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BANACH AND STATISTICAL CORES OF BOUNDED SEQUENCES

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Abstract. In this paper, we are mainly concerned with characterizing matrices that map every bounded sequence into one whose Banach core is a subset of the statistical core of the original sequence.

Keywords: almost convergent sequence, statistically convergent sequence, core of a sequence

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1. Introduction

If $T = (t_{nk})$ is an infinite matrix with real entries, and if $x = (x_k)$ is a sequence of real numbers, then Tx denotes the transformed sequence whose n-th term is given by $(Tx)_n = \sum\limits_{k=1}^{\infty} t_{nk}x_k$. In order to investigate the effect of such transformations upon the derived set, Knopp [14] introduced the idea of the core ($\mathcal{K}$-core) of a sequence and proved the well-known Core Theorem. That theorem asserts that $\mathcal{K}$-core$\{Tx\} \subseteq$ $\mathcal{K}$-core$\{x\}$, whenever Tx exists for the nonnegative regular matrix T. Some variants of the Core Theorem may be found in [4], [19], [23], [26].

Considering the method of almost convergence Loone [17] and Das [4] introduced the Banach core ($\mathcal{B}$-core) of a bounded sequence and proved some analogues of the assertions for the $\mathcal{K}$-core (see also [12], [23], [26], [27]).

In [10], [11], the notion of statistical core of a sequence is introduced and a statistical core theorem is proved.

Section 2 of the present paper presents a result which is complementary to [17] and [23], while Section 3 deals with characterizing matrices that map every bounded sequence into one whose $\mathcal{B}$-core is a subset of the statistical core of the original sequence. Before proceeding further we recall some notation and terminology. By l^∞ and c we denote the spaces of all bounded and convergent real sequences, respectively.

Let $T = (t_{nk})$ be an infinite matrix, and let X and Y be two sequence spaces. If Tx exists for each $x \in X$ and $Tx \in Y$ then we say that T maps X into Y. The set of matrices that map X into Y is denoted by (X, Y). The set of matrices that map X into Y and leave the limit or sum invariant is denoted by $(X, Y; p)$.

For example, if $T \in (c, c; p)$, then $\lim Tx = \lim x$ for every $x \in c$. In this case T is called regular (see [3], [24]). If it is regular and satisfies $\lim_n \sum_k |t_{nk} - t_{n,k+1}| = 0$, then T is called strongly regular [24].

2. $\mathcal{B}$-core and absolute equivalence

This section is complementary to [23] and [17]. It is well-known [18], [24] that the functional

$$q(x) = \inf_{n_1, n_2, \ldots, n_r} \limsup_k \frac{1}{r} \sum_{i=1}^{r} x_{k+n_i}$$

is sublinear on l^∞. We also consider the following functionals on l^∞:

$$\begin{aligned} L(x) &= \limsup x_n, \\ l^*(x) &= \liminf_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r, \\ L^*(x) &= \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r. \end{aligned}$$

It follows from the Corollary to Theorem 1 of [5] that $q(x) = L^*(x)$. If $q(x) = -q(-x) = s$, then x is called almost convergent to s [18], and in this case we write $F\text{-}\lim x = s$. By F we denote the set of all almost convergent sequences.

The Banach core ($\mathcal{B}$-core) of a bounded sequence x is defined to be the closed interval $[-q(-x), q(x)]$ (see Loone [17], Das [4]). Since $q(x) \leqslant L(x)$ for every $x \in l^\infty$, it follows that $\mathcal{B}\text{-core}\{x\} \subseteq \mathcal{K}\text{-core}\{x\}$ where $\mathcal{K}\text{-core}\{x\}$ is the Knopp core and it is given by $\mathcal{K}\text{-core}\{x\} = [\liminf x, \limsup x]$. It is shown in [23], [17] that

$$\mathcal{K}\text{-core}\{Ax\} \subseteq \mathcal{B}\text{-core}\{x\} \quad (\text{for every } x \in l^\infty)$$

if and only if A is strongly regular and $\lim_n \sum_k |a_{nk}| = 1$.

Now we have the following

Theorem 1. *Let $x \in l^\infty$ and let A be a strongly regular matrix. Then $\mathcal{K}$-core$\{Ax\} \subseteq \mathcal{B}$-core$\{x\}$ if and only if A is absolutely equivalent to a non-negative strongly regular matrix B for all bounded sequences.*

Proof. *Sufficiency.* Since A is absolutely equivalent to a nonnegative strongly regular matrix B, we have

$$\lim_n \{(Ax)_n - (Bx)_n\} = 0 \quad (\text{for every } x \in l^\infty). \tag{1}$$

Now Theorem 6.5.I of Cooke [3] implies that

$$\mathcal{K}\text{-core}\{Ax\} \subseteq \mathcal{K}\text{-core}\{x\}, \quad (\text{for every } x \in l^\infty). \tag{2}$$

Since B is a non-negative strongly regular matrix, it follows from Theorem 3 of [23] that, for every $x \in l^\infty$,

$$\mathcal{K}\text{-core}\{Bx\} \subseteq \mathcal{B}\text{-core}\{x\}. \tag{3}$$

Since (1) holds, Theorem 6.3.II of Cooke [3] implies that

$$\mathcal{K}\text{-core}\{Ax\} = \mathcal{K}\text{-core}\{Bx\}. \tag{4}$$

Now (3) and (4) imply $\mathcal{K}$-core$\{Ax\} \subseteq \mathcal{B}$-core$\{x\}$.

Necessity. Let $x \in l^\infty$ and let A be a strongly regular matrix. By hypothesis,

$$\mathcal{K}\text{-core}\{Ax\} \subseteq \mathcal{B}\text{-core}\{x\} \subseteq \mathcal{K}\text{-core}\{x\}. \tag{5}$$

Now, there is a non-negative regular matrix B such that A and B are absolutely equivalent on l^∞ (see Theorem 6.5.I of [3]). So, by Theorem 5.4.I of Cooke [3], we have

$$\lim_n \sum_k |b_{nk} - a_{nk}| = 0. \tag{6}$$

It remains to show that

$$\lim_n \sum_k |b_{nk} - b_{n,k+1}| = 0. \tag{7}$$

To see this, we first write

$$\begin{aligned}\sum_k |b_{nk} - b_{n,k+1}| &\leqslant \sum_k |b_{nk} - a_{nk}| + \sum_k |a_{n,k+1} - b_{n,k+1}| + \sum_k |a_{nk} - a_{n,k+1}| \\ &= c_n^1 + c_n^2 + c_n^3, \text{ say.}\end{aligned}$$

By (6), $c_n^1 \to 0$ $(n \to \infty)$. By the strong regularity of A, $c_n^3 \to 0$ $(n \to \infty)$, and by the absolute equivalence

$$c_n^2 = \sum_k |a_{n,k+1} - b_{n,k+1}| \leqslant \sum_k |a_{nk} - b_{nk}| \to 0 \quad (n \to \infty),$$

hence (7) holds. This proves the theorem. □

3. Statistical and Banach cores

If $K \subseteq \mathbb{N}$ then let $K_n := \{k \in K\colon\ k \leqslant n\}$; and $|K_n|$ will denote the cardinality of K_n. The natural density [22] of K is given by $\delta(K) := \lim_n n^{-1}|K_n|$, if it exists.

In [9] a statistical cluster point of a sequence x is defined as a number γ such that for every $\varepsilon > 0$ the set $\{k \in \mathbb{N}\colon\ |x_k - \gamma| < \varepsilon\}$ does not have density zero. In [10] the sequence x is defined to be statistically bounded if x has a bounded subsequence of density one; and the statistical core of such an x of real values is the closed interval $[\text{st-}\liminf x, \text{st-}\limsup x]$, where $\text{st-}\liminf x$ and $\text{st-}\limsup x$ are the least and greatest statistical cluster points of x (see [6], [10], [11], [16]). Recall [10] that, for a sequence x the number β is the $\text{st-}\limsup x$ if and only if for every $\varepsilon > 0$,

$$\delta\{k\colon\ x_k > \beta - \varepsilon\} \neq 0 \quad \text{and} \quad \delta\{k\colon\ x_k > \beta + \varepsilon\} = 0.$$

The dual statement for $\text{st-}\liminf x$ is as follows: The number α is the $\text{st-}\liminf x$ if and only if for every $\varepsilon > 0$,

$$\delta\{k\colon\ x_k < \alpha + \varepsilon\} \neq 0 \quad \text{and} \quad \delta\{k\colon\ x_k < \alpha - \varepsilon\} = 0.$$

A statistically bounded sequence x is statistically convergent if and only if $\text{st-}\limsup x = \text{st-}\liminf x$ [10]. Some results on statistical convergence may be found in [2], [8], [9], [10], [20], [21], [25].

In this section we are mainly concerned with characterizing matrices that map every bounded sequence into one whose $\mathcal{B}$-core is a subset of the statistical core of the original sequence. The final result follows a result of Choudhary [1] in giving conditions on matrices T and H so that the Banach core of Tx is contained in the statistical core of Hx.

We note that statistical convergence and almost convergence are incomparable [21].

By $\text{st}(b)$ we denote the set of all bounded statistically convergent sequences. It follows from Theorem 4.1 of [15] that $T \in (\text{st}(b), F; p)$ if and only if $T \in (c, F; p)$ and $T^{[K]} \in (l^\infty, F)$ for every K of density zero where $T^{[K]} = (d_{nk})$ is given by $d_{nk} = t_{nk}$ if $k \in K$ and $d_{nk} = 0$ otherwise.

By [13] and [7], this is equivalent to the following

Proposition 2. $T \in (\mathrm{st}(b), F; p)$ *if and only if*

(i) $\sup_n \sum_k |t_{nk}| < \infty$,

(ii) $F\text{-}\lim t_{nk} = 0$ *for every* k,

(iii) $F\text{-}\lim_n \sum_k t_{nk} = 1$, *and*

(iv) $\lim_r \sum_{k\in K} \left| \frac{1}{r+1} \sum_{i=1}^{r} t_{n+i,k} \right| = 0$, *uniformly in* n *for every* K *of density zero.*

Now we have

Theorem 3. *Let* $T\colon\ l^\infty \to l^\infty$ *and* $\beta(x) := \mathrm{st}\text{-}\limsup x$. *Then*

$$L^*(Tx) \leqslant \beta(x) \quad (\text{for every } x \in l^\infty), \tag{8}$$

if and only if

(a) $T \in (\mathrm{st}(b), F; p)$,

(b) $\lim_r \sum_{k=1}^{\infty} \left| \frac{1}{r+1} \sum_{i=1}^{r} t_{n+i,k} \right| = 1$, *uniformly in* n.

P r o o f. Assume (8) holds and $x \in l^\infty$. Then $Tx \in l^\infty$; and also we have

$$-\beta(-x) \leqslant -L^*(-Tx) \leqslant L^*(Tx) \leqslant \beta(x).$$

If $x \in \mathrm{st}(b)$, then $\beta(x) = -\beta(-x)$, hence T maps $\mathrm{st}(b)$ into F and $F\text{-}\lim Tx = \mathrm{st}\text{-}\lim x$, which proves (a). To prove (b), we first observe that Proposition 2 implies the conditions of Lemma 2 of [4]. Hence, from that Lemma, there is a bounded sequence x such that $\|x\|_\infty := \sup_k |x_k| \leqslant 1$ and

$$\limsup_n \sup_i \sum_k b_{nk}(i) x_k = \limsup_n \sup_i \sum_k |b_{nk}(i)| \tag{9}$$

where $b_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk}$.

Hence, by Proposition 2,

$$\begin{aligned} 1 = \liminf_n \sup_i \sum_k b_{nk}(i) &\leqslant \liminf_n \sup_i \sum_k |b_{nk}(i)| \\ &\leqslant \limsup_n \sup_i \sum_k |b_{nk}(i)| \\ &= \limsup_n \sup_i \sum_k b_{nk}(i) x_k, \quad \text{by (9)} \\ &\leqslant \beta(x), \quad \text{by hypothesis} \\ &\leqslant \|x\|_\infty \leqslant 1 \end{aligned}$$

from which we get (b).

Conversely assume (a) and (b) hold, and let $x \in l^\infty$. Then $Tx \in l^\infty$ and $\beta(x)$ is finite. Given $\varepsilon > 0$, let $E := \{k\colon\ x_k > \beta(x) + \varepsilon\}$. Hence $\delta(E) = 0$, and if $k \notin E$ then $x_k \leqslant \beta(x) + \varepsilon$. For any real number z we write $z^+ := \max\{z, 0\}$ and $z^- := \max\{-z, 0\}$, whence

$$|z| = z^+ + z^-, \quad z = z^+ - z^-, \quad |z| - z = 2z^-.$$

Letting

$$\begin{aligned}
b_{rk}(i) &:= \frac{1}{r+1}\sum_{n=i}^{i+r} t_{nk},\\
(b_{rk}(i))^+ &:= \frac{1}{r+1}\sum_{n=i}^{i+r} t_{nk}^+,\\
(b_{rk}(i))^- &:= \frac{1}{r+1}\sum_{n=i}^{i+r} t_{nk}^-,
\end{aligned}$$

then for a fixed positive integer m we write

$$\begin{aligned}
\frac{1}{r+1}\sum_{n=i}^{i+r}(Tx)_n = {} & \sum_{k<m} b_{rk}(i)x_k + \sum_{\substack{k\geqslant m\\ k\in E}} (b_{rk}(i))^+ x_k\\
& + \sum_{\substack{k\geqslant m\\ k\notin E}} (b_{rk}(i))^+ x_k - \sum_{k\geqslant m} (b_{rk}(i))^- x_k\\
\leqslant {} & \|x\|_\infty \sum_{k<m} |b_{rk}(i)| + (\beta(x)+\varepsilon) \sum_{k\geqslant m} |b_{rk}(i)|\\
& + \|x\|_\infty \sum_{k\geqslant m} |b_{rk}(i)| + \|x\|_\infty \sum_{k\geqslant m} (|b_{rk}(i)| - b_{rk}(i)).
\end{aligned}$$

On applying the operator $\limsup_r \sup_i$ and considering Proposition 2, we get

$$L^*(Tx) \leqslant \beta(x) + \varepsilon.$$

Since ε is arbitrary we conclude that (8) holds, whence the result. $\square$

Similarly we could get $\alpha(x) \leqslant l^*(Tx)$, and hence we have the following result.

Theorem 4. *If $T\colon\ l^\infty \to l^\infty$, then*

$$\mathcal{B}\text{-core}\{Tx\} \subseteq \text{st-core}\{x\} \quad \textit{for every } x \in l^\infty$$

if and only if conditions (a) *and* (b) *of Theorem 3 hold.*

Theorem 5. *Let H be a triangular matrix with non-zero diagonal entries, and denote its triangular inverse by H^{-1}. For an arbitrary matrix T, in order that, whenever $Hx \in l^\infty$, Tx should exist and be bounded and satisfy*

$$\mathcal{B}\text{-core}\{Tx\} \subseteq \text{st-core}\{Hx\}, \tag{10}$$

it is necessary and sufficient that

(i) $C := TH^{-1}$ *exists;*

(ii) $C \in (\text{st}(b), f; p)$*;*

(iii) $\lim_r \sum_{k=1}^{\infty} \left| \frac{1}{r+1} \sum_{i=1}^{r} c_{n+i,k} \right| = 1$*;*

(iv) *for any fixed* n,

$$\lim_\nu \sum_{k=0}^{\nu} \left| \sum_{j=\nu+1}^{\infty} t_{nj} h_{jk}^{-1} \right| = 0.$$

Proof. *Necessity.* If $(Tx)_n$ exists for each n whenever $Hx \in l^\infty$, then by Lemma 2 of Choudhary [1], (i) and (iv) hold. By the same Lemma, we also have $Tx = Cy$ where $y = Hx$. By hypothesis $Tx \in l^\infty$ hence $Cy \in l^\infty$. Now (10) implies that $\mathcal{B}$-core$\{Cy\} \subseteq$ st-core$\{y\}$. By Theorem 4, we get (ii) and (iii).

Sufficiency. Conditions (i)–(iv) imply the conditions of Lemma 2 of Choudhary [1]; so, it follows from that Lemma that $Cy \in l^\infty$, and hence $Tx \in l^\infty$. Now Theorem 4 yields that $\mathcal{B}$-core$\{Cy\} \subseteq$ st-core$\{y\}$, and since $y = Hx$ and $Cy = Tx$, we have $\mathcal{B}$-core$\{Tx\} \subseteq$ st-core$\{Hx\}$, whence the result. □

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