_k\frac{H_i(t-\sigma_k)}{Q_k(t-\tau_i)}\int_{t-\tau_l+\sigma_k}^{t} Q_k(s-\tau_i)x(s-\tau_i-\sigma_k)\,\mathrm{d}s\\
&\geqslant -\sum_{i=1}^{m} H_i(t)z(t-\tau_i) + hz'(t-r)\\
&\quad -\sum_{l=1}^{p}\sum_{k\in J_l} h_k\sum_{i=1}^{m}\int_{t-\tau_l+\sigma_k}^{t} H_i(s-\sigma_k)x(s-\tau_i-\sigma_k)\,\mathrm{d}s\\
&= -\sum_{i=1}^{m} H_i(t)z(t-\tau_i) + hz'(t-r) + \sum_{l=1}^{p}\sum_{k\in J_l} h_k\int_{t-\tau_l+\sigma_k}^{t} z'(s-\sigma_k)\,\mathrm{d}s\\
&= -\sum_{i=1}^{m} H_i(t)z(t-\tau_i) + hz'(t-r) + \sum_{j=1}^{n} h_j z(t-\sigma_j) - \sum_{l=1}\sum_{k\in J_l} h_k z(t-\tau_l).
\end{aligned}$$

Define $\overline{P}_i(t)$ by

$$\begin{aligned}
\overline{P}_i(t) &= H_i(t) + \sum_{k\in J_i} h_k, \quad i = 1,2,\ldots,p,\\
\overline{P}_i(t) &= H_i(t), \quad i = p+1, p+2, \ldots, m.
\end{aligned}$$

We obtain

$$[z(t) - hz(t-r)]' + \sum_{i=1}^{m}\overline{P}_i(t)z(t-\tau_i) - \sum_{j=1}^{n} h_j z(t-\sigma_j) \geqslant 0.$$

This implies that $-z(t)$ is a positive solution of the inequality

$$[y(t) - hy(t-r)]' + \sum_{i=1}^{m}\overline{P}_i(t)y(t-\tau_i) - \sum_{j=1}^{n} h_j y(t-\sigma_j) \leqslant 0,$$

which yields a contradiction by Lemmas 2.1 and 2.2. The proof is complete. □

Next we give a criterion for nonoscillation.

**Theorem 3.4.** *Assume that* (2.12) *holds with* $\delta > 0$ *and that*

$$t\int_t^{\infty}\sum_{i=1}^{m} H_i(s)\,\mathrm{d}s \leqslant \frac{\delta}{4} \quad \text{for large } t. \tag{3.17}$$

*Then Eq.* (1.1) *has an eventually positive solution.*

Proof. The conclusion of Theorem 3.4 is an immediate consequence of Lemma 2.4 and Lemma 2.5. □

**Example** ([14]). Consider the equation

$$[x(t) - (1-\alpha)x(t-r)]' + (\alpha + t^{-\beta})x(t-\tau) - \alpha x(t-\sigma) = 0, \quad t \geqslant 1, \tag{3.18}$$

where $0 \leqslant \alpha < 1$, $-\infty < \beta \leqslant 2$, $\tau = \sigma + 1$, $\sigma > 0$ and $r > 0$. All conditions of Theorem 3.2 are satisfied when $-\infty < \beta < 2$ or $\beta = 2$ and $\varrho < 4$. Thus, all solutions of (3.18) oscillate when $-\infty < \beta < 2$ or $\beta = 2$ and $\varrho < 4$. On the other hand, by Theorem 3.4, Eq. (3.18) has an eventually positive solution when $\beta > 2$ or $\beta = 2$ and $\delta \geqslant 4$.

**Remark 3.2.** It should be noted that condition (11) in [14] is not satisfied for Eq. (3.18) when $3/2 < \beta \leqslant 2$. Thus our condition (3.11) is better than condition (11) in [14], and so Theorem 3.2 and Theorem 3.3 improve and extend Theorem 1 and Theorem 3 in [14], respectively.

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